On some applications of the Eisenstein series in String Theory. Mathematical connections with some sectors of Number Theory and with Φ and π.

"A mathematician is a person who can see analogies between theorems; a good mathematician is a person who can see the analogies between the demonstrations and a very good mathematician can see the analogies between the theories. We can surmise that the best mathematician is one who sees analogies between the analogies".
(Stefan Banach)

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Introduction

Universal music system based on Phi (Lange Christian)

Christian Lange has studied with much interest mathematical proportion showed by nature noticing especially the golden mean relationship Phi=1,6180339... because it's present everywhere in nature beginning from the atom going up to the galaxies representing the divine proportion of universal harmony. This relationship represent even the beauty of natural harmonic geometries. The question is, why this proportion is not present in the scale of western music when we consider that the music should be the highest artistic expression of natural harmony ? The official answer is, that a string of a music instrument divided by the proportion of Phi never would match with the natural spectrum of overtones based on fractions of little natural numbers like 2/1, 3/1 ,3/2, 4/3. Accords or music intervals according to this fractions would be considered harmonic and graceful by our ears. Under a physic profile, the Phi-Interval is very far away from corresponding with numbers based on fractions of little numbers and because of this fact, who goes to built up a Phi-based music system is obstacle by the opinion of the "experts" in the field of musical harmonics. It's possible to create a Phi based music system ? It does make sense from a musical point of view ?

The answer is yes if we are able to jump the obstacles of superficial evidence entering the miraculous connections that the number of golden mean section offer. The first element to take care of is, that every music system has to be an exponential system (on the keyboard of a piano every 12 keys we get music notes with double frequency) because our hear- sense is based on a logarithmic scale.

In a very simple way we can use the powers of Phi to get a series of Phi-based numbers following an exponential curve

1,618x1,618=Phi^2=2,618;1,618x1,618x1,618=Phi^3=4,236;1,618x1,618x1,618x1,618=Phi^4=6,854…

Now we have to fill the spaces between a Phi-power and the next because under a musical point of view the Phi-interval is almost large and we some more keys in between to compose music. Under a mathematician point of view this means that we have to divide the space of a Phi-Interval in a natural number of parts that we can decide freely. For example we can divide this space in 9 parts using $1,618^{(n/9)}$ where n is a natural number, even negative. For n=9 the exponent is 9/9=1 getting Phi^1=1,618. For n=13 happens an interesting thing: $Phi^{13/9}$=2,003876 a value very near of 2 that is corresponding to the harmonic music interval that doubles the frequency of a base frequency called Octave in traditional music.

This little example shows, using Phi as a base for an exponential music system, that we can obtain a proportion that respects the natural over-or undertones of a vibrating string but we have to go beyond the single interval of the octave.

If we choose to divide the Phi interval in 7 parts, we obtain optimal connection with the number 3 (3 times the frequency of basic sound) and with number 2 because $Phi^{16/7}$=3,0039 and $Phi^{10/7}$=1,9886. In addition we get connections based on the combination of 2 and 3: 2/3=0,666 e 3/2=1,5 corresponding to natural harmonic music interval of the vibrating string. The system with 7 parts in a Phi-interval permits us to get these music-intervals 2, 3, 1.5, 0.666 in an approximated manner.

Would it be possible to get the exact numbers of natural harmonic intervals of a vibrating string ? Observing in an accurate way the powers of Phi, we notice that it's possible to obtain every natural number exactly by adding powers of Phi remembering that the traditional harmonic intervals are based on fractions of little entire numbers. More little are the natural numbers creating the fraction(1,2,3,4..) , more harmonic sounds the music interval like ¼, ½, 1/3, 2/3, 3/4,4/3, 3/2, 2/1, 3/1, 4/1.

Here we see how to built the first 4 numbers by adding Phi-Powers:

Phi^{-1}+Phi^{-2}=Phi^0=0,61803399+0,38196601=1,00000000

Phi^1+Phi^{-2}=1,61803399+0,38196601=2,00000000

Phi^2+Phi^{-2}=2,61803399+0,38196601=3,00000000

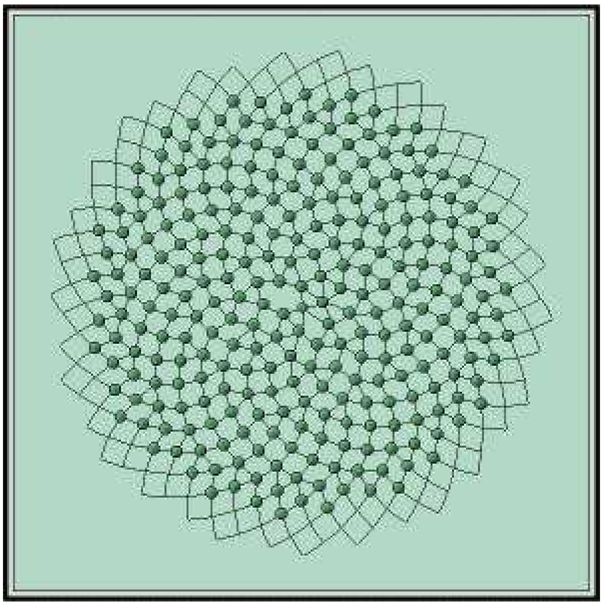
Phi^2+Phi^{-2}+Phi^0=2,61803399+0,38196601+1,00000000=4,00000000

We can use the technique of adding Phi-powers to get natural numbers with absolute precision, no approximation and we can use these precise numbers systematically instead of the approximated values of the curve getting by $Phi^{(n/7)}$ creating a perfect system of embedding all the harmonic intervals of music with perfect Phi-based logic for every music note of the system. The difference of the exact frequencies from the original curve $Phi^{(n/7)}$ is less than 1%.

Obviously we can represent the values of the Phi-system like a golden mean spiral that would be like a Nautilus.

In his book "*432 Hertz: the music revolution. Golden mean tuning for biological music*", Riccardo Tristano Tuis writes: "*If we could hear the music based on the golden mean spiral it would be in a certain manner the music for life, on a biological level but even on a perceptive level, because it would use the same math of both*". Tuis continues citing LaRouche from the Schiller Institute :"*There is nothing mysterious or mystic around the introduction of the golden section as absolute value of the life process*" in reference to music. Then he writes: "*The perfect music scale* (the moderated scale is it not) *is the one with the proportions of the frequencies of the music notes one from another based exactly on the golden section with the intonation register based also on it*". In the same book Tuis public a music scale based on 12 notes per octave but he doesn't found the perfect Phi interval for all the music notes. In every case it is appreciable to see his honest effort to search the truth about universal music. In the last version of the Phi based music system we have the natural frequencies indicated by Tuis like 432Hz, 288Hz, 216Hz, 144Hz, 72Hz according to this choose.

After discussing the harmonic intervals of the Phi based music system, we have to examine the Phi interval closer. We understand that it's possible to create the harmonic intervals by adding powers of Phi but the Phi-interval itself is pleasant ? Considering the harmonic laws of overtones related to the vibrating string, the Phi-interval should be horrible for the hearer but in praxis it isn't. At contrary, it's very pleasant e we will try to find an explanation. For this we will take the seeds of the sunflower like an example. The positions of the seeds were chosen to fill out the whole area of the circle without leaving empty spaces. Beginning in the center of the circle, turning around, at which angles of the whole circle of 360° we have to position the seeds to fill out the circle at best ? In order to avoid empty space, a single seed should never happens exactly behind another one respect of the center of the circle creating beams like in a bicycle wheel because the space between a beam an another ones is growing from the center to the border. Using angles based on fraction composed by little numbers we would get beams inevitably. The beam like distribution of the seeds is corresponding to a harmonic music interval based on a fraction of natural number multiplied with 360°. In this way the "seeds" will match exactly one behind another but this kind of distribution is not indicated to fill out the whole area of the circle with a maximum number of seeds. To position the seeds, the sunflower uses the so called golden angle of $360°/Phi^2=137{,}5077°$. In this way a seed will never happens exactly behind another one. Transforming something like this in music intervals we will expect a horrible sound but it isn't for the same reason why the positioning of the sunflower seeds is not ugly but highly beautiful with his embedded spirals turning clockwise and counterclockwise and you cannot stick your easily from it because the beauty is so fascinating. The same happens hearing the Phi interval.



Another kind of explanation more technical would be the following: the Phi proportion divides the time axis in a fractal manner creating infinitely all the powers of Phi itself and the human ear recognize the perfect repeating of these values. Probably our brain calculates the sums of the powers of Phi creating perfectly natural numbers harmonic to our ears. To the Phi interval we can add another note that corresponds to another power of Phi or a natural number or a fraction of natural numbers pushing the keys on our Phi-intonated piano keyboard (because they are part of the music system we could do so), our brain recognized the perfect embedding of this musical agreement of all these values interpreting it “harmonic” even if this kind of harmonic is fractal like nature and not like the beams of a bicycle.

The following figure shows how the duration of the Phi power based oscillations are creating other durations corresponding to powers of Phi and the creating the natural numbers of 1,2 and 3. You can observe the frequent presence of embedded powers of Phi in a fractal manner on time axis. Of course the same principle is valid for frequencies that have the inverse value of oscillation duration.

Phi^{-4}=0,145898

Phi^{-3}=0,236068

Phi^{-2}=0,381966

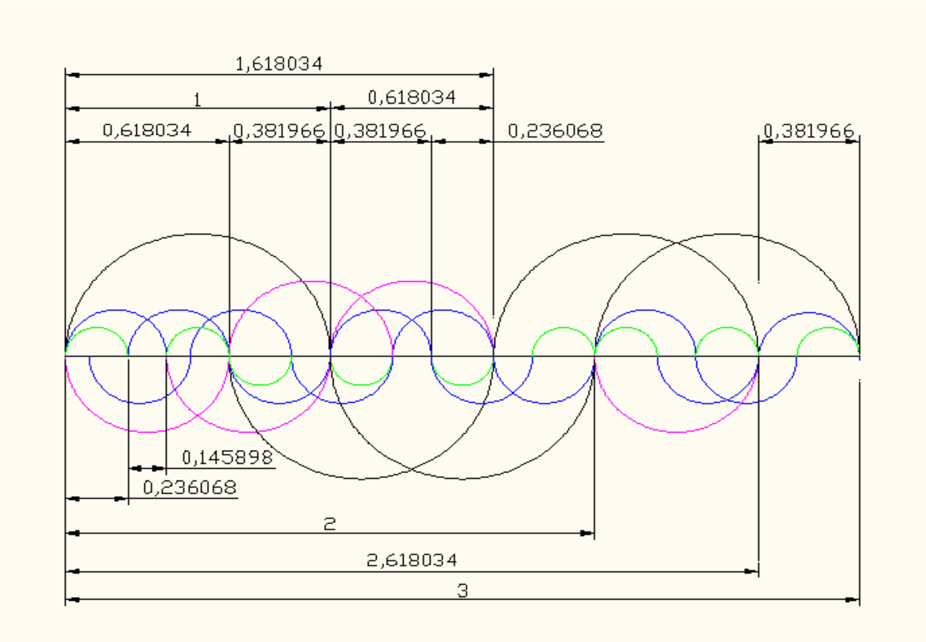
Phi^{-1}=0,618034

Phi^{-1}+Phi^{-2}=Phi^{0}=0,618034+0,381966=1,000000

Phi^{1}=1,618034

Phi^{1}+Phi^{-2}=1,618034+0,381966=2,000000

Phi^{2}+Phi^{-2}=2,618034+0,381966=3,000000



In the Phi music system we have to follow a repeating pattern made of positions. We can occupy a single position only one times with a frequency. A key of a music instrument can't have two frequencies at the same time. Adding the powers of Phi we can create every number and because of this we have to choice the numbers for the positions. How we can know if we made the right choice ? At this point we get some help from another mathematical constant present in nature, Pigreco =3,14159… . In order to this control system, we introduce the concept of half - tone with n=0.5, 1.0, 1.5, 2.0, 2.5. It's possible to connect every tone with a halftone by Pigreco when we did the right choices of numbers. If not, we wouldn't get correspondence with Pigreco for every note.

This kind of research is limited only on artistic sector of music? Absolutely no. The writer Alessio Di Benedetto, says that "*we are dipped into an infinitely oscillating field, like countless music harmonies going out from a single basic sound. From this field we recognize only the frequencies near to us*".

In the book "*In the beginning was the vibrating*" Alessio Di Benedetto said: "*Is there a way to produce energy by frequency devices, connected with systems of anti-gravitation and time travelling ? Would it be possible to influence our DNA by resonance with modulated frequencies in order to correct genetic errors? If we block the new research based on the love for the secrets of universe, we will poor the whole humanity condemn her to produce energy with obsolete systems that will destruct our planet.*" At this point we have to consider that according to the String Theory the universe is not empty but full of continuous vibrating dark energy and matter corresponding together at 96% of all what exist in the universe, the matter we know is corresponding only to 4% of all. If we get in resonance with this kind of dark energy, may be in future we can produce energy from space without pollution for ever.

The alternative medicine takes care about bio-physical aspects in our organism and in front of a disease speaks about disharmony on an energetic level. A sick liver don't vibrate at his sane frequency of 40 Hz. All our organs have specific frequencies but the question is, how they are connected together ? Today it's possible to "hear" the sound of a DNA sequence. If the patient has cancer, the sound would be disharmonic and in medicine this is used for diagnostic. What kind of music scale use our DNA to generate the sound researchers can make hear ? The DNA molecule is made of Phi-based geometry and this make us think that the DNA music is Phi based.

Probably there is a universal music scale connection all our organs, our DNA and in future it would be possible to heal using these frequencies harmonizing our cells but only future research could confirm that.

We think that a music corresponding to our biology and all what exist could have harmonizing effects on us. We think about the Mozart effect and music therapy. This kind of music combine the pleasure that music can give with benefits for our health. May be in future it would be possible even to heal cancer by using frequencies. In her book "When music heals" Fabien Maman talk about experiments made with tumor cells where subject to vary music intervals and she found out, that the most disharmonic interval of the seventh made explode the tumor cells but not the sane cells.

At this point we have to begin to discuss about the String Theory. In few words, this theory says that all what exist, is pure energy that is vibrating. It depends on the way the energy is vibrating at a certain frequency if there would be a manifestation like a force or a subatomic particle. In the moment, the String Theory is the favorite candidate to be able to predict and explain with the same physical model the 4 forces in universe (electromagnetism, gravity, strong and weak force of nucleus) and the favorite for the Theory of All. To reach this goal, the String Theory needs mathematical connections (theory of numbers including prime numbers). The fact, that in this theory all is vibrating give us the idea of an universal symphony using always the same music scale. Which one? In the String Theory connections of Phi and Pigreco together with harmonic relationships are leading to important results. The mathematician Michele Nardelli have used (also in this paper) the numbers of the Phi based on the music system in the String Theory and we have already deposited a first scientific paper in 2008 at CNR (Lange, Christian and Nardelli, Michele and Bini, Giuseppe (2008) *Sistema Musicale Aureo Phi^(n/7) e connessioni matematiche tra Numeri Primi e "Paesaggio" della Teoria delle Stringhe.*) where we have discussed the original curve of Phi$^{(n/7)}$ together with the musician Giuseppe Bini who had care of the musical aspects of the Phi System.

We don't wonder about that the genius Creator of the universe based the creation on a fractal Phi music system with big sense of beauty and arts. Probably we found the music scale used by the Creator to compose his creation like a symphony. Galileo said: "Mathematics is the language with which God composed the universe."

In this paper Nardelli have examined and described the harmonic relationships of the exact system in combination of Phi and Pigreco inside some sectors of String Theory. Furthermore, we want remember that there exist an fundamental simple algebraic relationship that link π and Φ. This is the following:

$$\sqrt{\pi \cdot \frac{5}{6}} = \Phi \quad \text{or} \quad \pi \cdot \frac{5}{6} = (\Phi)^2$$

where $$\Phi = \frac{\sqrt{5}+1}{2} = 1{,}618033989...$$

Abstract

In this paper in the **Section 1**, we have described some equations concerning the duality and higher derivative terms in M-theory. In the **Section 2,** we have described some equations concerning the moduli-dependent coefficients of higher derivative interactions that appear in the low energy expansion of the four-supergraviton amplitude of maximally supersymmetric string theory

compactified on a d-torus. Thence, some equations regarding the automorphic properties of low energy string amplitudes in various dimensions. In the **Section 3**, we have described some equations concerning the Eisenstein series for higher-rank groups, string theory amplitudes and string perturbation theory. In the **Section 4**, we have described some equations concerning U-duality invariant modular form for the $D^6\mathcal{R}^4$ interaction in the effective action of type IIB string theory compactified on T^2. Furthermore, in the **Section 5**, we have described various possible mathematical connections between the arguments above mentioned and some sectors of Number Theory, principally the Aurea Ratio $\Phi = \left(\sqrt{5}+1\right)/2$, some equations concerning the Ramanujan's modular equations that are related to the physical vibrations of the bosonic strings and of the superstrings, some Ramanujan's identities concerning π and the zeta strings. In conclusion, in the **Appendix A**, we have analyzed some pure numbers concerning various equations described in the present paper. Thence, we have obtained some useful mathematical connections with some sectors of Number Theory. In the **Appendix B**, we have showed the column "system" concerning the **universal music system based on Phi** and the table where we have showed the difference between the values of Phi^(n/7) and the values of the column "system"

1. On some equations concerning the duality and higher derivative terms in M-theory. [1]

The first term in the derivative expansion beyond the Einstein-Hilbert term that contributes to four-graviton scattering has the form

$$l_s^{-2}\int d^{10}x\sqrt{-g}\,e^{-\phi/2}Z_{\frac{3}{2}}^{(0,0)}\mathcal{R}^4\ , \quad (1.1)$$

in string frame. The dilaton factor $e^{-\phi/2}$ is again absent in Einstein frame. The symbol $\mathcal{R}^4$ denotes a specific contraction of four Weyl tensors that arises from the leading behaviour in the low energy expansion of the four-graviton amplitude. The function $Z_{3/2}^{(0,0)}\left(\Omega,\overline{\Omega}\right)$ is a modular form with holomorphic and anti-holomorphic weights $(0,0)$. It is a function of the complex coupling $\Omega = \Omega_1 + i\Omega_2$, where $\Omega_2 = e^{-\phi}$ and $\Omega_1 = C^{(0)}$. The leading term in the low energy limit determined the dilaton dependent function $Z_{5/2}^{(0,0)}\left(\Omega,\overline{\Omega}\right)$ of the

$$l_s^{-2}\int d^{10}x\sqrt{-g}\,e^{\phi/2}Z_{\frac{5}{2}}^{(0,0)}D^4\mathcal{R}^4 \quad (1.2)$$

interaction, is again expressed in string frame. The dilaton-dependent functions $Z_{3/2}^{(0,0)}$ and $Z_{5/2}^{(0,0)}$ in (1.1) and (1.2) are non-holomorphic Eisenstein series that are special cases of the series

$$Z_s^{(w,w')} = \sum_{(m,n)\neq(0,0)}\frac{\Omega_2^s}{\left(m+n\Omega\right)^{s+w}\left(m+n\overline{\Omega}\right)^{s+w'}}\ . \quad (1.3)$$

Interactions have $w = -w' = q/2$ where q denotes the $U(1)$ R-symmetry charge of the interaction. For example, there is an interaction of the form

$$\int d^{10}x\sqrt{-g}e^{-\phi/2}Z_{3/2}^{(12,-12)}\lambda^{16}$$

where the dilatino λ transforms with weights $(-3/4,3/4)$. The series $Z_s^{(0,0)}$ is an eigenfunction of the Laplace operator on the fundamental domain of $SL(2,Z)$ with eigenvalue $s(s-1)$,

$$\Delta_\Omega Z_s^{(0,0)} \equiv 4\Omega_2^2\partial_\Omega\partial_{\overline{\Omega}}Z_s^{(0,0)} = s(s-1)Z_s^{(0,0)} . \quad (1.4)$$

This equation is a consequence of supersymmetry. For general values of s Z_s has the large-Ω_2 (weak coupling) expansion

$$Z_s(\Omega,\overline{\Omega}) = 2\zeta(2s)\Omega_2^s + 2\sqrt{\pi}\Omega_2^{1-s}\frac{\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)} + \\ + \frac{2\pi^s}{\Gamma(s)}\sum_{k\neq 0}\mu(k,s)e^{-2\pi(|k|\Omega_2 - ik\Omega_1)}|k|^{s-1}\left(1+\frac{s(s-1)}{4\pi|k|\Omega_2}+\dots\right), \quad (1.5)$$

where the last term comes from the asymptotic expansion of a modified Bessel function, ζ is the Riemann's zeta function and $\mu(k,s)=\sum_{d|k}1/d^{2s-1}$. This expression contains precisely two power behaved terms proportional to Ω_2^s and Ω_2^{1-s}, which should be identified with tree-level and $(s-1/2)$-loop term in the IIB string perturbation expansion of the four graviton amplitude. In addition, there is an infinite sequence of D-instanton terms in Z_s, which have a characteristic phase of the form $e^{2\pi ik\Omega}$, where k is the instanton number. Thus, with $s=3/2$ (the $\mathcal{R}^4$ term) there are tree-level and one-loop terms, as well as the infinite series of D-instanton contributions.
The objective of this section is to extend the analysis of the dilaton dependence of higher derivative interactions to the $D^6\mathcal{R}^4$ interaction. This has the form (in string frame)

$$l_s^4\int d^{10}x\sqrt{-g}e^{\phi}\xi_{(3/2,3/2)}D^6\mathcal{R}^4 , \quad (1.6)$$

where the function $\xi_{(3/2,3/2)}(\Omega,\overline{\Omega})$ is a new $(0,0)$ modular form that depends on the complex coupling, Ω. The function $\xi_{(3/2,3/2)}$ satisfies a Laplace equation on moduli space with a source term,

$$\Delta_\Omega\xi_{(3/2,3/2)} = 12\xi_{(3/2,3/2)} - 6Z_{3/2}Z_{3/2} , \quad (1.7)$$

and determine its solution.
To separate perturbative and non-perturbative contributions we write $\xi_{(3/2,3/2)}(\Omega,\overline{\Omega})$ in terms of a Fourier expansion of the form

$$\xi_{(3/2,3/2)}(\Omega,\overline{\Omega}) = \tilde{\xi}_{(3/2,3/2)}^{(0)}(\Omega_2) + \sum_{k\neq 0}\tilde{\xi}_{(3/2,3/2)}^{(k)}(\Omega_2)e^{2ik\pi\Omega_1} . \quad (1.8)$$

The dependence on Ω_1 enters through the phase factor $e^{2ik\pi\Omega_1}$, that accompanies the non-zero mode. This is characteristic of a D-instanton contribution which comes from the double sum of D-instantons with charges k_1 and k_2, where $k_1+k_2=k$. There is a corresponding exponentially

decreasing coefficient $\tilde{\xi}^{(k)}_{(3/2,3/2)}$, that should behave as $e^{-2\pi(|k_1|+|k_2|)\Omega_2}$ at weak coupling $(\Omega_2 \to \infty)$. The zero mode, $\tilde{\xi}^{(0)}_{(3/2,3/2)}$, contains the piece that is a power-behaved function of the inverse string coupling constant, Ω_2 which is interpreted as a perturbative string contribution. There will also be an exponentially decreasing contribution to the zero mode piece, which is interpreted as a double D-instanton contribution in which the instanton charges are equal and opposite in sign $(k_1 = -k_2)$.
The zero mode in (1.8) satisfies the equation

$$\left(\Omega_2^2 \partial_{\Omega_2}^2 - 12\right)\tilde{\xi}^{(0)}_{(3/2,3/2)}(\Omega_2) = -6\left(\left(2\zeta(3)\Omega_2^{\frac{3}{2}} + 4\zeta(2)\Omega_2^{-\frac{1}{2}}\right)^2 + (8\pi)^2 \Omega_2 \sum_{k \neq 0} k^2 \mu^2\left(k, \frac{3}{2}\right)\mathcal{K}_1^2\left(2\pi|k|\Omega_2\right)\right), \tag{1.9}$$

where the right-hand side comes from a Fourier expansion of $Z_{3/2}^2$. The factor $\left(2\zeta(3)\Omega_2^{3/2} + 4\zeta(2)\Omega_2^{-1/2}\right)^2$ comes from the square of the zero mode of $Z_{3/2}$ (defined by the first line in (1.5) with $s = 3/2$) whereas the term involving the square of Bessel functions $\mathcal{K}_1^2$ comes from the modes with non-zero k, which arise as a sum over D-instanton anti D-instanton pairs with $k_1 = -k_2$ and $|k_1| + |k_2| = k$. The quantity $\mu(k, 3/2) = \sum_{d|k} d^{-2}$ is the D-instanton measure factor.
Consider first the solution for the perturbative part of $\tilde{\xi}^{(0)}_{(3/2,3/2)}$, which is a sequence of power-behaved terms. The general solution for the power behaved terms that satisfy (1.9) is

$$\tilde{\xi}^{(0)pert}_{(3/2,3/2)} = 4\zeta(3)^2 \Omega_2^3 + 8\zeta(3)\zeta(2)\Omega_2 + \frac{48}{5}\zeta(2)^2 \Omega_2^{-1} + \alpha\Omega_2^4 + \beta\Omega_2^{-3}, \tag{1.10}$$

where the coefficients α and β are not determined directly by (1.9) because the terms Ω_2^4 and Ω_2^{-3} individually satisfy the homogeneous equation.
The coefficient β represents a three-loop contribution in string perturbation theory. The remaining coefficients in (1.10) are determined directly by (1.9). These correspond to the tree-level, one-loop and two-loop contributions to the $D^6 R^4$ interaction. The leading Ω_2^3 term in (1.10) represents the tree-level contribution and has precisely the expected coefficient that matches the string tree-level calculation.
We have the following Laplace equation:

$$\Delta_\Omega \xi_{(3/2,3/2)} = 12\xi_{(3/2,3/2)} - 6\mathcal{Z}_{(3/2,3/2)}, \tag{1.11}$$

where $\mathcal{Z}_{(3/2,3/2)} \equiv Z_{3/2} Z_{3/2}$. Expanding (1.11) in Fourier modes gives an equation for each mode of the form

$$\left[\Omega_2^2\left(\partial_{\Omega_2}^2 - 4\pi^2 k^2\right) - 12\right]\tilde{\xi}^{nonpert(k)}_{(3/2,3/2)}(\Omega_2) = -384\pi^2 \Omega_2 \sum_{\substack{k_1 \neq 0, k_2 \neq 0 \\ k_1 + k_2 = k}} |k_1 k_2| \mu\left(k_1, \frac{3}{2}\right)\mu\left(k_2, \frac{3}{2}\right)\mathcal{K}_1\left(2\pi|k_1|\Omega_2\right)\mathcal{K}_1\left(2\pi|k_2|\Omega_2\right)$$
$$- 96\pi\left(2\zeta(3)\Omega_2^{3/2} + 4\zeta(2)\Omega_2^{-1/2}\right)\sum_{k_1 \neq 0} |k_1| \mu\left(k_1, \frac{3}{2}\right)\mathcal{K}_1\left(2\pi|k_1|\Omega_2\right), \tag{1.12}$$

where $\tilde{\xi}^{nonpert(k)}_{(3/2,3/2)}(\Omega_2)$ are the non-perturbative terms.

Using the asymptotic form for the modified Bessel function $\mathcal{K}_1(z) \approx \sqrt{\pi/2z}e^{-z}$, the large-$\Omega_2$ limit of the solution is easy to determine. For a general value of $k = k_1 + k_2$ it has the form

$$\sum_{k_1} P_{k_1}(\Omega_2) e^{-2\pi(|k_1|+|k-k_1|)\Omega_2} e^{2\pi i k \Omega_1} \,, \quad (1.13)$$

where the functions $P_k \approx \Omega_2^{-p_k}$ with positive p_k. When k_1 and k_2 $(= k - k_1)$ both have the same sign the action is equal to the charge $(|k_1| + |k - k_1| = k)$. There is a $k = 0$ contribution to $\xi^{(0)}_{(3/2,3/2)}$ due to D-instanton – anti D-instanton pairs, that has the form

$$-64\pi^2 \sum_k |\hat{k}| \mu\left(\hat{k}, \frac{3}{2}\right)^2 \left(\frac{1}{4\pi|\hat{k}|\Omega_2} + \ldots\right) e^{-4\pi|\hat{k}|\Omega_2} \,. \quad (1.14)$$

We will now determine the three-loop coefficient, β, of the Ω_2^{-3} term in (1.10). First we should note that a general solution of the Laplace equation (1.7) can be written as the sum of a particular solution and a multiple of Z_4, which is the solution of the homogeneous Laplace equation, $\Delta Z_4 = 12 Z_4$. Recall also that $Z_4 = \sum_{(m,n)\neq(0,0)} \Omega_2^4 / |m + n\Omega|^8$ has the large-Ω_2 expansion

$$Z_4 = 2\zeta(8)\Omega_2^4 + \frac{5\pi}{8}\zeta(7)\Omega_2^{-3} + \ldots \quad (1.15)$$

where … denotes exponentially suppressed terms. However, the special solution $\xi_{(3/2,3/2)}$ that we obtained from the two-loop supergravity expression is known not to have a Ω_2^4 piece, so that the coefficient of Z_4 in the general solution must be zero. The question remains as to whether $\xi_{(3/2,3/2)}$ contains a $\beta\Omega_2^{-3}$ term. To study this we multiply the left-hand and right-hand sides of the inhomogeneous Laplace equation (1.7) by the Eisenstein series Z_4 and integrate over a fundamental domain of Ω. Since the relevant integrals diverge at the boundary $\Omega_2 \to \infty$, we will introduce a cut-off at $\Omega_2 = L$ and consider the $L \to \infty$ limit. Denoting the cut-off fundamental domain by $\mathcal{F}_L$, the resulting equation is

$$\int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} Z_4 \Delta \xi_{(3/2,3/2)} = 12 \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} Z_4 \xi_{(3/2,3/2)} - 6 \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} Z_4 Z_{3/2}^2 \,. \quad (1.16)$$

Integrating the left-hand side by parts and using the fact that $\Delta Z_4 = 12 Z_4$, gives

$$\int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} Z_4 \Delta \xi_{(3/2,3/2)} = \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} \Delta Z_4 \xi_{(3/2,3/2)} + \int_{-1/2}^{1/2} d\Omega_1 \left(Z_4 \partial_{\Omega_2} \xi_{(3/2,3/2)} - \partial_{\Omega_2} Z_4 \xi_{(3/2,3/2)}\right)\Big|_{\Omega_2 = L} =$$

$$= 12 \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} Z_4 \xi_{(3/2,3/2)} + \int_{-1/2}^{1/2} d\Omega_1 \left(Z_4 \partial_{\Omega_2} \xi_{(3/2,3/2)} - (\partial_{\Omega_2} Z_4) \xi_{(3/2,3/2)}\right)\Big|_{\Omega_2 = L} \,. \quad (1.17)$$

Comparing (1.17) with (1.16) we see that

$$\int_{-1/2}^{1/2} d\Omega_1 \left(Z_4 \partial_{\Omega_2} \xi_{(3/2,3/2)} - \left(\partial_{\Omega_2} Z_4\right) \xi_{(3/2,3/2)} \right)\Big|_{\Omega_2 = L \to \infty} = -6 \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} Z_4 Z_{3/2}^2 \,. \quad (1.18)$$

The left-hand side of this equation is simply a surface time that is easy to evaluate

$$\int_{-1/2}^{1/2} d\Omega_1 \left(Z_4 \partial_{\Omega_2} \xi_{(3/2,3/2)} - \left(\partial_{\Omega_2} Z_4\right) \xi_{(3/2,3/2)} \right)\Big|_{\Omega_2 = L \to \infty} = -\zeta(8)\left(8\zeta(3)^2 L^6 + 48\zeta(3)\zeta(2) L^4 + 96\zeta(2)^2 L^2 + 14\beta\right) \quad (1.19)$$

The right-hand side of (1.18) may be evaluated by unfolding the integral onto the strip using $Z_4 = 2\zeta(8)\sum_{\gamma \in Sl(2,Z)} \Im m(\gamma \cdot \Omega)^4$ and the fact that $Z_{3/2}^2$ is modular invariant, which gives

$$\frac{1}{2\zeta(8)} \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^4} Z_4 Z_{3/2}^2 = \int_0^L \frac{d\Omega_2}{\Omega_2^2} \Omega_2^4 \int_{-1/2}^{1/2} d\Omega_1 Z_{3/2}^2 = \frac{2}{3}\zeta(3)^2 L^6 + \zeta(3)\zeta(2) L^4 + 2\zeta(2)^2 L^2 +$$
$$+ (8\pi)^2 \int_0^L d\Omega_2 \Omega_2^3 \sum_{k \neq 0} k^2 \mu\left(|k|, \frac{3}{2}\right)^2 \mathcal{K}_1^2\left(2\pi |k| \Omega_2\right). \quad (1.20)$$

Using the integral representation for the Bessel function, we find that

$$\beta = \frac{384\pi^2}{7} \int_0^\infty d\Omega_2 \Omega_2^3 \sum_{k \neq 0} k^2 \mu\left(|k|, \frac{3}{2}\right)^2 \mathcal{K}_1^2\left(2\pi |k| \Omega_2\right) = \frac{32}{7\pi^2} \sum_{k \geq 1} \frac{\mu\left(k, \frac{3}{2}\right)^2}{k^2}, \quad (1.21)$$

which gives a non-zero value for the three-loop term. Recalling that $\mu(n,s) = \sum_{m|n} n^{1-2s}$ and using an identity by Ramanujan

$$\sum_{n=1}^{\infty} \frac{\mu(n,s)\mu(n,s')}{n^r} = \frac{\zeta(r)\zeta(r+2s-1)\zeta(r+2s'-1)\zeta(r+2s+2s'-2)}{\zeta(2r+2s+2s'-2)} \quad (1.22)$$

we find that the three-loop coefficient has the value

$$\beta = \frac{16}{189}\pi^2 \zeta(4). \quad (1.23)$$

This number is in complete agreement with the calculation of the three-loop coefficient in type IIA string theory. The one-loop four-graviton amplitude of eleven-dimensional supergravity compactified on a two-torus gives rise to a series of higher-derivative terms in the nine-dimensional type IIA effective action of the form

$$A_4^{(1)} = \left(4\pi^8 l_{11}^{15} r_A^{-1}\right) \hat{K} r_A \Bigg[2\zeta(3) e^{-2\phi^A} + \frac{2\pi^2}{3r_A^2} + \frac{2\pi^2}{3} - 8\pi^2 r_A l_s \left(-\mathcal{W}^s\right)^{1/2} + 8\pi^{3/2} \sum_{n=2}^{\infty} \Bigg(\Gamma\left(n - \frac{1}{2}\right)\zeta(2n-1)$$
$$\frac{r_A^{2(n-1)}}{n!}\left(l_s^2 \mathcal{W}^s\right)^n + \sqrt{\pi}\,\Gamma(n-1)\zeta(2n-2)\frac{e^{2(n-1)\phi^A}}{n!}\left(l_s^2 \mathcal{W}^s\right)^n \Bigg)\Bigg] + \text{non-perturbative}, \quad (1.24)$$

where

$$\left(\mathcal{W}^s\right)^n = \left(\mathcal{G}^s_{ST}\right)^n + \left(\mathcal{G}^s_{TU}\right)^n + \left(\mathcal{G}^s_{US}\right)^n , \quad (1.25)$$

and

$$\left(\mathcal{G}^s_{ST}\right)^n = \int_0^1 d\omega_3 \int_0^{\omega_3} d\omega_2 \int_0^{\omega_2} d\omega_1 \left(s\omega_1(\omega_3-\omega_2)+t(\omega_2-\omega_1)(1-\omega_3)\right)^n . \quad (1.26)$$

The terms in the third line of (1.24) give higher-loop contributions to the ten-dimensional effective action of the type IIA theory. The term with $n=2$ gives the two-loop D^4R^4 term in the IIA theory that matches the same term in the type IIB theory. The term with $n=3$ in the third line of (1.24) contributes to the three-loop D^6R^4 term in the IIA theory and has the value

$$S^{(IIA)}_{D^6R^4} = l_s^4 \frac{1}{4\cdot 96\cdot(4\pi)^7}\frac{16}{189}\pi^2\zeta(4)\int d^{10}x\sqrt{-g^A}e^{4\phi^A}D^6R^4 . \quad (1.27)$$

Including the absolute normalisation this type IIA expression and the following type IIB expression

$$S^{(IIB)}_{D^6R^4} = l_s^4 \frac{\pi^6}{4\cdot 96\cdot(4\pi)^7}\int d^{10}x\sqrt{-g^B}e^{\phi^B}\xi_{(3/2,3/2)}(\Omega,\overline{\Omega})D^6R^4 , \quad (1.28)$$

we find a perfect match between the two values for the three-loop coefficient for the D^6R^4 in superstring theory.

2. On some equations regarding the automorphic properties of low energy string amplitudes. [2]

The simplest non-trivial examples of automorphic functions arise in the ten-dimensional IIB theory, where the coset is $SO(2)\backslash SL(2)$, so there is a single complex modulus, $\Omega = \Omega_1 + i\Omega_2$, and the duality group is $SL(2,Z)$. In this case the first two terms in the expansion beyond the classical term are given by particular examples of non-holomorphic Eisenstein series for $SL(2,Z)$

$$E_s(\Omega) = \sum_{(m,n)\neq(0,0)} \frac{\Omega_2^s}{|m+n\Omega|^{2s}} , \quad (2.1)$$

which satisfies the Laplace equation

$$\Delta_\Omega E_s(\Omega) \equiv \Omega_2^2\left(\partial^2_{\Omega_1}+\partial^2_{\Omega_2}\right)E_s(\Omega) = s(s-1)E_s(\Omega). \quad (2.2)$$

The Fourier expansion of E_s

$$E_s(\Omega) = 2\zeta(2s)\Omega_2^s + 2\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta(2s-1)\Omega_2^{1-s} + \frac{2\pi^s}{\Gamma(s)}\Omega_2^{\frac{1}{2}}\sum_{n\neq 0}|n|^{s-\frac{1}{2}}\sum_{\substack{0<d\\ n/d\in N}}\frac{1}{d^{2s-1}}K_{s-\frac{1}{2}}\left(2\pi|n|\Omega_2\right)e^{2i\pi n\Omega_1} ,$$

has a zero mode or "constant term" that consists of the sum of two powers,

$$\int_{-1/2}^{1/2} d\Omega_1 E_s = 2\zeta(2s)\Omega_2^s + 2\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta(2s-1)\Omega_2^{1-s}\ , \quad (2.3)$$

which correspond to a tree-level and genus-$(s-1/2)$ contribution to the interaction in string perturbation theory.

The minimal parabolic Eisenstein series for a group G is defined by

$$E_\lambda^G(g) = \sum_{\gamma\in G(Q)/B(Q)} e^{\langle \lambda+\rho, H(g\gamma)\rangle}\ . \quad (2.3a)$$

We consider $H = \gamma g \gamma^T$, where $\gamma \in SL(d,Z)$ and g is the $SL(d)$ matrix parametrizing the coset space $SO(d)\backslash SL(d)$. Letting H_k be the bottom right $k\times k$ minor of H the general minimal parabolic Eisenstein series associated with the minimal parabolic subgroup $P(1,...,1)$,

$$E^{SL(d)}_{[\varepsilon_1,...,\varepsilon_{d-1}];s_1,...,s_d} = \sum_{\gamma\in SL(n,Z)/B(Z)} \prod_{k=1}^{d-1} \left(\det H_k\right)^{\frac{\lambda_{d-k+1}-\lambda_{d-k}-1}{2}}\ , \quad (2.3b)$$

which is a special case of the general formula (2.3a). Here we have set $2s_k = \lambda_{d-k+1} - \lambda_{d-k} - 1$ for $1 \le k \le d-1$, and $\varepsilon_k = 1$ if $s_k \ne 0$ and $\varepsilon_k = 0$ if $s_k = 0$. The $SL(d)$ series that are studied in this paper are:

$$E^{SL(d)}_{[1,0^{d-2}];s}\ , \quad E^{SL(d)}_{[0,1,0^{d-3}];s}\ , \quad E^{SL(d)}_{[0^{d-2},1];s}\ ,$$

thence

$$E^{SL(d)}_{[1,0^{d-2}];s} = \sum_{(m^1,...,m^d)\in Z^d\backslash(0,...,0)} \frac{1}{\left(m^i g_{ij} m^j\right)^s}\ ; \quad E^{SL(d)}_{[0^{d-2},1];s} = \sum_{(n_1,...,n_d)\in Z^d\backslash(0,...,0)} \frac{1}{\left(n_i \left(g^{-1}\right)^{ij} n_j\right)^s}\ ;$$

$$E^{SL(d)}_{[0,1,0^{d-3}];s} = \sum_{\substack{1\le k\le d-1 \\ [M_{0,k}]}} \frac{1}{\left(g_{ij} g_{kl} d^{il} d^{jk}\right)^s}\ . \quad (2.3c)$$

Now we analyze the following order in the analytic part of the momentum expansion of the amplitude that is encoded into the local effective action

$$S_{\partial^6 R^4} = l_D^{14-D} \int d^D x \sqrt{-G^{(D)}}\, \xi^{(D)}_{(0,1)} \partial^6 \mathcal{R}^4\ . \quad (2.4)$$

At this order in the low energy expansion the structure of the equation satisfied by the coefficient functions changes, as is evident from the following $D=10$ $SL(2,Z)$ case

$$\left(\Delta_\Omega - 12\right)\xi^{(10)}_{(0,1)}(\Omega) = -\left(\xi^{(10)}_{(0,0)}(\Omega)\right)^2\ , \quad (2.5)$$

which has a source term on the right-hand side

$$\left(\Delta_{SO(2)\backslash SL(2)}-12\right)\xi^{(10)}_{(0,1)}=-\left(\xi^{(10)}_{(0,0)}\right)^2. \quad (2.6)$$

The constant term is given by

$$l_{10}^4\int_{-1/2}^{1/2}d\Omega_1\xi^{(10)}_{(0,1)}=l_s^4\left(\frac{2\zeta(3)^2}{3}\Omega_2^2+\frac{4\zeta(2)\zeta(3)}{3}+\frac{8\zeta(2)^2}{5}\Omega_2^{-2}+\frac{4\zeta(6)}{27}\Omega_2^{-4}+O\left(e^{-4\pi\Omega_2}\right)\right), \quad (2.7)$$

which has perturbative contributions up to genus three and has contributions from D-instanton/anti-D-instanton pairs with zero net instanton number.

With regard the nine dimensions case, the effective action (2.4), with $D=9$, contains the coefficient function that is

$$\xi^{(9)}_{(0,1)}=\nu_1^{-\frac{6}{7}}\xi^{(10)}_{(0,1)}+\frac{2\zeta(2)}{3}\nu_1^{\frac{1}{7}}E_{3/2}+\frac{2\zeta(2)}{63}\nu_1^{\frac{15}{7}}E_{5/2}+\frac{4\zeta(2)\zeta(5)}{63}\nu_1^{-\frac{20}{7}}+\frac{8\zeta(2)^2}{5}\nu_1^{\frac{8}{7}}. \quad (2.8)$$

The function $\xi^{(10)}_{(0,1)}$ is the ten-dimensional coefficient that satisfies the inhomogeneous Laplace equation, 2.4. It is readily checked that $\xi^{(9)}_{(0,1)}$ satisfies

$$\left(\Delta^{(9)}-\frac{90}{7}\right)\xi^{(9)}_{(0,1)}=-\left(\xi^{(9)}_{(0,0)}\right)^2. \quad (2.9)$$

The source term is again quadratic in the modular function that arises for the coefficient of the $\mathcal{R}^4$ interaction, as it was for $D=10$ in (2.5).

The contribution (2.8) can be re-expressed in ten-dimensional units recalling that $\ell_9=\ell_{10}^{\frac{8}{7}}r_B^{-\frac{1}{7}}$ and $\nu_1=\left(r_B/\ell_{10}\right)^{-2}$, giving

$$\ell_9^5\xi^{(9)}_{(0,1)}=\ell_{10}^4r_B\left(\xi^{(10)}_{(0,1)}+\frac{2\zeta(2)}{3}\left(\frac{\ell_{10}}{r_B}\right)^2\xi^{(10)}_{(0,0)}+\frac{4\zeta(2)}{63}\left(\frac{\ell_{10}}{r_B}\right)^6\xi^{(10)}_{(1,0)}+\frac{4\zeta(2)\zeta(5)}{63}\left(\frac{r_B}{\ell_{10}}\right)^4+\right.$$
$$\left.+\frac{8\zeta(2)^2}{5}\left(\frac{\ell_{10}}{r_B}\right)^4+O\left(e^{-r_B}\right)\right). \quad (2.10)$$

The term proportional to r_B gives the ten-dimensional expression in the $r_B\to\infty$ limit. Once again, there is growing term with the expected power of r_B^5, which contributes a term proportional to $\left(sr_B^2\right)^2\mathcal{R}^4$ to the expansion of the ten-dimensional $s\mathcal{R}^4\log\left(-\ell_{10}^2s\right)$ threshold in the limit $sr_B^2\to\infty$.

The perturbative expansion of this coefficient is given by expanding in powers of the string coupling,

$$\ell_9^5\int_{-\frac{1}{2}}^{\frac{1}{2}}d\Omega_1\xi^{(9)}_{(0,1)}=\ell_s^5r_B\left(\frac{\zeta(3)^2}{3g_B^2}+\frac{\zeta(2)\zeta(3)}{9}\left(1+\frac{\ell_s^2}{r_B^2}\right)+\frac{\zeta(5)\zeta(2)}{189}\left(\frac{r_B^2}{\ell_s^4}+\frac{\ell_s^6}{r_B^6}\right)+\frac{5\zeta(4)g_B^2}{9}\frac{\ell_s^2}{r_B^2}+\right.$$

$$+\frac{\zeta(4)g_B^2}{3}\left(1+\frac{\ell_s^4}{r_B^4}\right)+\frac{7\zeta(6)}{576}g_B^4\left(1+\frac{\ell_s^6}{r_B^6}\right)+O\left(e^{-1/g_B}\right)\Bigg). \quad (2.11)$$

This expression is symmetric under T-duality transformation $r_B \to 1/r_A$ and $g_B \to g_A/r_A$. The symbol $O\left(e^{-1/g_B}\right)$ indicates schematically the presence of instanton/anti-instanton pairs in the zero D-instanton sector.

Collecting the $L=2$ and $L=1$ modular functions along with the genus-one terms of the following equation

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d\Omega_1 dB_{RR}\xi^{(8)}_{(0,0)}=\frac{2\zeta(3)}{y_8}+2\left(\hat{E}_1(T)+\hat{E}_1(U)\right)+\frac{2\pi}{3}\log(y_8/\tilde{\mu}_8), \quad (2.11b)$$

we find the modular invariant expression,

$$\ell_9^5\xi^{(9)}_{(0,1)}=\ell_{11}^5\hat{\mathcal{V}}_2\left(\frac{\xi_{(0,1)}}{12}\frac{1}{\hat{\mathcal{V}}_2^3}+\frac{\zeta(5)\zeta(2)}{189}\frac{1}{\hat{\mathcal{V}}_2^6}+\frac{\zeta(4)}{3}+\hat{\mathcal{V}}_2^{\frac{7}{2}}\frac{\zeta(2)}{378}E_{5/2}+\frac{\zeta(2)}{9}\hat{\mathcal{V}}_2^{\frac{1}{2}}E_{3/2}\right). \quad (2.12)$$

This expression sums all the contributions determined from the analysis of the $L=1$ and $L=2$ loop amplitude on a torus, to which has been added the contribution $\zeta(5)\zeta(2)/\hat{\mathcal{V}}_2^6$, which arises from a Λ^3 divergence of the $L=3$ amplitude.

Now we analyze the eight-dimensional $\partial^6\mathcal{R}^4$ interaction, which has an effective action (2.6) that is invariant under the U-duality group $E_{3(3)}=SL(3)\times SL(2)$. We will show that the modular function satisfies the differential equation

$$\Delta^{(8)}\xi^{(8)}_{(0,1)}=12\xi^{(8)}_{(0,1)}-\left(\xi^{(8)}_{(0,0)}\right)^2, \quad (2.13)$$

where $\Delta^{(8)}$ is the $SL(3)\times SL(2)$ Laplacian. The source term appearing in this equation again involves the square of the eight-dimensional $\mathcal{R}^4$ coefficient. The solution is close to the one on the basis of consistency with the higher-dimensional interaction,

$$\xi^{(8)}_{(0,1)}=\xi^{SL(3)}_{(0,1)}+\frac{40}{9}E^{SL(3)}_{[10];-3/2}E_3(U)+\frac{1}{3}\hat{E}^{SL(3)}_{[10];3/2}\hat{E}_1(U)+f(U)+\frac{\pi}{36}\hat{E}^{SL(3)}_{[10];3/2}+\frac{\pi}{9}\hat{E}_1(U)+\frac{\zeta(2)}{9}, \quad (2.14)$$

where the function $f(U)$ is defined as the solution of the equation

$$(\Delta_U-12)f(U)=-4\hat{E}_1^2(U), \quad (2.15)$$

where $\Delta_U=U_2^2\left(\partial_{U_1}^2+\partial_{U_2}^2\right)$. It is straightforward to extract the power-behaved terms in its expansion. We have also introduced $\xi^{SL(3)}_{(0,1)}$ satisfying

$$\left(\Delta_{SO(3)\backslash SL(3)}-12\right)\xi^{SL(3)}_{(0,1)}=-\left(\hat{E}^{SL(3)}_{[10];3/2}\right)^2. \quad (2.16)$$

The last three terms in (2.14) arises from the regularisation of the $\mathcal{R}^4$ interaction. In the decompactification limit $r_2/\ell_9\to\infty$ the $SL(3,Z)$ modular functions in (2.14) have the form

$$\int_{-1/2}^{1/2} dB_{RR} dB_{NS} E^{SL(3)}_{[10];-3/2} = \frac{9}{16\pi^4} \nu_2^{\frac{1}{2}} E_{5/2}(\Omega) + \frac{\pi}{315} \nu_2^{-2} , \quad (2.17)$$

$$\int_{-1/2}^{1/2} dB_{RR} dB_{NS} \hat{E}^{SL(3)}_{[10];3/2} = \nu_2^{-\frac{1}{2}} E_{3/2}(\Omega) + \pi \log \nu_2 . \quad (2.18)$$

Substituting the latter expansion into the source term in the following equation

$$\left(\Delta_{SO(3)\backslash SL(3)} - 12\right) A^{SL(3)} = -\left(\hat{E}^{SL(3)}_{[10];3/2}\right)^2 ,$$

one finds that the interaction coefficient becomes

$$\int_{-1/2}^{1/2} dB_{RR} dB_{NS} \xi^{SL(3)}_{(0,1)} = \frac{1}{\nu_2} \xi^{(10)}_{(0,1)} + \left(\frac{2\pi}{9} \nu_2^{-\frac{1}{2}} \log(\nu_2) + c_1 \nu_2^{\frac{3}{2}} + c_2 \nu_2^{-\frac{5}{2}} \right) E_{3/2}(\Omega) +$$
$$+ \frac{\zeta(2)}{9} \left(5 + 4\log(\nu_2) + 8\log^2(\nu_2)\right) + O\left(e^{-\Omega_2^{1/2} \nu_2^{-1/2}}, e^{-(\Omega_2 \nu_2)^{-1/2}}\right), \quad (2.19)$$

where c_1, c_2 are integration constants. We have that $c_1 = \zeta(5)/(12\pi)$ and $c_2 = 0$. In this case the zero instanton sector contains instanton/anti-instanton pairs consisting of D-instantons and wrapped (p,q)-string world-sheets as indicated by the last term. The $SL(2,Z)$ modular functions have the expansions

$$\int_{-1/2}^{1/2} dU_1 E_3(U) = 2\zeta(6) U_2^3 + \frac{3\pi\zeta(5)}{4} U_2^{-2} , \quad (2.20)$$

$$\int_{-1/2}^{1/2} dU_1 \hat{E}_1(U) = 2\zeta(2) U_2 - \pi \log(U_2) , \quad (2.21)$$

and the expansion of the function $f(U)$ is

$$6 f(U) = \frac{\pi^2}{180}\left(65 - 20\pi U_2 + 48\pi^2 U_2^2\right) + \frac{\zeta(3)\zeta(5)}{\pi U_2^3} - 2\zeta(2) \log U_2 \left(4\pi U_2 - 6\log U_2 + 1\right) + O\left(e^{-U_2}\right). \ (2.22)$$

Therefore, the constant term associated with decompactifying to nine dimensions is

$$\ell_8^6 \int_{-1/2}^{1/2} dB_{RR} dB_{NS} \xi^{(8)}_{(0,1)} = \ell_9^5 r_2 \xi^{(9)}_{(0,1)} + \ell_9^6 \left(\frac{\pi}{36} \xi^{(9)}_{(0,0)} + \left(\frac{\ell_9}{r_2}\right)^4 \frac{15\zeta(5)}{4\pi^3} \xi^{(9)}_{(1,0)} + \frac{16\pi\zeta(6)}{567} \left(\frac{r_2}{\ell_9}\right)^6 \right) +$$
$$- \ell_9^6 \frac{\pi}{9} \log\left(\frac{r_2}{\ell_9}\right) \left(7\xi^{(9)}_{(0,0)} - 4\zeta(2) \nu_1^{\frac{4}{7}} \right) - \ell_9^6 \nu_1^{\frac{4}{7}} \frac{4\pi\zeta(2)}{21} \log(\nu_1) + \frac{\ell_9^7}{r_2} \zeta(2) \left(\frac{37}{36} + \frac{86}{9} \log^2\left(\frac{r_2}{\ell_9}\right) - \frac{20}{9} \log\left(\frac{r_2}{\ell_9}\right) \right) +$$
$$- \frac{\ell_9^7}{r_2} \frac{\zeta(2)}{21} \log(\nu_1) \left(1 + 4\log\left(\frac{r_2}{\ell_9}\right) - \frac{48}{7} \log(\nu_1) \right) + O\left(e^{-r_2}\right). \quad (2.23)$$

The term linear in r_2 reproduces the nine-dimensional $\partial^6 \mathcal{R}^4$ interaction, the term independent of r_2 is proportional to the nine-dimensional $\mathcal{R}^4$ interaction, and the term proportional to r_2^{-4} is proportional to the nine-dimensional $\partial^6\mathcal{R}^4$ interaction. The term proportional to r_2^2 is needed to reproduce the $D=9$ threshold of the form $(-s)^{1/2}\mathcal{R}^4$.

The perturbative expansion of the coefficient $\xi_{(0,1)}^{(8)}$ in increasing powers of $y_8=\left(\Omega_2^2 T_2\right)^{-1}$ is given by the following equation:

$$\xi_{(0,1)}^{(8)}=\xi_{(0,1)}^{SL(3)}+\frac{40}{9}E_{[10];-3/2}^{SL(3)}E_3(U)+\frac{1}{3}\hat{E}_{[10];3/2}^{SL(3)}\hat{E}_1(U)+f(U)+\frac{\pi}{36}\left(\hat{E}_{[10];3/2}^{SL(3)}+4\hat{E}_1(U)\right)+\frac{\zeta(2)}{9}. \quad (2.24)$$

The function $j_h^{(p,q)}$ is the expansion of the integrand of the genus-h string loop diagram to order $\sigma_2^p\sigma_3^q\mathcal{R}^4$

$$\ell_8^6\int_{-1/2}^{1/2}d\Omega_1 dB_{RR}\xi_{(0,1)}^{(8)}=\ell_s^6\left(\frac{2\zeta(3)^2}{3y_8}+\frac{64\pi}{3}I_1^{(2)}\left(j_1^{(0,1)}\right)+\frac{2\pi\zeta(3)}{9}\log(y_8)+\frac{2}{3}y_8 I_2^{(2)}\left(j_2^{(0,1)}\right)+\frac{\pi}{9}\left(\frac{\pi}{2}+I_1^{(2)}\left(j_1^{(0,0)}\right)\right)\right.$$
$$\left.y_8\log(y_8)+\frac{\pi^2}{27}y_8\log(y_8)^2+20y_8^2 I_3^{(2)}\left(j_3^{(0,1)}\right)+O\left(e^{-(T_2 y_8)^{-\frac{1}{2}}},e^{-T_2^{1/2}y_8^{-1/2}}\right)\right). \quad (2.25)$$

The genus-one contribution to this expression has the form

$$I_1^{(2)}\left(j_1^{(0,1)}\right)=\frac{10}{32\pi^6}E_3(T)E_3(U)+\frac{\zeta(3)}{32\pi}\left(\hat{E}_1(T)+\hat{E}_1(U)+\log\mu\right). \quad (2.26)$$

Comparing (2.25) with the expansion of $\xi_{(0,1)}^{(8)}$, we see that the genus-two contribution is given by

$$I_2^{(2)}\left(j_2^{(0,1)}\right)=\frac{2}{3}\hat{E}_1(T)\hat{E}_1(U)+\frac{\pi}{9}\left(\hat{E}_1(T)+\hat{E}_1(U)\right)+f(T)+f(U)+\frac{11\zeta(2)}{36}. \quad (2.27)$$

The genus-three contribution in (2.25) extracted from the expansion of $\xi_{(0,1)}^{(8)}$ is

$$I_3^{(2)}\left(j_3^{(0,1)}\right)=\frac{1}{270}\left(E_3(T)+E_3(U)\right). \quad (2.28)$$

The modular function multiplying the $\partial^6\mathcal{R}^4$ interaction in $D=7$ is determined by

$$\left(\Delta^{(7)}-\frac{42}{5}\right)\xi_{(0,1)}^{(7)}=-\left(\xi_{(0,0)}^{(7)}\right)^2, \quad (2.29)$$

where

$$\xi_{(0,0)}^{(7)}=E_{[1000];3/2}^{SL(5)}. \quad (2.30)$$

The solution can be written as

$$\xi_{(0,1)}^{(7)}=\xi_{(0,1)}^{SL(5)}+\frac{25}{2\pi^5}E_{[0010];7/2}^{SL(5)}, \quad (2.31)$$

where $\xi_{(0,1)}^{SL(5)}$ is a particular solution and $E_{[0010];7/2}^{SL(5)}$ is the only solution of the homogeneous equation that has perturbative terms consistent with string theory.
In the limit $r_3/\ell_8 \to \infty$, for the following equation

$$\int_{P(3,2)} E_{[0010];s}^{SL(5)} = 2r^{\frac{24s}{5}}\zeta(2s-1)\zeta(2s)+\frac{\sqrt{\pi}}{2}r^{2+\frac{4s}{5}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}E_{[1];s-\frac{1}{2}}^{SL(2)}E_{[01];s-\frac{1}{2}}^{SL(3)}+$$
$$+(2\pi)^2 r^{8-\frac{16s}{5}}\frac{\Gamma(2s-3)}{\Gamma(2s-1)}\zeta(2s-3)E_{[10];s-1}^{SL(3)}, \quad (2.32)$$

after setting $r_3 = r^2$, we obtain

$$\int_{P(3,2)} E_{[0010];\frac{7}{2}}^{SL(5)} = 2\zeta(6)\zeta(7)\left(\frac{r_3}{\ell_8}\right)^{\frac{42}{5}} + \frac{\pi^2\zeta(2)}{5}\left(\frac{\ell_8}{r_3}\right)^{\frac{8}{5}}E_{[10];\frac{5}{2}}^{SL(3)} + \frac{8}{15}\left(\frac{r_3}{\ell_8}\right)^{\frac{12}{5}}E_{[01];3}^{SL(3)}E_3^{SL(2)}. \quad (2.33)$$

From this expression we recognise the term $E_{[01];3}^{SL(3)}E_3^{SL(2)}$ that decompactifies to eight dimensions. The $r_3^{42/5}$ term in (2.33) contributes to the $D=8$ threshold. Comparing with the eight-dimensional expression for $\xi_{(0,1)}^{(8)}$ and using $E_{[01];3}^{SL(3)} = 2\pi^5/3E_{[01];-3/2}^{SL(3)}$, fixes the relative coefficient in (2.31), as follows. In addition, we recognise the term $E_{[10];\frac{5}{2}}^{SL(3)}$ in (2.33), multiplied by $r_3^{-8/5}$, which is part of the $\partial^4\mathcal{R}^4$ interaction in eight dimensions. The other part of the $\partial^4\mathcal{R}^4$ interaction is a term $r_3^{-8/5}E_{[01];2}^{SL(3)}E_2^{SL(2)}$, which does not show up in (2.33), but arises from $\xi_{(0,1)}^{SL(5)}$, as follows. The large-r_3 limit of the source term is obtained with the use of

$$\int_{P(3,2)} E_{[1000];\frac{3}{2}}^{SL(5)} = \left(\frac{r_3}{\ell_8}\right)^{\frac{6}{5}}\xi_{(0,0)}^{(8)} - 4\pi\left(\frac{r_3}{\ell_8}\right)^{\frac{6}{5}}\log\left(\frac{r_3}{\ell_8\mu_7}\right). \quad (2.34)$$

In this limit, the constant term of the particular solution $\xi_{(0,1)}^{SL(5)}$ contains the contributions

$$\int_{P(3,2)} \xi_{(0,1)}^{SL(5)} = \left(\frac{r_3}{\ell_8}\right)^{\frac{12}{5}}\left(\xi_{(0,1)}^{SL(3)} + \frac{1}{3}\hat{E}_{[10];\frac{3}{2}}^{SL(3)}\hat{E}_1(U) + f(U) + \left(\frac{\ell_8}{r_3}\right)^4\xi_h + \ldots\right). \quad (2.35)$$

The first three terms reproduce the eight-dimensional result. Since the source term does not contain the power $r_3^{-8/5}$, ξ_h solves a homogeneous equation for the $SL(3)\times SL(2)$ Laplacian with eigenvalue 10/3, which is the same as the eigenvalue of $E_{[10];5/2}^{SL(3)}$ in (2.33). The term we are expecting is of the form $kE_{[01];2}^{SL(3)}E_2^{SL(2)}$, where the coefficient k is fixed by comparing with the $\partial^4\mathcal{R}^4$ interaction, which gives $k = -8\pi^2\zeta(2)/5$.
We will now find the constant part of the particular solution, $\xi_{(0,1)}^{SL(5)}$, in the parabolic subgroup of relevance to the limit of perturbative string theory. In this limit, the result is expressed in terms of functions invariant under $SO(3,3) \approx SL(4)$, the T-duality group. We will need the expansions

$$\int_{P(4,1)} E^{SL(5)}_{[1000];\frac{3}{2}} = 2\zeta(3) y_7^{-\frac{6}{5}} + 2 y_7^{-\frac{1}{5}} E^{SL(4)}_{[100];1}, \qquad (2.36)$$

$$\int_{P(4,1)} E^{SL(5)}_{[0010];\frac{7}{2}} = y_7^{-\frac{7}{5}} E^{SL(4)}_{[010];\frac{7}{2}} + \frac{8\pi\zeta(4)}{15} y_7^{\frac{3}{5}} E^{SL(4)}_{[001];3}. \qquad (2.37)$$

Thus the homogeneous solution provides part of the genus-one and genus-three contributions. In order to study the perturbative string theory limit we will also need the decomposition of the $SL(5)$ Laplace operator into the $SL(4)$ Laplace operator plus the second-order differential operator associated with y_7,

$$\Delta^{(7)} = \Delta_{SO(5)\backslash SL(5)} \to \Delta_{SO(4)\backslash SL(4)} + \frac{5}{2}(y_7\partial y_7)^2 + 5(y_7\partial y_7). \qquad (2.38)$$

The coefficients 5/2 and 5 in this equation have been determined by using the known $D = 8,7\,\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ interaction coefficients. The $\mathcal{R}^4$ coefficient is given in (2.36), whereas the $\partial^4\mathcal{R}^4$ case can be checked using

$$\int_{P(4,1)} E^{SL(5)}_{[1000];\frac{5}{2}} = 2\zeta(5) y_7^{-2} + \frac{4}{3} E^{SL(4)}_{[100];2}, \qquad (2.39)$$

$$\int_{P(4,1)} E^{SL(5)}_{[0010];\frac{5}{2}} = y_7^{-1} E^{SL(4)}_{[010];\frac{5}{2}} + \frac{4\pi\zeta(2)}{3} E^{SL(4)}_{[001];2}. \qquad (2.40)$$

The constant term of the particular solution associated with the parabolic subgroup of relevance to the perturbative expansion is a series of the form

$$\ell_7^7 \int_{P(4,1)} \xi^{SL(5)}_{(0,1)} = \ell_s^7 \sum_{n=0}^{3} \xi_n^{SL(4)} y_7^{n-1}. \qquad (2.41)$$

The coefficient functions $\xi_n^{SL(4)}$ can be determined by substituting this genus expansion into the Laplace equation (2.29) and using (2.31), which gives

$$6\xi_0^{SL(4)} = 4\zeta(3)^2, \quad (2.42) \qquad \left(\Delta_{SO(4)\backslash SL(4)} - \frac{21}{2}\right)\xi_1^{SL(4)} = -8\zeta(3) E^{SL(4)}_{[100];1}, \quad (2.43)$$

$$\left(\Delta_{SO(4)\backslash SL(4)} - 10\right)\xi_2^{SL(4)} = -4\left(E^{SL(4)}_{[100];1}\right)^2, \quad (2.44) \qquad \left(\Delta_{SO(4)\backslash SL(4)} - \frac{9}{2}\right)\xi_3^{SL(4)} = 0. \quad (2.45)$$

Equation (2.42) gives the tree level contribution. The genus-one coefficient is determined by (2.43), which is solved by

$$\xi_1^{SL(4)} = a E^{SL(4)}_{[100];1+2\sqrt{2}} + a' E^{SL(4)}_{[001];1+2\sqrt{2}} + b E^{SL(4)}_{[010];7/2} + \frac{2\zeta(3)}{3} E^{SL(4)}_{[100];1}, \qquad (2.46)$$

for any a,a',b. The constants a,a' must be zero to match the genus-one contribution in $D=8$, and b can be fixed by the decompactification limit. Equation (2.44) defines the genus-two function $\xi_2^{SL(4)}$ which, by construction, in the decompactification limit becomes the genus-two contribution $\hat{E}_1(T)\hat{E}_1(U)+f(T,\overline{T})+f(U,\overline{U})$ of the $\partial^4\mathcal{R}^4$ interaction in eight dimensions. Finally, (2.45) has two independent admissible solutions $E^{SL(4)}_{[001];3}$ and $E^{SL(4)}_{[100];3}$. The first one combines with the solution of the homogeneous equation. Thus, the complete perturbative expansion of the modular function $\xi^{(7)}_{(0,1)}$ is given by

$$\ell_7^7\int_{P(4,1)}\xi^{(7)}_{(0,1)}=\ell_s^7\left(\frac{2\zeta(3)^2}{3}\frac{1}{y_7}+\left(\frac{2\zeta(3)}{3}E^{SL(4)}_{[100];1}+(1+b)E^{SL(4)}_{[010];\frac{7}{2}}\right)+y_7\xi_2^{SL(4)}+2y_7^2\left(E^{SL(4)}_{[001];3}+E^{SL(4)}_{[100];3}\right)+n.p.\right), \tag{2.47}$$

where $n.p.$ indicates non-perturbative contributions. By construction this reproduces (2.25) in the decompactification limit since, as discussed above, in this limit the differential equation becomes the eight-dimensional one. The genus-one contribution in string perturbation theory is given by $I_1^{(3)}\left(j_1^{(0,1)}\right)$, i.e.:

$$I_1^{(3)}\left(j_1^{(0,1)}\right)=\frac{25}{8!}E^{SL(4)}_{[010];\frac{7}{2}}+\frac{\zeta(3)}{16\pi}E^{SL(4)}_{[100];1}, \tag{2.48}$$

which determines the value of $b=5\pi/756-1$. Interestingly, as in $D=8$, the value of the genus-three contribution is given by integrating the three-dimensional lattice factor over the Siegel fundamental domain for $Sp(3,Z)$,

$$\int_{\mathcal{F}_{Sp(3,Z)}}\frac{\left|d^6\tau\right|^2}{(\det\Im m\,\tau)^5}\Gamma_{(3,3)}=\frac{1}{270}\left(E^{SL(4)}_{[100];3}+E^{SL(4)}_{[001];3}\right). \tag{2.49}$$

3. On some equations concerning the Eisenstein series for higher-rank groups, string theory amplitudes and string perturbation theory. [3]

It is useful to translate the terms in the low energy expansion of the analytic part of the following scattering amplitude

$$A_D^{analytic}(s,t,u)=\sum_{p,q=0}^{\infty}\xi^{(D)}_{(p,q)}\left(\phi_{E_{d+1}/K}\right)\sigma_2^p\sigma_3^q\mathcal{R}^4, \tag{3.1}$$

(where $3\le D=10-d\le 10$, $\sigma_n=\left(s^n+t^n+u^n\right)\left(\ell_D^2/4\right)^n$, and ℓ_D is the D-dimensional Planck length) into local terms in an effective action, so that the first three terms beyond classical Einstein theory in D dimensions are

$$S_{R^4}=\ell_D^{8-D}\int d^Dx\sqrt{-G^{(D)}}\,\xi^{(D)}_{(0,0)}\mathcal{R}^4, \tag{3.2}$$

and

$$S_{\partial^4 R^4} = \ell_D^{12-D} \int d^D x \sqrt{-G^{(D)}} \xi_{(1,0)}^{(D)} \partial^4 \mathcal{R}^4 \ , \quad (3.3)$$

and

$$S_{\partial^6 R^4} = \ell_D^{14-D} \int d^D x \sqrt{-G^{(D)}} \xi_{(0,1)}^{(D)} \partial^6 \mathcal{R}^4 \ . \quad (3.4)$$

The automorphic coefficients in (3.2) and (3.3) are given by the simple expressions,

$$\xi_{(0,0)}^{(D)} = 2\zeta(3) E_{\alpha_1;3/2}^{G} := E_{[10^d];3/2}^{E_{d+1}} , \quad (3.5)$$

and

$$\xi_{(1,0)}^{(D)} = \zeta(5) E_{\alpha_1;5/2}^{G} := \frac{1}{2} E_{[10^d];5/2}^{E_{d+1}} , \quad (3.6)$$

for $3 \le D \le 5$ (or $7 \ge d \ge 5$).

The decompactification from D to $D+1$, is the limit associated with the parabolic subgroup $P_{\alpha_{d+1}}$, for $d = 10 - D$. Consistency under decompactification in this limit $r_d / \ell_{D+1} >> 1$ requires

$$\int_{P_{\alpha_{d+1}}} \xi_{(0,0)}^{(D)} \cong \frac{\ell_{D+1}^{8-D}}{\ell_D^{8-D}} \left(\frac{r_d}{\ell_{D+1}} \xi_{(0,0)}^{(D+1)} + \left(\frac{r_d}{\ell_{D+1}} \right)^{8-D} \right), \quad (3.7)$$

and

$$\int_{P_{\alpha_{d+1}}} \xi_{(1,0)}^{(D)} \cong \frac{\ell_{D+1}^{12-D}}{\ell_D^{12-D}} \left(\frac{r_d}{\ell_{D+1}} \xi_{(1,0)}^{(D+1)} + \left(\frac{r_d}{\ell_{D+1}} \right)^{6-D} \xi_{(0,0)}^{(D+1)} + \left(\frac{r_d}{\ell_{D+1}} \right)^{12-D} \right). \quad (3.8)$$

The symbol $\cong$ means that constant factors multiplying each term have been suppressed. For $D = 3,4,5$, i.e. for the duality groups E_6, E_7 and E_8, the automorphic coefficients in these expressions are simply given by the Eisenstein series shown in (3.5) and (3.6).

The perturbative limit is associated with the parabolic subgroup P_{α_1} and is given by $y_D \to 0$ with ℓ_s fixed. In this limit the expansions of the interactions (3.2) and (3.3) are given by the constant terms,

$$\ell_D^{8-D} \int_{P_{\alpha_1}} \xi_{(0,0)}^{(D)} \cong \ell_s^{8-D} \left(\frac{2\zeta(3)}{y_D} + E_{[10^{d-1}];\frac{d}{2}-1}^{SO(d,d)} \right), \quad (3.9)$$

and

$$\ell_D^{12-D} \int_{P_{\alpha_1}} \xi_{(1,0)}^{(D)} \cong \ell_s^{12-D} \left(\frac{\zeta(5)}{y_D} + E_{[10^{d-1}];\frac{d}{2}+1}^{SO(d,d)} + y_D E_{[0^{d-2}10];2}^{SO(d,d)} \right). \quad (3.10)$$

The semi-classical M-theory is the limit associated with the parabolic subgroup P_{α_2}. In this limit the volume $\mathcal{V}_{d+1} \to \infty$ of the M-theory torus becomes large and the semi-classical, or Feynman diagram, approximation to eleven-dimensional supergravity is useful. The constant term of the coefficients in

(3.2) and (3.3) in this parabolic subgroup is given by (using the relation $\ell_D^{D-2} = \ell_{11}^9 / \mathcal{V}_{d+1}$, as well as $r^{1+d} = \left(\mathcal{V}_{d+1} / \ell_{11}^{d+1}\right)^{3/2}$ with $d = 10 - D$),

$$\ell_D^{8-D} \int_{P_{\alpha_2}} \xi_{(0,0)}^{(D)} \cong \frac{\mathcal{V}_{d+1}}{\ell_{11}^3} \left(4\zeta(2) + \left(\frac{\ell_{11}^{d+1}}{\mathcal{V}_{d+1}} \right)^{\frac{3}{d+1}} E^{SL(d+1)}_{[10^{d-1}];\frac{3}{2}} \right), \quad (3.11)$$

and

$$\int_{P_{\alpha_d}} \xi_{(1,0)}^{(D)} \cong \frac{\ell_{11} \mathcal{V}_{d+1}}{\ell_D^{12-D}} \left(\left(\frac{\mathcal{V}_{d+1}}{\ell_{11}^{d+1}} \right)^{\frac{1}{d+1}} E^{SL(d+1)}_{[10^{d-1}];-\frac{1}{2}} + \left(\frac{\ell_{11}^{d+1}}{\mathcal{V}_{d+1}} \right)^{\frac{5}{d+1}} E^{SL(d+1)}_{[10^{d-1}];\frac{5}{2}} + \left(\frac{\ell_{11}^{d+1}}{\mathcal{V}_{d+1}} \right)^{\frac{8}{d+1}} E^{SL(d+1)}_{[010^{d-2}10];2} \right). \quad (3.12)$$

The various contributions in (3.11) agree with the expressions obtained by evaluating the sum of one-loop and two-loop Feynman diagram contributions to the amplitude in eleven-dimensional supergravity compactified on a $(d+1)$-torus. The two terms in the $\mathcal{R}^4$ coefficient (3.11) arise from the compactified one-loop diagrams together with the counterterm diagram, while the terms in $\partial^4\mathcal{R}^4$ coefficient (3.12) arise from the sum of the compactified two-loop diagrams and the one-loop diagram that includes a vertex for the one-loop counterterm.
The constant term of the E_6 Eisenstein series, corresponding to the decompactification from D = 5 to D = 6 results in the sum of two $SO(5,5)$ series in the combination $\frac{1}{2}\hat{E}^{SO(5,5)}_{[10000];\frac{5}{2}} + \frac{4}{45}\hat{E}^{SO(5,5)}_{[00001];3}$ multiplying $r^{20/3}$, where the hats indicate the finite part of the series after subtraction of an ε pole. Although the individual $SO(5,5)$ series have poles in s, the residues of these poles cancel and the sum is finite. This is seen by using the relations $\ell_5^3 = \ell_6^4 / r_5$ and $r = (r_5 / \ell_6)^{1/2}$, leading to

$$\int_{P_{\alpha_6}} \xi_{(1,0)}^{(5)} = \frac{\ell_6^6 r_5}{\ell_5^7} \left(\frac{1}{2}\hat{E}^{SO(5,5)}_{[10000];\frac{5}{2}} + \frac{4}{45}\hat{E}^{SO(5,5)}_{[00001];3} + 2\log\left(\frac{r_5}{\ell_6 \mu}\right)\xi_{(0,0)}^{(6)} + \frac{\zeta(7)}{6}\left(\frac{r_5}{\ell_6}\right)^6 \right). \quad (3.13)$$

where μ is a constant scale factor. The term linear in r_5 is the one that multiplies the $D = 6$ coefficient, $\xi_{(1,0)}^{(6)}$ so that

$$\xi_{(1,0)}^{(6)} = \frac{1}{2}\hat{E}^{SO(5,5)}_{[10000];\frac{5}{2}} + \frac{4}{45}\hat{E}^{SO(5,5)}_{[00001];3} . \quad (3.14)$$

The expression for $E^{SO(d,d)}_{[10^{d-1}];s}$ is expressible by a Siegel-Weil formula relating the integral over the moduli space of genus-one Riemann surfaces of $SL(2,Z)$ Eisenstein series times lattice sums and $SO(d,d)$ Eisenstein series,

$$E^{SO(d,d)}_{[10^{d-1}];s} = \frac{\pi^s}{2\zeta(2s+2-d)\Gamma(s)} \int_{\mathcal{F}_{SL(2,Z)}} \frac{d^2\tau}{\tau_2^2} E_{s+1-\frac{d}{2}}(\tau)\left(\Gamma_{(d,d)}(\tau) - V_d\right), \quad (3.15)$$

where $E_s(\tau) = \sum_{(m,n)\neq(0,0)} y^s / |m + n\tau|^{2s}$ is the usual $SL(2,Z)$ Eisenstein series and $\Gamma_{(d,d)}(\tau)$ is defined in the following expression

$$\Gamma_{(d,d)}(\tau)=V_d\sum_{(m^i,n^i)\in Z^{2d}}e^{-\frac{\pi}{\tau_2}(m^i-n^i\tau)G_{ij}(m^j-n^j\bar{\tau})}. \quad (3.16)$$

It follows from this definition that the series satisfies the functional equation

$$E^{SO(d,d)}_{[10^{d-1}];s}=\frac{\xi(2s-2d+3)\xi(2s-d+1)}{\xi(2s)\xi(2s-d+2)}\frac{\zeta(2s)}{\zeta(2d-2-2s)}E^{SO(d,d)}_{[10^{d-1}];d-1-s}, \quad (3.17)$$

where, in this equation, $\xi(s)$ is the completed Riemann ζ -function $\xi(s)=\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$.

Based on input from string theory the constant term in the parabolic subgroup $P_{\alpha_{d+1}}$, the decompactification limit (decompactification to D = 4), should consist of five components with distinct powers of r

$$\int_{P_{\alpha_{d+1}}}\xi^{(D)}_{(0,1)}\cong\left(\frac{\ell_{D+1}}{\ell_D}\right)^{14-D}\left(\frac{r_d}{\ell_{D+1}}\xi^{(D+1)}_{(0,1)}+\left(\frac{r_d}{\ell_{D+1}}\right)^{14-D}+\left(\frac{r_d}{\ell_{D+1}}\right)^{8-D}\xi^{(D+1)}_{(0,0)}+\left(\frac{r_d}{\ell_{D+1}}\right)^{4-D}\xi^{(D+1)}_{(1,0)}+\right.$$
$$\left.+\left(\frac{r_d}{\ell_{D+1}}\right)^{15-2D}+O\left(e^{-r_d/\ell_{D+1}}\right)\right). \quad (3.18)$$

We now consider the constant term that arises from the solution of the following equation

$$\left(\Delta^{(3)}+198\right)\xi^{(3)}_{(0,1)}=-\left(\xi^{(3)}_{(0,0)}\right)^2, \quad (3.19)$$

which describes the limit of decompactification to $D=4$ in the E_8 case. We are interested in the limit associated with the parabolic subgroup P_{α_8}. We note that the constant term of the source term can be expressed as

$$\int_{P_{\alpha_8}}\left(\xi^{(3)}_{(0,0)}\right)^2=\frac{\ell_4^{11}}{\ell_3^{11}}\left(\frac{r_7}{\ell_4}\left(E^{E_7}_{[10^6];\frac{3}{2}}\right)^2+\left(\frac{r_7}{\ell_4}\right)^5\frac{6\zeta(5)}{\pi}E^{E_7}_{[10^6];\frac{3}{2}}+\left(\frac{r_7}{\ell_4}\right)^9\frac{9\zeta(5)^2}{\pi^2}+O\left(e^{-r_7/\ell_4}\right)\right). \quad (3.20)$$

The perturbative expansion is given by the constant term associated with the maximal parabolic subgroup associated with the node P_{α_1} with Levi subgroup $GL(1)\times SO(7,7)$. String perturbation theory (an expansion in powers of y_3 starting at y_3^{-1}) requires this to have the form

$$\ell_3^{11}\int_{P_{\alpha_1}}\xi^{(3)}_{(0,1)}=\ell_s^{11}\left(\sum_{k=0}^{3}y_3^{k-1}F_k^{SO(7,7)}+O\left(e^{-1/y_3}\right)\right). \quad (3.21)$$

The coefficients $F_k^{SO(7,7)}$ can be determined by a procedure analogous to the one in the previous limit, as follows. First the Laplacian $\Delta^{(3)}$ on $E_8/SO(16)$ is decomposed in this limit into a sum of the Laplacian on $SO(7,7)/(SO(7)\times SO(7))$ and a Laplacian along the y_3 direction

$$\Delta^{(3)} \to \Delta^{SO(7,7)/(SO(7)\times SO(7))} + \frac{1}{2}\left(y_3\partial_{y_3}\right)^2 + 23 y_3\partial_{y_3}\,. \quad (3.22)$$

Next, the constant term of the source is obtained by substituting the expansion of $\xi^{(3)}_{(0,0)}$, resulting in

$$\int_{P_{\alpha_1}} \left(\xi^{(3)}_{(0,0)}\right)^2 = \frac{\ell_s^{11}}{\ell_3^{11}}\left[\frac{4\zeta(3)^2}{y_3} + \frac{6}{\pi}E^{SO(7,7)}_{[10^6];\frac{3}{2}} + \frac{9y_3}{4\pi^2}\left(E^{SO(7,7)}_{[10^6];\frac{3}{2}}\right)^2 + O\left(e^{-1/y_3}\right)\right]. \quad (3.23)$$

The structure of this expression is consistent with (3.21), which we may use as an ansatz for the solution. Substituting (3.21), (3.22) and (3.23) into (3.19) results in equations that determine the coefficients F_k (using $\ell_3 = \ell_s y_3$),

$$\left(\Delta^{SO(7,7)/SO(7)\times SO(7)} - 6\right)F_0^{SO(7,7)} = -4\zeta(3)^2\,, \quad (3.24)$$

$$\left(\Delta^{SO(7,7)/SO(7)\times SO(7)} + \frac{11}{2}\right)F_1^{SO(7,7)} = -\frac{6\zeta(3)}{\pi}E^{SO(7,7)}_{[10^6];\frac{5}{2}}\,, \quad (3.25)$$

$$\left(\Delta^{SO(7,7)/SO(7)\times SO(7)} + 18\right)F_2^{SO(7,7)} = -\frac{9}{4\pi^2}\left(E^{SO(7,7)}_{[10^6];\frac{5}{2}}\right)^2, \quad (3.26)$$

$$\left(\Delta^{SO(7,7)/SO(7)\times SO(7)} + \frac{63}{2}\right)F_3^{SO(7,7)} = 0\,. \quad (3.27)$$

A solution to (3.24) that is compatible with string perturbation theory is the constant

$$F_0^{SO(7,7)} = \frac{2\zeta(3)}{3}\,, \quad (3.28)$$

which is precisely the genus zero (tree-level) contribution. A solution to the homogeneous equation ((3.25) with no source term) that is consistent with string theory is $E^{SO(7,7)}_{[10^6];\frac{11}{2}}$, resulting in a solution of (3.25) given by

$$F_1^{SO(7,7)} = \frac{1}{12}E^{SO(7,7)}_{[10^6];\frac{11}{2}} + \frac{\zeta(3)}{2\pi}E^{SO(7,7)}_{[10^6];\frac{5}{2}}\,. \quad (3.29)$$

The function, $F_3^{SO(7,7)}$, satisfies the source-free (homogeneous) equation (3.27) since there is no y_3^2 term in the constant term of the source. A solution of relevance to string theory is given by the linear combination of maximal parabolic Eisenstein series,

$$F_3^{SO(7,7)} = \alpha E^{SO(7,7)}_{[0^61];3} + \beta E^{SO(7,7)}_{[0^510];3} + \gamma E^{SO(7,7)}_{[0^210^4];\frac{3}{2}}\,, \quad (3.30)$$

where α, β and γ are constants that are determined from the boundary conditions. We find that these Eisenstein series satisfy the relations

$$E^{SO(7,7)}_{[0000001];3} = E^{SO(7,7)}_{[0000010];3}\,, \qquad\qquad E^{SO(7,7)}_{[0010000];\frac{3}{2}} = 0\,. \qquad (3.31)$$

Therefore the expression (3.30) takes the form

$$F_3^{SO(7,7)} = (\alpha+\beta)E^{SO(7,7)}_{[0^61];3}\,. \quad (3.32)$$

The normalisation is fixed by comparison with the genus three contribution in string theory in the limit in which the volume of the 7-torus, $\mathcal{T}^7$, is large, resulting in

$$\alpha+\beta = \frac{vol\left(\mathcal{F}_{Sp(3,Z)}\right)}{2\zeta(6)} = \frac{1}{270}\,, \quad (3.33)$$

where $vol\left(\mathcal{F}_{Sp(3,Z)}\right) = \zeta(6)/135$ is the volume of the Siegel fundamental domain for $Sp(3,Z)$. Thence, we have determined the constant terms of the solution of equation (3.19) for $\xi^{(3)}_{(0,1)}$ in the parabolic subgroup P_{α_1} that agree with the results of the explicit evaluation of string perturbation theory.

4. On some equations concerning U-duality invariant modular form for the $D^6\mathcal{R}^4$ interaction in the effective action of type IIB string theory compactified on T^2. [4]

The complete perturbative part of the modular form is given by the following expression:

$$\zeta(3)^2\left(\tau_2^2V_2\right)^2 + \zeta(3)\tau_2^2V_2E_1\left(T,\overline{T}\right)^{SL(2,Z)} + f\left(T,\overline{T}\right) + \frac{1}{9}\left(\tau_2^2V_2\right)^{-1}E_3\left(T,\overline{T}\right)^{SL(2,Z)} + f\left(U,\overline{U}\right) +$$
$$+\frac{20}{3}E_{-3/2}(M)^{SL(3,Z)}_{pert}E_3\left(U,\overline{U}\right)^{SL(2,Z)} + \frac{1}{2}E_{3/2}(M)^{SL(3,Z)}_{pert}E_1\left(U,\overline{U}\right)^{SL(2,Z)}. \quad (4.1)$$

In ten dimensions (4.1) without the $f\left(T,\overline{T}\right) + f\left(U,\overline{U}\right)$ term, gives all the contributions of the following expression:

$$S \approx l_s^4\int d^{10}x\sqrt{-g}\left(\zeta(3)^2e^{-2\phi} + 2\zeta(3)\zeta(2) + 6\zeta(4)e^{2\phi} + \frac{2}{9}\zeta(6)e^{4\phi} + \ldots\right)D^6\mathcal{R}^4\,, \quad (4.1a)$$

(i.e. the interaction concerning the low energy effective action for type IIB superstring theory in ten dimensions, where the … involve contributions form D-instantons), except the genus two contribution. We first decompactify to nine dimensions by defining

$$T_2 = r_\infty r_B\,, \qquad U_2 = \frac{r_\infty}{r_B}\,,$$

where r_∞ is the direction that is being decompactified. Here r_∞ and r_B are the radii of T^2 in the string frame. Now let us take the limit $r_\infty \to \infty$, so that $T_2, U_2 \to \infty$. This leads to the nine dimensional interaction

$$l_s^5\int d^9x\sqrt{-g_9}\left[\left(r_B e^{-2\phi}\right)\zeta(3)^2+\left\{\frac{15}{\pi^4}\zeta(5)\zeta(6)\left(r_B^5+\frac{1}{r_B^5}\right)+2\zeta(2)\zeta(3)\left(r_B+\frac{1}{r_B}\right)\right\}+\right.$$
$$\left.+4\zeta(2)^2\left(r_B e^{-2\phi}\right)^{-1}+\frac{2}{9}\zeta(6)\left(r_B e^{-2\phi}\right)^{-2}\left(r_B^3+\frac{1}{r_B^3}\right)\right]D^6\mathcal{R}^4, \quad (4.1b)$$

where we have set $l_s\int d^8x\sqrt{-g_8}r_\infty=\int d^9x\sqrt{-g_9}$. Taking the limit $r_B\to\infty$, we get the term in the ten dimensional effective action

$$l_s^4\int d^{10}x\sqrt{-g}\left(\zeta(3)^2e^{-2\phi}+2\zeta(3)\zeta(2)+\frac{2}{9}\zeta(6)e^{4\phi}\right)D^6\mathcal{R}^4, \quad (4.1c)$$

where we have set $l_s\int d^9x\sqrt{-g_9}r_B=\int d^{10}x\sqrt{-g}$.

Given the expression (4.1) for the perturbative part of the modular form, it is natural to propose that the exact expression for the modular form is given by

$$\xi_{(3/2,3/2)}(M)+\frac{20}{3}E_{-3/2}(M)^{SL(3,Z)}E_3(U,\overline{U})^{SL(2,Z)}+f(U,\overline{U})+\frac{1}{2}E_{3/2}(M)^{SL(3,Z)}E_1(U,\overline{U})^{SL(2,Z)}, \quad (4.2)$$

where

$$\xi_{(3/2,3/2)}(M)_{pert}=\zeta(3)^2\left(\tau_2^2V_2\right)^2+\zeta(3)\tau_2^2V_2E_1(T,\overline{T})^{SL(2,Z)}+f(T,\overline{T})+\frac{1}{9}\left(\tau_2^2V_2\right)^{-1}E_3(T,\overline{T})^{SL(2,Z)}. \quad (4.3)$$

We now construct $f(T,\overline{T})$, and also obtain the non-perturbative completion of (4.3). Now, the modular form $\xi_{(3/2,3/2)}(\tau,\overline{\tau})$ for the $D^6\mathcal{R}^4$ interaction in ten dimensions satisfies a Poisson equation

$$\Delta_{SL(2,Z)}\xi_{(3/2,3/2)}(\tau,\overline{\tau})=12\xi_{(3/2,3/2)}(\tau,\overline{\tau})-6\left(E_{3/2}(\tau,\overline{\tau})\right)^2 \quad (4.4)$$

on the fundamental domain of $SL(2,Z)_\tau$. The source term in (4.4) is the square of the modular form for the $\mathcal{R}^4$ interaction, which can be understood based on considerations of supersymmetry. Because $SL(2,Z)_\tau\subset SL(3,Z)_M$, and the U dependence in the expression (4.2) is already fixed, it is natural to propose that $\xi_{(3/2,3/2)}(M)$ satisfies a Poisson equation on the fundamental domain of $SL(3,Z)_M$ given by

$$\Delta_{SL(3,Z)}\xi_{(3/2,3/2)}(M)=\alpha\xi_{(3/2,3/2)}(M)+\beta\left(E_{3/2}(M)\right)^2, \quad (4.5)$$

where α and β are numbers. Again, the source term in (4.5) is the square of the modular form for the $\mathcal{R}^4$ interaction in eight dimensions. Let us first consider the perturbative content of (4.5). We use the relation

$$\Delta_{SL(3,Z)}^{pert}=\Delta_{SL(2,Z)_T}+3\mu^2\frac{\partial^2}{\partial\mu^2}, \quad (4.6)$$

where $\mu = \tau_2^2 V_2$ is the eight dimensional dilaton. Now (4.6) can be obtained based on symmetries alone. From (4.3), we see that every term in the perturbative part of $\xi_{(3/2,3/2)}(M)$ is of the form $\mu^k g_k(T,\overline{T})$, where $g_k(T,\overline{T})$ is $SL(2,Z)_T$ invariant. Thus $\Delta^{pert}_{SL(3,Z)}$ must have the form

$$\Delta^{pert}_{SL(3,Z)} = \xi_1 \Delta_{SL(2,Z)_T} + \xi_2 \mu^2 \frac{\partial^2}{\partial \mu^2} + \xi_3 \mu \frac{\partial}{\partial \mu}, \quad (4.7)$$

where, in this equation, ξ_1, ξ_2, and ξ_3 are numbers. In order to determine them, we act with $\Delta^{pert}_{SL(3,Z)}$ on $E_s(M)^{pert}_{SL(3,Z)}$, such that $\Delta^{pert}_{SL(3,Z)} E_s(M)^{pert}_{SL(3,Z)} = 2s(2s/3-1) E_s(M)^{pert}_{SL(3,Z)}$. Using (4.6), (4.3) and

$$E_{3/2}(M)_{pert} = 2\mu\zeta(3) + 2E_1(T,\overline{T})^{SL(2,Z)}, \quad (4.8)$$

we see that (4.5) gives us the set of equations

$$\alpha + 4\beta = 6, \qquad \alpha + 8\beta = 0, \qquad \frac{\alpha}{9} = \frac{4}{3}, \quad (4.9)$$

and

$$\Delta_{SL(2,Z)_T} f(T,\overline{T}) = \alpha f(T,\overline{T}) + 4\beta \left(E_1(T,\overline{T})\right)^2. \quad (4.10)$$

So (4.9) is solved by

$$\alpha = 12, \qquad \beta = -\frac{3}{2}, \quad (4.11)$$

thus (4.10) reduces to

$$\Delta_{SL(2,Z)_T} f(T,\overline{T}) = 12 f(T,\overline{T}) - 6\left(E_1(T,\overline{T})\right)^2. \quad (4.12)$$

Thus (4.12) gives us the equation for $f(T,\overline{T})$ (and $f(U,\overline{U})$ as well), while (4.5) reduces to

$$\Delta_{SL(3,Z)} \xi_{(3/2,3/2)}(M) = 12 \xi_{(3/2,3/2)}(M) - \frac{3}{2}\left(E_{3/2}(M)\right)^2, \quad (4.13)$$

thus giving us an explicit equation satisfied by the modular form $\xi_{(3/2,3/2)}(M)$. The structure of (4.12) is very similar to (4.4), and our analysis is along similar lines. In (4.12) we substitute

$$f(T,\overline{T}) = f_0(T_2) + \sum_{k \neq 0} f_k(T_2) e^{2\pi i k T_1}. \quad (4.14)$$

Substituting the regularized expression for $E_1(T,\overline{T})$ given by:

$$E_1(T,\overline{T})^{SL(2,Z)} = \frac{\pi^2}{3} T_2 - \pi \ln T_2 + 2\pi\sqrt{T_2} \sum_{m\neq 0, n\neq 0} \left|\frac{m}{n}\right|^{1/2} K_{1/2}(2\pi T_2 |mn|) e^{2\pi i m n T_1} = -\pi \ln\left(T_2 |\eta(T)|^4\right), \quad (4.14b)$$

we get the equation satisfied by $f_0(T_2)$

$$\left(T_2^2\frac{\partial^2}{\partial T_2^2}-12\right)f_0(T_2)=-6\left[(2\zeta(2)T_2-\pi\ln T_2)^2+4\pi^2\sum_{k\neq 0}\mu^2(k,1)e^{-4\pi|k|T_2}\right]. \quad (4.15)$$

Now writing

$$f_0(T_2)=\hat{f}_0(T_2)+\sum_{k\neq 0}\hat{f}_k(T_2)e^{-4\pi|k|T_2}, \quad (4.16)$$

where $\hat{f}_0(T_2)$ is the contribution from the zero worldsheet instanton sector, and $\hat{f}_k(T_2)$ is the contribution from the worldsheet instanton anti-instanton sector with vanishing NS-NS charge, from (4.15) we get differential equations for $\hat{f}_0(T_2)$ and $\hat{f}_k(T_2)$. For $\hat{f}_0(T_2)$ we get

$$\left(T_2^2\frac{\partial^2}{\partial T_2^2}-12\right)\hat{f}_0(T_2)=-6(2\zeta(2)T_2-\pi\ln T_2)^2, \quad (4.17)$$

which has the solution

$$\hat{f}_0(T_2)=\frac{\pi^2}{720}\left[65-20\pi T_2+48\pi^2T_2^2\right]+\pi^2\ln T_2\left[-\frac{\pi T_2}{3}+\frac{1}{2}\ln T_2-\frac{1}{12}\right]+\lambda_1T_2^4+\frac{\lambda_2}{T_2^3}, \quad (4.18)$$

where λ_1 and λ_2 are arbitrary constants.
For $\hat{f}_k(T_2)$, we get

$$\left[T_2^2\left(\frac{\partial^2}{\partial T_2^2}-8\pi|k|\frac{\partial}{\partial T_2}+(4\pi|k|)^2\right)-12\right]\hat{f}_k(T_2)=-24\pi^2\mu^2(k,1), \quad (4.19)$$

which has the solution

$$\hat{f}_k(T_2)=-\frac{\mu^2(k,1)}{448|k|^3\pi T_2^3}\Big[24(4\pi|k|T_2+1)^2+\left((4\pi|k|T_2)^3-3\right)^2+15+(4\pi|k|T_2)^4(2-4\pi|k|T_2)+$$
$$+(4\pi|k|T_2)^7e^{4\pi|k|T_2}Ei(-4\pi|k|T_2)\Big], \quad (4.20)$$

where $Ei(x)$ is the exponential integral function. Using the relation

$$Ei(-x)=e^{-x}\left[-\frac{1}{x}+\int_0^\infty dt\frac{e^{-t}}{(t+x)^2}\right], \qquad x>0, \quad (4.21)$$

we see that the last term in (4.20) has the correct structure to be a worldsheet instanton contribution. Thence, we can rewrite the eq. (4.20) also as follows:

$$\hat{f}_k(T_2)=-\frac{\mu^2(k,1)}{448|k|^3\pi T_2^3}\Big[24(4\pi|k|T_2+1)^2+\left((4\pi|k|T_2)^3-3\right)^2+15+(4\pi|k|T_2)^4(2-4\pi|k|T_2)+$$

$$+\left(4\pi|k|T_2\right)^7 e^{4\pi|k|T_2}\cdot e^{-4\pi|k|T_2}\left(-\frac{1}{4\pi|k|T_2}+\int_0^\infty dt\frac{e^{-t}}{\left(t+4\pi|k|T_2\right)^2}\right)\Bigg]. \quad (4.21b)$$

For the worldsheet instantons with non-vanishing NS-NS charge, we get the equation

$$\left[T_2^2\left(\frac{\partial^2}{\partial T_2^2}-4\pi^2k^2\right)-12\right]f_k(T_2)=-24\pi\left(2\zeta(2)T_2-\pi\ln T_2\right)\mu(k,1)e^{-2\pi|k|T_2}+$$
$$-24\pi^2\sum_{k_1\neq 0,k_2\neq 0,k_1+k_2=k}\mu(k_1,1)\mu(k_2,1)e^{-2\pi(|k_1|+|k_2|)T_2}, \quad (4.22)$$

which in principle can be solved iteratively by expanding in large T_2. Substituting (4.18) and the corresponding expression for $\hat{f}_0(U_2)$ into (4.2), we can easily study the decompactification limit as before. Only the T_2^2 term in the expression for $\hat{f}_0(T_2)$ contributes in this limit. In nine dimensions, in addition to (4.1b) it also gives a term

$$6\zeta(4)l_s^5\int d^9x\sqrt{-g_9}\left(r_Be^{-2\phi}\right)^{-1}\left(r_B^2+\frac{1}{r_B^2}\right)D^6\mathcal{R}^4, \quad (4.23)$$

where we have used $\zeta(4)=\pi^4/90$. Thence, we can rewrite the eq. (4.23) also as follows:

$$6\left(\pi^4/90\right)l_s^5\int d^9x\sqrt{-g_9}\left(r_Be^{-2\phi}\right)^{-1}\left(r_B^2+\frac{1}{r_B^2}\right)D^6\mathcal{R}^4. \quad (4.23b)$$

However, it also gives a divergent contribution

$$\lambda_1 l_s^5\int d^9x\sqrt{-g_9}\left(r_Be^{-2\phi}\right)^{-1}\left(r_B^4+\frac{1}{r_B^4}\right)r_\infty^2 D^6\mathcal{R}^4. \quad (4.24)$$

Further decompactifying to ten dimension, this gives an additional contribution to (4.1c) which is equal to

$$6\zeta(4)l_s^4\int d^{10}x\sqrt{-g}e^{2\phi}D^6\mathcal{R}^4. \quad (4.25)$$

This is a non-trivial consistency check on our proposed modular form. Note that we can send

$$f\left(T,\overline{T}\right)\to f\left(T,\overline{T}\right)+\lambda E_4\left(T,\overline{T}\right)^{SL(2,Z)}, \quad (4.26)$$

for arbitrary λ in (4.12) because $E_4\left(T,\overline{T}\right)^{SL(2,Z)}$ satisfies the homogeneous equation

$$\Delta_{SL(2,Z)}E_4\left(T,\overline{T}\right)^{SL(2,Z)}=12E_4\left(T,\overline{T}\right)^{SL(2,Z)}. \quad (4.27)$$

In the zero worldsheet instanton sector, this involves shifting the coefficient of the T_2^4 term

$$\lambda_1 \to \hat{\lambda}_1 \equiv \lambda_1 + 2\lambda\zeta(8), \quad (4.28)$$

and the T_2^{-3} term

$$\lambda_2 \to \hat{\lambda}_2 \equiv \lambda_2 + \frac{5\pi}{8}\lambda\zeta(7). \quad (4.29)$$

In the sector with world sheet instanton charge k, the extra terms are automatically solutions of the homogeneous equation in (4.22).

Multiplying (4.12) by $E_4(T,\overline{T})^{SL(2,Z)}$ and integrating over the restricted fundamental domain of $SL(2,Z)_T$, we get that

$$\int_{\mathcal{F}_L} \frac{d^2T}{T_2^2} E_4(T,\overline{T})^{SL(2,Z)} \Delta_{SL(2,Z)_T} f(T,\overline{T}) = 12\int_{\mathcal{F}_L} \frac{d^2T}{T_2^2} E_4(T,\overline{T})^{SL(2,Z)} f(T,\overline{T}) +$$
$$-6\int_{\mathcal{F}_L} \frac{d^2T}{T_2^2} E_4(T,\overline{T})^{SL(2,Z)} \left(E_1(T,\overline{T})\right)^2. \quad (4.30)$$

We have restricted the integral to be over $\mathcal{F}_L$ as the integrals diverge and we regulate them, and finally take $L \to \infty$. Integrating by parts, and using (4.27), from (4.30) we get that

$$\int_{-1/2}^{1/2} dT_1 \left(E_4^{SL(2,Z)} \frac{\partial f}{\partial T_2} - f \frac{\partial E_4^{SL(2,Z)}}{\partial T_2} \right)_{T_2 = L} = -6\int_{\mathcal{F}_L} \frac{d^2T}{T_2^2} E_4(T,\overline{T})^{SL(2,Z)} \left(E_1(T,\overline{T})\right)^2. \quad (4.31)$$

Using (4.18) with λ_2 replaced by $\hat{\lambda}_2$, the left hand side of (4.31) yields

$$\zeta(8)\left(-14\hat{\lambda}_2 - \frac{4\pi^4}{15}L^5 - \frac{\pi^3}{2}L^4 - \frac{8\pi^2}{9}L^3 + 2\pi^3 L^4 \ln L - 4\pi^2 L^3 (\ln L)^2 + \frac{8\pi^2}{3}L^3 \ln L \right). \quad (4.32)$$

Using the Poincare series representation for $E_4^{SL(2,Z)}$, and the Rankin-Selberg formula the right hand side of (4.31) yields

$$\zeta(8)\left(-\frac{48}{5}\zeta(2)^2 L^5 - 3\pi\zeta(2)L^4 - \frac{8\pi^2}{9}L^3 + 12\zeta(2)L^4 \ln L - 4\pi^2 L^3 (\ln L)^2 + \frac{8\pi^2}{3}L^3 \ln L \right) +$$
$$-48\pi^2\zeta(8)\int_0^L dT_2 T_2^2 \sum_{k\neq 0} \mu^2(k,1) e^{-4\pi|k|T_2}, \quad (4.33)$$

leading to

$$\hat{\lambda}_2 = \frac{3}{14\pi}\sum_{k=1}^{\infty} \frac{\mu^2(k,1)}{k^3} = \frac{1}{4}\zeta(3)\zeta(5), \quad (4.34)$$

using an identity due to Ramanujan.

The analytic part of the amplitude relevant for the $D^{2k}\mathcal{R}^4$ interaction is given by

$$I(S,T)_{anal} = \frac{2\pi^4 \mathcal{G}^k_{ST}}{k! l_{11}^3 V_3} \sum_{(l_1,l_2,l_3)\neq(0,0,0)} \int_0^\infty d\sigma \sigma^{k-1} e^{-G^{IJ} l_I l_J \sigma / l_{11}^2} = \frac{2\pi^{4+k} \mathcal{G}^k_{ST}}{k!} \sum_{(\hat{l}_1,\hat{l}_2,\hat{l}_3)\neq(0,0,0)} \int_0^\infty d\sigma \sigma^{k-5/2} e^{-\pi G_{IJ} \hat{l}_I \hat{l}_J l_{11}^2/\sigma} , \quad (4.35)$$

where

$$\mathcal{G}^k_{ST} = \int_0^1 d\omega_3 \int_0^{\omega_3} d\omega_2 \int_0^{\omega_2} d\omega_1 \left(-Q(S,T;\omega_r)\right)^k . \quad (4.36)$$

The two perturbative contributions are given by

$$I(S,T)^1_{anal} = 4\pi^{2k+5/2} \Gamma\left(\frac{3}{2}-k\right) \zeta(3-2k) l_{11}^{2k-3} e^{2(2k-3)\phi^A/3} \frac{\mathcal{G}^k_{ST}}{k!} , \quad (4.37)$$

and

$$I(S,T)^2_{anal} = \frac{2\pi^4 l_{11}^{2k-3}}{k R_{11}^k} \left(T_2^A\right)^{k-1} \mathcal{G}^k_{ST} E_k\left(U^A, \overline{U}^A\right)^{SL(2,Z)} . \quad (4.38)$$

This leads to

$$A_4 = \frac{\kappa_{11}^4 \hat{K}}{(2\pi)^{11} l_{11}^3} \left[\frac{2\pi^4}{k} \left(T_2^A\right)^{k-1} E_k\left(U^A,\overline{U}^A\right)^{SL(2,Z)} + \frac{4\pi^{2k+5/2}}{k!} \Gamma\left(\frac{3}{2}-k\right) \zeta(3-2k) e^{2(k-1)\phi^A} \right] l_s^{2k} \mathcal{W}^k , \quad (4.39)$$

where

$$\mathcal{W}^k = \mathcal{G}^k_{ST} + \mathcal{G}^k_{SU} + \mathcal{G}^k_{UT} . \quad (4.40)$$

The expression (4.39) leads to terms in the IIB effective action given by

$$l_s^{2k} \int d^8x \sqrt{-g_8} \left[\frac{2\pi}{k} \left(U_2^B\right)^k E_k\left(T^B,\overline{T}^B\right)^{SL(2,Z)} + \frac{4\pi^{2k-1/2}}{k!} \Gamma\left(\frac{3}{2}-k\right) \zeta(3-2k) \left(e^{-2\phi^B} T_2^B\right)^{1-k} \left(U_2^B\right)^k \right] D^{2k} \mathcal{R}^4 . \quad (4.41)$$

Given the perturbative equality of the amplitude in the two type II theories and the Eisenstein series of order s for $SL(2,Z)$, defined by

$$E_s\left(T,\overline{T}\right)^{SL(2,Z)} = \sum_{(p,q)\neq(0,0)} \frac{T_2^s}{|p+qT|^{2s}} = 2\zeta(2s) T_2^s + 2\sqrt{\pi} T_2^{1-s} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1) +$$

$$+ \frac{2\pi^s \sqrt{T_2}}{\Gamma(s)} \sum_{m_1\neq 0, m_2\neq 0} \left|\frac{m_1}{m_2}\right|^{s-1/2} K_{s-1/2}\left(2\pi T_2 |m_1 m_2|\right) e^{2\pi i m_1 m_2 T_1} = 2\zeta(2s) T_2^s + 2\sqrt{\pi} T_2^{1-s} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1) +$$

$$+ \frac{4\pi^s \sqrt{T_2}}{\Gamma(s)} \sum_{k\neq 0} |k|^{s-1/2} \mu(k,s) K_{s-1/2}\left(2\pi T_2 |k|\right) e^{2\pi i k T_1} , \quad (4.41b)$$

it is natural to enhance the $\left(U_2^B\right)^k$ factors to $E_k\left(U^B,\overline{U}^B\right)^{SL(2,Z)}$, and symmetrize in U^B and T^B. Thus (4.41) gets enhanced to

$$l_s^{2k}\int d^8x\sqrt{-g_8}\left[\frac{(2k)!}{(2\pi)^{2k-1}|B_{2k}|k}E_k\left(T^B,\bar{T}^B\right)^{SL(2,Z)}E_k\left(U^B,\bar{U}^B\right)^{SL(2,Z)}+\frac{4\Gamma\left(k+\frac{1}{2}\right)\Gamma(k-1)\zeta(2k-2)}{\pi^{2k-3/2}|B_{2k}|}\right.$$
$$\left.\left(e^{-2\phi^B}T_2^B\right)^{1-k}\times\left(E_k\left(T^B,\bar{T}^B\right)^{SL(2,Z)}+E_k\left(U^B,\bar{U}^B\right)^{SL(2,Z)}\right)\right]D^{2k}\mathcal{R}^4, \quad (4.42)$$

where we have used the relations

$$\zeta(2k)=\frac{2^{2k-1}\pi^{2k}|B_{2k}|}{(2k)!}, \quad (4.43)$$

where k is a positive integer, B_{2k} are the Bernoulli numbers, and

$$\Gamma(2x)=\frac{2^{2x-1/2}}{\sqrt{2\pi}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right). \quad (4.44)$$

Decompactifying to nine dimensions, we see that (4.42) gives the interaction

$$l_s^{2k-1}\int d^9x\sqrt{-g_9}\left[\frac{4\pi^{3/2}}{k!}\zeta(2k-1)\Gamma\left(k-\frac{1}{2}\right)\left(r_B^{2k-1}+\frac{1}{r_B^{2k-1}}\right)+4\pi^2\frac{\zeta(2k-2)}{k(k-1)}\left(e^{-2\phi^B}r_B\right)^{1-k}\left(r_B^k+\frac{1}{r_B^k}\right)\right]D^{2k}\mathcal{R}^4 \quad (4.45)$$

which contributes at genus one and at genus k. It also gives the divergent contribution

$$\frac{4\pi}{k}\zeta(2k)l_s^{2k-1}\int d^9x\sqrt{-g_9}r_\infty^{2k-1}D^{2k}\mathcal{R}^4 \quad (4.46)$$

which leads to the threshold singularities. Further decompactifying (4.45) to ten dimensions, this leads to the interaction

$$4\pi^2\frac{\zeta(2k-2)}{k(k-1)}l_s^{2k-2}\int d^{10}x\sqrt{-g}e^{-2(1-k)\phi^B}D^{2k}\mathcal{R}^4, \quad (4.47)$$

which contributes at genus k, while the genus one contribution vanishes. It also gives the divergent contribution

$$\frac{4\pi^{3/2}}{k!}\zeta(2k-1)\Gamma\left(k-\frac{1}{2}\right)l_s^{2k-2}\int d^{10}x\sqrt{-g}r_B^{2(k-1)}D^{2k}\mathcal{R}^4, \quad (4.48)$$

corresponding to the threshold singularities.

5. Mathematical connections

Now in this Section, we have described various possible mathematical connections between the arguments above mentioned and some sectors of Number Theory, principally with some equations concerning the Ramanujan's modular equations that are related to the physical vibrations of the bosonic strings and of the superstrings, some Ramanujan's identities concerning π and the zeta strings.

Now we want to show an interesting equation concerning the gauge fields as described in the **Jormakka's paper** "*Solutions to Yang-Mills equations* **[5]** and connected with an Ramanujan's identity concerning π in the my recent paper: "*On some equations concerning quantum electrodynamics coupled to quantum gravity, the gravitational contributions to the gauge couplings and quantum effects in the theory of gravitation: mathematical connections with some sector of String Theory and Number Theory*". **[6]**

$$\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\int dy_1 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_1^2} = \int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dx_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dy_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$= \left[\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2 = 8\frac{\pi^3}{4}\times\frac{1}{\left(\sqrt{2}\beta\right)^6} \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3\frac{\cos\pi x^2}{\sinh\pi x}dx + \int_0^\infty x^3\frac{\sin\pi x^2}{\sinh\pi x}dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right); \quad (5.1)$$

This equation can be connected with the eq. (1.21) that we have multiplied for $\frac{56\pi^5}{128}$ as follows:

$$\frac{56\pi^5}{128}\cdot\frac{384\pi^2}{7}\int_0^\infty d\Omega_2\Omega_2^3\sum_{k\neq 0}k^2\mu\left(|k|,\frac{3}{2}\right)^2\mathcal{K}_1^2\left(2\pi|k|\Omega_2\right) = \frac{56\pi^5}{128}\cdot\frac{32}{7\pi^2}\sum_{k\geq 1}\frac{\mu\left(k,\frac{3}{2}\right)^2}{k^2} \Rightarrow$$

$$24\pi^7\int_0^\infty d\Omega_2\Omega_2^3\sum_{k\neq 0}k^2\mu\left(|k|,\frac{3}{2}\right)^2\mathcal{K}_1^2\left(2\pi|k|\Omega_2\right) = 2\pi^3\sum_{k\geq 1}\frac{\mu\left(k,\frac{3}{2}\right)^2}{k^2}. \quad (5.2)$$

Thence, we obtain the following expression:

$$\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\int dy_1 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_1^2} = \int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dx_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dy_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$= \left[\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2 = 8\frac{\pi^3}{4}\times\frac{1}{\left(\sqrt{2}\beta\right)^6} \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3 \frac{\cos \pi x^2}{\sinh \pi x}dx + \int_0^\infty x^3 \frac{\sin \pi x^2}{\sinh \pi x}dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right) \Rightarrow$$

$$\Rightarrow 24\pi^7 \int_0^\infty d\Omega_2 \Omega_2^3 \sum_{k \neq 0} k^2 \mu\left(|k|, \frac{3}{2}\right)^2 \mathcal{K}_1^2\left(2\pi|k|\Omega_2\right) = 2\pi^3 \sum_{k \geq 1} \frac{\mu\left(k, \frac{3}{2}\right)^2}{k^2}. \quad (5.3)$$

With regard the mathematical connections with π and Φ, thence with the **universal music system based on Phi**, we have the following value in the eq. (1.21) $\frac{32}{7\pi^2} = 0{,}463182$. Now, this number is very near to the following $0{,}46440787 = \frac{1{,}458980337}{\pi}$, where 1,458980337 is a value of the system, that is about $(\Phi)^{n/7} = (\Phi)^{5.50/7} = 1{,}459501596$. We note also that $\frac{1{,}459501596}{\pi} = 0{,}464573$. Thence, we have $0{,}46457 \cong 0{,}46440 \cong 0{,}46318$, a very good approximation!

With regard the eq. (2.25)

$$\ell_8^6 \int_{-1/2}^{1/2} d\Omega_1 dB_{RR} \xi_{(0,1)}^{(8)} = \ell_s^6 \left(\frac{2\zeta(3)^2}{3y_8} + \frac{64\pi}{3} I_1^{(2)}\left(j_1^{(0,1)}\right) + \frac{2\pi\zeta(3)}{9}\log(y_8) + \frac{2}{3} y_8 I_2^{(2)}\left(j_2^{(0,1)}\right) + \frac{\pi}{9}\left(\frac{\pi}{2} + I_1^{(2)}\left(j_1^{(0,0)}\right)\right)\right.$$
$$\left. y_8 \log(y_8) + \frac{\pi^2}{27} y_8 \log(y_8)^2 + 20 y_8^2 I_3^{(2)}\left(j_3^{(0,1)}\right) + O\left(e^{-(T_2 y_8)^{-\frac{1}{2}}}, e^{-T_2^{1/2} y_8^{-1/2}}\right)\right),$$

that we have multiplied for $\frac{27\pi}{4}$, we obtain the following expression:

$$\frac{27\pi}{4} \ell_8^6 \int_{-1/2}^{1/2} d\Omega_1 dB_{RR} \xi_{(0,1)}^{(8)} = \ell_s^6 \left(\frac{9\pi\zeta(3)^2}{2y_8} + 144\pi^2 I_1^{(2)}\left(j_1^{(0,1)}\right) + \frac{3\pi^2\zeta(3)}{2}\log(y_8) + 9\pi y_8 I_2^{(2)}\left(j_2^{(0,1)}\right) + \frac{3\pi^2}{4}\left(\frac{\pi}{2} + I_1^{(2)}\left(j_1^{(0,0)}\right)\right)\right.$$
$$\left. y_8 \log(y_8) + \frac{\pi^3}{4} y_8 \log(y_8)^2 + 135\pi y_8^2 I_3^{(2)}\left(j_3^{(0,1)}\right) + O\left(e^{-(T_2 y_8)^{-\frac{1}{2}}}, e^{-T_2^{1/2} y_8^{-1/2}}\right)\right). \quad (5.4)$$

This equation can be connected with the (5.1), and we obtain:

$$\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)} \frac{1}{\sqrt{3}} \int dy_1 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_1^2} = \int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)} \frac{1}{\sqrt{3}} \sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} \int dx_3 e^{-\beta^2 y_3^2} \int dx_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} \int dx_3 e^{-\beta^2 y_3^2} \int dy_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2} \int dx_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2} \int dy_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$= \left[\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)} \frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2 = 8\frac{\pi^3}{4} \times \frac{1}{\left(\sqrt{2}\beta\right)^6} \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3 \frac{\cos \pi x^2}{\sinh \pi x}dx + \int_0^\infty x^3 \frac{\sin \pi x^2}{\sinh \pi x}dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right) \Rightarrow$$

$$\Rightarrow \frac{27\pi}{4}\ell_8^6\int_{-1/2}^{1/2} d\Omega_1 dB_{RR}\xi_{(0,1)}^{(8)} = \ell_s^6\left(\frac{9\pi\zeta(3)^2}{2y_8}+144\pi^2 I_1^{(2)}\left(j_1^{(0,1)}\right)+\frac{3\pi^2\zeta(3)}{2}\log(y_8)+9\pi y_8 I_2^{(2)}\left(j_2^{(0,1)}\right)+\frac{3\pi^2}{4}\left(\frac{\pi}{2}+I_1^{(2)}\left(j_1^{(0,0)}\right)\right.\right.$$

$$\left.\left. y_8\log(y_8)+\frac{\pi^3}{4}y_8\log(y_8)^2+135\pi y_8^2 I_3^{(2)}\left(j_3^{(0,1)}\right)+O\left(e^{-(T_2 y_8)^{-\frac{1}{2}}}, e^{-T_2^{1/2}y_8^{-1/2}}\right)\right)\right). \quad (5.5)$$

With regard the mathematical connections with π and Φ, thence with the **universal music system based on Phi**, we have the following values in the eq. (2.25):

$$\frac{\pi}{9} = 0{,}3490 \cong 0{,}34830590; \quad \frac{\pi^2}{27} = 0{,}365540 \cong 0{,}3647506; \quad \frac{64\pi}{3} = 67{,}02064 \cong 67{,}157.$$

Now, we note that these values, i.e. 0,34830590; 0,3647506; 67,157 are all connected with $1/\pi$ and π. Indeed, we have: $\frac{1{,}094235253}{\pi} = 0{,}348305898$; $\frac{1{,}145898033}{\pi} = 0{,}364750672$; $(12{,}13525491\cdot\pi)+(9{,}241808286\cdot\pi) = 38{,}12402767 + 29{,}03399702 = 67{,}158024$. We want to evidence that the numbers **1,094235253**; **1,145898033**; **12,13525491** and **9,241808286** are all belonging to the column "system".

With regard the equation (2.33):

$$\int_{P(3,2)} E_{[0010];\frac{7}{2}}^{SL(5)} = 2\zeta(6)\zeta(7)\left(\frac{r_3}{\ell_8}\right)^{\frac{42}{5}} + \frac{\pi^2\zeta(2)}{5}\left(\frac{\ell_8}{r_3}\right)^{\frac{8}{5}} E_{[10];\frac{5}{2}}^{SL(3)} + \frac{8}{15}\left(\frac{r_3}{\ell_8}\right)^{\frac{12}{5}} E_{[01];3}^{SL(3)} E_3^{SL(2)},$$

if we multiply for $\frac{5\pi}{4}$, we obtain the following expression:

$$\frac{5\pi}{4}\int_{P(3,2)} E_{[0010];\frac{7}{2}}^{SL(5)} = \frac{5\pi}{4}2\zeta(6)\zeta(7)\left(\frac{r_3}{\ell_8}\right)^{\frac{42}{5}} + \frac{\pi^3}{4}\zeta(2)\left(\frac{\ell_8}{r_3}\right)^{\frac{8}{5}} E_{[10];\frac{5}{2}}^{SL(3)} + \frac{2\pi}{3}\left(\frac{r_3}{\ell_8}\right)^{\frac{12}{5}} E_{[01];3}^{SL(3)} E_3^{SL(2)}. \quad (5.6)$$

This equation can be connected with the (5.1), and we obtain:

$$\int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\int dy_1 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_1^2} = \int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dx_2 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_2^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dy_2 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_2^2} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$= \left[\int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2 = 8\frac{\pi^3}{4}\times\frac{1}{\left(\sqrt{2}\beta\right)^6} \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3\frac{\cos\pi x^2}{\sinh\pi x}dx + \int_0^\infty x^3\frac{\sin\pi x^2}{\sinh\pi x}dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right) \Rightarrow$$

$$\Rightarrow \frac{5\pi}{4}\int_{P(3,2)} E^{SL(5)}_{[0010];\frac{7}{2}} = \frac{5\pi}{4} 2\zeta(6)\zeta(7)\left(\frac{r_3}{\ell_8}\right)^{\frac{42}{5}} + \frac{\pi^3}{4}\zeta(2)\left(\frac{\ell_8}{r_3}\right)^{\frac{8}{5}} E^{SL(3)}_{[10];\frac{5}{2}} + \frac{2\pi}{3}\left(\frac{r_3}{\ell_8}\right)^{\frac{12}{5}} E^{SL(3)}_{[01];3} E^{SL(2)}_{3}. \quad (5.7)$$

With regard the mathematical connections with π and Φ, thence with the **universal music system based on Phi**, we have the following values in the eq. (5.6):

$$\frac{2\pi}{3} = 2{,}094395102 \cong 2{,}100899497; \qquad \frac{5\pi}{4} = 3{,}926990817 \cong 3{,}922506639;$$

$$\frac{\pi^3}{4} = 7{,}75156917 \cong 7{,}766444154; \quad \frac{\pi^2}{5} = 1{,}97392088 \cong 1{,}96726328.$$

Now, we note that these values, i.e. 2,100899497; 3,922506639; 7,766444154; 1,96726328 are all connected with $1/\pi$, π and 1/1,375 (where 1,375 is the constant regarding the number of the partitions). Indeed, we have:

$$0{,}66873708 \cdot \pi = 2{,}100899497; \quad 5{,}393446629 \cdot \frac{1}{1{,}375} = 3{,}922506639;$$

$$2{,}472135954 \cdot \pi = 7{,}76644415; \quad 6{,}1803398874 \cdot \frac{1}{\pi} = 1{,}96726328.$$

We want to evidence that the numbers **0,66873708**; **5,393446629**; **2,472135954** and **6,1803398874** are all belonging to the column "system".

With regard the eqs (4.19) and (4.21b), we have the following mathematical connections:

$$\left[T_2^2\left(\frac{\partial^2}{\partial T_2^2} - 8\pi|k|\frac{\partial}{\partial T_2} + (4\pi|k|)^2\right) - 12\right]\hat{f}_k(T_2) = -24\pi^2\mu^2(k,1) \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} = \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}, \quad (5.8)$$

$$\hat{f}_k(T_2) = -\frac{\mu^2(k,1)}{448|k|^3\pi T_2^3}\Big[24(4\pi|k|T_2+1)^2 + \left((4\pi|k|T_2)^3 - 3\right)^2 + 15 + (4\pi|k|T_2)^4(2 - 4\pi|k|T_2) +$$

$$+ (4\pi|k|T_2)^7 e^{4\pi|k|T_2}\cdot e^{-4\pi|k|T_2}\left(-\frac{1}{4\pi|k|T_2} + \int_0^\infty dt\frac{e^{-t}}{(t+4\pi|k|T_2)^2}\right)\Big] \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} = \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}, \quad (5.9)$$

i.e. with the Ramanujan's modular equations regarding the physical vibrations of the bosonic strings that are connected also with π (thence with Φ by the simple expression $\sqrt{\pi\cdot\frac{5}{6}}=\Phi$).

With regard the mathematical connections with π and Φ, thence with the **universal music system based on Phi**, we have the following values in the eqs. (4.18) and (4.23b):

$$\hat{f}_0(T_2)=\frac{\pi^2}{720}\left[65-20\pi T_2+48\pi^2 T_2^2\right]+\pi^2\ln T_2\left[-\frac{\pi T_2}{3}+\frac{1}{2}\ln T_2-\frac{1}{12}\right]+\lambda_1 T_2^4+\frac{\lambda_2}{T_2^3},$$

$$6\left(\pi^4/90\right)l_s^5\int d^9x\sqrt{-g_9}\left(r_B e^{-2\phi}\right)^{-1}\left(r_B^2+\frac{1}{r_B^2}\right)D^6\mathcal{R}^4 .$$

$$\frac{\pi^2}{720}=0,013707783\cong 0,013551238=\frac{0,042572472}{\pi};$$

$$\frac{\pi^4}{90}=1,082323234\cong 1,082022746=0,344418537\cdot\pi$$

We want to evidence that the numbers **0,042572472** and **0,344418537** are all belonging to the column "system".

Also in the eq. (4.29) we have a value connected with the **universal music system based on Phi:**

$$\lambda_2\to\hat{\lambda}_2\equiv\lambda_2+\frac{5\pi}{8}\lambda\zeta(7).$$

Indeed, we have that

$$\frac{5\pi}{8}=1,963495409\cong\frac{6,180339887}{\pi}=1,967263286;$$

and the number **6,180339887** is belonging to the column "system".

With regard the eqs. (4.32), (4.33) and (4.34), we have the following values connected with the **universal music system based on Phi:**

$$\zeta(8)\left(-14\hat{\lambda}_2-\frac{4\pi^4}{15}L^5-\frac{\pi^3}{2}L^4-\frac{8\pi^2}{9}L^3+2\pi^3L^4\ln L-4\pi^2L^3(\ln L)^2+\frac{8\pi^2}{3}L^3\ln L\right);$$

$$\hat{\lambda}_2 = \frac{3}{14\pi}\sum_{k=1}^{\infty}\frac{\mu^2(k,1)}{k^3} = \frac{1}{4}\zeta(3)\zeta(5).$$

$\frac{4\pi^4}{15} = 25{,}97575761 \cong 8{,}090169943 \cdot \pi = 25{,}41601846$; or $= 0{,}36067977 \cdot 72 = 25{,}9689437$;

$\frac{\pi^3}{2} = 15{,}50313834 \cong 19{,}41640786 \times \frac{4}{5} = 15{,}53312629$;

$\frac{8\pi^2}{9} = 8{,}772981689 \cong \frac{27{,}41640786}{\pi} = 8{,}726913665$;

$\frac{8\pi^2}{3} = 26{,}31894507 \cong \frac{82{,}24922359}{\pi} = 26{,}180741$;

$\frac{3}{14\pi} = 0{,}068209261 \cong 0{,}02192602 \cdot \pi = 0{,}068882623$.

We want to evidence that the numbers **0,02192602; 0,36067977; 8,090169943; 19,41640786; 27,41640786; 82,24922359** are all belonging to the column "system".

With regard the eq. (4.33), if we multiply it for $\frac{\pi}{32}$ we have the following mathematical connection with the eq. (5.1):

$$\frac{\pi}{32}\zeta(8)\left(\frac{\pi}{32}\cdot -\frac{48}{5}\zeta(2)^2 L^5 - \frac{3\pi^2}{32}\zeta(2)L^4 - \frac{\pi^3}{4}\cdot\frac{1}{9}L^3 + \frac{\pi}{32}\cdot 12\zeta(2)L^4\ln L - \frac{\pi^3}{8}L^3(\ln L)^2 + \frac{\pi^3}{4}\cdot\frac{1}{3}L^3\ln L\right) +$$

$$-\frac{6\pi^3}{4}\zeta(8)\int_0^L dT_2 T_2^2 \sum_{k\neq 0}\mu^2(k,1)e^{-4\pi|k|T_2} \Rightarrow$$

$$\Rightarrow \int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\int dy_1 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_1^2} = \int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dx_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dy_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$= \left[\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2 = 8\frac{\pi^3}{4}\times\frac{1}{\left(\sqrt{2}\beta\right)^6} \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3\frac{\cos\pi x^2}{\sinh\pi x}dx + \int_0^\infty x^3\frac{\sin\pi x^2}{\sinh\pi x}dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right). \quad (5.10)$$

Furthermore, we have also the mathematical connection between the eq. (4.33) and the Ramanujan's modular equations regarding the physical vibrations of the bosonic strings that are connected also with π (thence with Φ by the simple expression $\sqrt{\pi \cdot \frac{5}{6}} = \Phi$):

$$\zeta(8)\left(-\frac{48}{5}\zeta(2)^2 L^5 - 3\pi\zeta(2)L^4 - \frac{8\pi^2}{9}L^3 + 12\zeta(2)L^4 \ln L - 4\pi^2 L^3 (\ln L)^2 + \frac{8\pi^2}{3}L^3 \ln L\right) +$$

$$-48\pi^2 \zeta(8)\int_0^L dT_2 T_2^2 \sum_{k \neq 0} \mu^2(k,1) e^{-4\pi|k|T_2} \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} = \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (5.11)$$

Now we want to describe the mathematical connections between some equations described in the present paper and the most important equations concerning the zeta strings. **[7]**
We remember that the equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \quad (5.12)$$

which has an evident solution $\phi = 0$.
For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0| > \sqrt{2}+\varepsilon} e^{-ik_0 t} \zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (5.13)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk} \zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n \geq 1} \theta^{\frac{n(n-1)}{2}} \phi^n , \quad (5.14)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta = \frac{1}{(2\pi)^D}\int e^{ixk} \zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n \geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1} - 1\right)\right], \quad (5.15)$$

and one can easily see trivial solution $\phi = \theta = 0$.

We note that the eqs. (2.25) and (2.33) can be related with the eq. (5.13). We obtain:

$$\ell_8^6 \int_{-1/2}^{1/2} d\Omega_1 dB_{RR} \xi_{(0,1)}^{(8)} = \ell_s^6 \left(\frac{2\zeta(3)^2}{3y_8} + \frac{64\pi}{3} I_1^{(2)}\left(j_1^{(0,1)}\right) + \frac{2\pi\zeta(3)}{9}\log(y_8) + \frac{2}{3} y_8 I_2^{(2)}\left(j_2^{(0,1)}\right) + \frac{\pi}{9}\left(\frac{\pi}{2} + I_1^{(2)}\left(j_1^{(0,0)}\right)\right)\right.$$

$$y_8 \log(y_8) + \frac{\pi^2}{27} y_8 \log(y_8)^2 + 20 y_8^2 I_3^{(2)}\left(j_3^{(0,1)}\right) + O\left(e^{-(T_2 y_8)^{-\frac{1}{2}}}, e^{-T_2^{1/2} y_8^{-1/2}}\right)\Bigg) \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}; \quad (5.16)$$

$$\int_{P(3,2)} E^{SL(5)}_{[0010];\frac{7}{2}} = 2\zeta(6)\zeta(7)\left(\frac{r_3}{\ell_8}\right)^{\frac{42}{5}} + \frac{\pi^2\zeta(2)}{5}\left(\frac{\ell_8}{r_3}\right)^{\frac{8}{5}} E^{SL(3)}_{[10];\frac{5}{2}} + \frac{8}{15}\left(\frac{r_3}{\ell_8}\right)^{\frac{12}{5}} E^{SL(3)}_{[01];3} E_3^{SL(2)} \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (5.17)$$

Also the eqs. (3.20) and (3.23) can be related with the eq (5.13). We obtain:

$$\int_{P_{\alpha_8}} \left(\xi^{(3)}_{(0,0)}\right)^2 = \frac{\ell_4^{11}}{\ell_3^{11}}\left(\frac{r_7}{\ell_4}\left(E^{E_7}_{[10^6];\frac{3}{2}}\right)^2 + \left(\frac{r_7}{\ell_4}\right)^5 \frac{6\zeta(5)}{\pi} E^{E_7}_{[10^6];\frac{3}{2}} + \left(\frac{r_7}{\ell_4}\right)^9 \frac{9\zeta(5)^2}{\pi^2} + O\left(e^{-r_7/\ell_4}\right)\right) \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}; \quad (5.18)$$

$$\int_{P_{\alpha_1}} \left(\xi^{(3)}_{(0,0)}\right)^2 = \frac{\ell_s^{11}}{\ell_3^{11}}\left[\frac{4\zeta(3)^2}{y_3} + \frac{6}{\pi} E^{SO(7,7)}_{[10^6];\frac{3}{2}} + \frac{9y_3}{4\pi^2}\left(E^{SO(7,7)}_{[10^6];\frac{3}{2}}\right)^2 + O\left(e^{-1/y_3}\right)\right] \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (5.19)$$

In conclusion, also the eqs. (4.42) and (4.45) can be connected with the eq. (5.14). We obtain:

$$l_s^{2k}\int d^8x\sqrt{-g_8}\left[\frac{(2k)!}{(2\pi)^{2k-1}|B_{2k}|k} E_k\left(T^B,\bar{T}^B\right)^{SL(2,Z)} E_k\left(U^B,\bar{U}^B\right)^{SL(2,Z)} + \frac{4\Gamma\left(k+\frac{1}{2}\right)\Gamma(k-1)\zeta(2k-2)}{\pi^{2k-3/2}|B_{2k}|}\right.$$

$$\left.\left(e^{-2\phi^B}T_2^B\right)^{1-k} \times \left(E_k\left(T^B,\bar{T}^B\right)^{SL(2,Z)} + E_k\left(U^B,\bar{U}^B\right)^{SL(2,Z)}\right)\right] D^{2k}\mathcal{R}^4 \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n, \quad (5.20)$$

$$l_s^{2k-1}\int d^9x\sqrt{-g_9}\left[\frac{4\pi^{3/2}}{k!}\zeta(2k-1)\Gamma\left(k-\frac{1}{2}\right)\left(r_B^{2k-1} + \frac{1}{r_B^{2k-1}}\right) + 4\pi^2\frac{\zeta(2k-2)}{k(k-1)}\left(e^{-2\phi^B}r_B\right)^{1-k}\left(r_B^k + \frac{1}{r_B^k}\right)\right] D^{2k}\mathcal{R}^4 \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n. \quad (5.21)$$

Also here, we can observe the mathematical connections with π and Φ, thence with the **universal music system based on Phi**. Indeed, we have the following values in the eqs. (2.33) and (4.45)

$$\frac{\pi^2}{5} = 1{,}97392088 \cong \frac{6{,}180339887}{\pi} = 1{,}967263286\,;$$

$$4\pi^{3/2} = 44{,}54662397 \cong 14{,}12022659 \cdot \pi = 44{,}36000012\,.$$

We want to evidence that the numbers **6,180339887** and **14,12022659;** are all belonging to the column "system".
With regard the mathematical connections with the Ramanujan's modular equations regarding the physical vibrations of the bosonic strings that are connected also with π (thence with Φ by the simple expression $\sqrt{\pi \cdot \frac{5}{6}} = \Phi$), we have that also the eqs. (1.19) and (2.11) can be related with them. Indeed, we have that:

$$\int_{-1/2}^{1/2} d\Omega_1 \left(Z_4 \partial_{\Omega_2} \xi_{(3/2,3/2)} - \left(\partial_{\Omega_2} Z_4 \right) \xi_{(3/2,3/2)} \right) \Big|_{\Omega_2 = L \to \infty} = -\zeta(8) \left(8\zeta(3)^2 L^6 + 48\zeta(3)\zeta(2) L^4 + 96\zeta(2)^2 L^2 + 14\beta \right)$$

$$\Rightarrow \frac{4\left[anti\log \dfrac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \dfrac{\sqrt{142}}{t^2 w'}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]} = \frac{\pi \sqrt{142}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]}\,; \quad (5.22)$$

$$\ell_9^5 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\Omega_1 \xi_{(0,1)}^{(9)} = \ell_s^5 r_B \left(\frac{\zeta(3)^2}{3 g_B^2} + \frac{\zeta(2)\zeta(3)}{9} \left(1 + \frac{\ell_s^2}{r_B^2} \right) + \frac{\zeta(5)\zeta(2)}{189} \left(\frac{r_B^2}{\ell_s^4} + \frac{\ell_s^6}{r_B^6} \right) + \frac{5\zeta(4) g_B^2}{9} \frac{\ell_s^2}{r_B^2} + \right.$$

$$\left. + \frac{\zeta(4) g_B^2}{3} \left(1 + \frac{\ell_s^4}{r_B^4} \right) + \frac{7\zeta(6)}{24^2} g_B^4 \left(1 + \frac{\ell_s^6}{r_B^6} \right) + O\left(e^{-1/g_B} \right) \right) \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log \dfrac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \dfrac{\sqrt{142}}{t^2 w'}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]} = \frac{\pi \sqrt{142}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]}\,. \quad (5.23)$$

Appendix A (Di Noto Francesco) [8]

In this Appendix, we have analyzed some pure numbers concerning various equations described in the present paper. We have obtained some useful mathematical connections with some sectors of Number Theory.

Analysis n first series

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 21, 23, 24, 25, 27, 32, 36, 37, 40, 42, 45, 48.

(the numbers in red are already triangular numbers)

Analysis N' second series

63, 64, 65, 86, 90, 180, 189, 198, 270, 315, 378, 448, 567, 576, 720.

In the second series , only 378 is a triangular number

Analysis first series

List of triangular numbers

A list of some triangular numbers is the following:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, 1225, 1275, 1326, 1378, 1431, 1485, 1540, 1596, 1653, 1711, 1770, 1830, 1891, 1953, 2016, 2080, 2145, 2211, 2278, 2346, 2415, 2485, 2556, 2628, 2701, 2775, 2850, 2926, 3003, 3081, 3160, 3240.

We have: (in **red** the numbers **n** of the first series)

TABELLA 2T ± 6

T	2T-6	2T-5	2T-4	2T-3	2T-2	2T-1	2T	2T +1	2T+2	2T+3	2T+4	2T+5	2T+6
1							2	3	4	5	6	7	8
3	0	1	2	3	4	5	6	7	8	9	10	11	12
6	6	7	8	9	10	11	12	13	14	15	16	17	18
10	14	15	16	17	18	19	20	21	22	23	24	25	26
15	24	25	26	27	28	29	30	31	32	33	34	35	36
21	36	37	38	39	40	41	42	43	44	45	46	47	48

We note that aren't the n prime numbers (13, 17, 19, 29,31,41,43 e 47, but also 22=2*11, 26=2*13, 28=4*7. 33=3*11, 34= 2*17, 35=7*5, 38=2*19, 44*4*11, 46=2*23). In other words, among the numbers n of the first series there aren't some prime numbers and their small

multiples, while there are only the prime numbers 5, 7 , 11, 23 e 37 with progressive differences 2, 4, 12 e 14; there are all the Fibonacci's numbers except the smallest 1, and the numbers 13 and 34 the largest up to 48 (last number of the series) thence three Fibonacci's numbers of eight.

POSSIBLE CONNECTIONS WITH (p(n)), THE NUMBERS OF PARTITIONS OF n

p(n) = 1, 1, 2, 3, 5, 7, 11, 15, **22**, **30**, 42, 56, 77, 101, 135, 176....
n = 0 1 **2 3 4 5 6 7** 8 9 **10** 11 12 13 14 15

N = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 21, 23, 24, 25, 27, 32, 36, 37, 40, 42, 45, 48.

We have that seven of the numbers N analyzed (i.e. 2,3,5,7,11,15, and 42 are also partitions of numbers p(n) with n = 2, 3, 4, 5, 6, 7, 10 that are also among the numbers **N** analyzed (N now uppercase for distinguish it from n of p(n)); while the 22 = p(8) is among the numbers N 21 and 23, and 30 is among 27 and 32 about a as arithmetic mean (27+32)/2 = 29,5 ≈30

We know that the number of partitions of n p(n) come out in the Nature about with the same frequency of the Fibonacci's numbers, and they are link from the equation p(n) ≈ F(n) ≈ n^2 +n ± c' (see the paper: "L'equazione preferita dalla natura"), and, as the Fibonacci's numbers and the aurea ratio, they are present also in the string theory; thence, we conclude that there is an important and fundamental relationship between the numbers N, the partitions p(n) and the string theory.

Now we observe the following Table

n	p(n) = N	N	n = N
2	2	*2*	***2***
3	3	*3*	*3*
		4	
4	5	*5*	*4*
		6	
5	7	*7*	*5*
		8	
		9	
		10	

6	11	11	*6*
		12	
		14	
7	15	15	*7*
		16	
		18	
		20	
8	22 ≈	21	
		23	
		24	
		25	
		27	
9	30 ≈	32	
		36	
		37	
		40	
10	42	42	*10*
		45	
		48	

We note that some **n** of p(n) and some p(n) are also N = numbers of the series that are come out from the calculation on the strings.

Now we see with the triangle of the even numbers (2T)

0
2 4
6 **8** 10
12 14 16 18
20 22 **24** 26 28
30 32 34 36 38 40
42 44 46 **48** 50 52 54

where the even numbers N are on the top, then thin out, skipping one or two even numbers. The middle column, consisting of squares of odd numbers dispari – 1, instead, is complete : **0** = 1^2-1, 8=3^2-1, 24= 5^2-1 48 =7^2-1 (**0** not part of the numbers N, but is of the form 1^2 -1 = **0**)

Triangle of the odd numbers:

1
3 5
7 9 11
13 15 17 19
21 23 25 27 29
31 33 35 37 39 41
43 45 47 49 51 53 55

Also here the numbers N are on the top, then thin out, skipping some odd numbers; the middle column consisting of squares of the odd numbers. The sums of each row are cubes (1,8,27,64)

Triangle mixed (even and odd)

1
2 *3*
4 *5* 6
7 8 9 10
11 12 13 14 *15*
16 17 18 19 20 21
22 23 24 25 26 27 28
29 *30* 31 32 33 34 35 36
37 38 39 40 41 *42* 43 44 45
46 47 48 49 50 51 52 53 54 55
56 … … … … … ……. …

The number of partitions p(n) are in *italic*. They are on the outside, with a few exceptions (for example, 42, almost to half of their line).
Also here we note that the external diagonal of right consists of triangular numbers, with six number N also triangular numbers T, and other six numbers N in the diagonal left that are of the form T+1. And since also the internal diagonals containing many numbers N, it follows that many of them are of the form T ±1 , T±2, as from the following Table

T-2	T-1	T	T+1	T+2
		1	*2*	*3*
1	*2*	**3**	4	5
4	*5*	**6**	*7*	8
13	14	**15**	16	17
19	20	**21**	22	23
26	27	28	29	30
34	35	**36**	37	38
43	44	**45**	46	47
53	54	55	56	57

Only 11 numbers N of 28 beyond to the above forms, but the connection is obvious and evident (the missing numbers moving away from them, but they are all included in T±3, T±4, ecc. (see Table 2T±6)

Analysis second series

Now we see with the Table concerning 2T±-a, 2T± b, the other series of numbers, that are N' = **63, 64, 65, 86, 90, 180, 189, 198, 270, 315, 378, 448, 567, 576, 720.**
We note that the root of the number 576 is 24, that 720 = 24 x **30**, where 24 is connected to the vibrations of the bosonic strings, 448 = **56** x 8, where 8 is connected to the vibrations of the superstrings and is a Fibonacci's number and 378 = **21** x 2 x 9 where 21 is a Fibonacci's number.
We note that 30 +1 =**31** = Lie's number (5^2+5+1=**31**), 56+1 = **57** = Lie's number (7^2+7+1=**57**), 21 already Lie's number (4^2*4*1=**21**),
and that also 56 =**7***8 is a multiple of 8, and 7 is also a Lie's number 2^2+2+1 =**7; furthermore also 90 +1 = 91 is a Lie's number (9^2+9+1=91) and 90 is in 90=90*1, 180=90*2, 90*3=270, 90*8 = 720 , all multiples of 90 and numbers N'; and 90 = also number of the form 2T =2*45 and about triangular number (90 ≈ 91 triangular number)**

TABLE for the number **N'**

2T-a	2T- b	2T	2T+a	2T+b
2T -9 = 63	72-8 =**64** 72 -7= **65**	72		
	90 - 4 = 86	**90**		
	182-2=180	182	182 +7 =189	182+16=198
		210		
	272 -2 =270	272		
		306	306 + 9=315	
	380 -2 =378	380		
462 -14 = 448		462		
		552	552 + 15=567	552+24= 576
		702		702+18=720

Thence, the number of the second series are present about in the columns 2T±a and 2T±b, with a and b small numbers, that aren't > of √N'.

Now we take the two series

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 21, 23, 24, 25, 27, 32, 36, 37, 40, 42, 45, 48.

63, 64, 65, 86, 90, 180, 189, 198, 270, 315, 378, 448, 567, 576, 720.

and the following partitions of numbers

56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792

We observe that:

56 = 48 + 8; **77** = 63 + 14; 101 = 90 + 11; 135 = 90 + 45; 176 = 90 + 86;

231 = 189 + 42; 297 = 270 +27; 385 = 378 + 7; 490 = 448 + 42;

627 = 378 + 180 + 48 + 21; 792 = 378 + 378 + 36.

Thence, partitions of numbers that are sums of numbers of the two series.

Now we analyze the Lie's numbers and the Witten's numbers, searching the mathematical connections with the numbers of the two series.

The Lie's numbers of the parabolic form n^2 + n + 1, of the projective geometries.

TABLE 1 - LIE's NUMBERS SERIES (in green)

(in ***block*** are the Fibonacci's numbers and in **red** the Lie's groups)

n   n^2 + n + 1 = L(n) ≈ ***F*** **rapporto Ln/Ln-1**
----
0   0   ***0***   1   **1** = ***1*** -
1   1   1   1   **3** = ***3***   3
2   4   ***2***   1   **7** ~ ***8***   2,333 ≈ 2,61 = 1,618^2 (7/3)
(7* **2** = **14** = G2 (7= piano di Fano)
3   9   3   1   **13** = ***13***   1,857 (13/7)
(***13*****4 = 52 = F4; 13*6 = 78 = E6 Lie's group)**
4   16   4   1   **21** = ***21***   1,615 **≈** ***1,618*** (21/13)
5   25   5   1   **31** ~ ***34***   1,476 (31/21)
6   36   6   1   **43** ≈ 44,5   1,387 (44,5 =(***34***+***55***)/2
7   49   7   1   **57** ~ ***55***   1,325
8   64   8   1   **73** ≈ 72   1,280 ~ √Φ = 1,272; 72 =(55+89)/2
9   81   9   1   **91** ≈ ***89***   1,246
10   100   10   1   **111** ≈ 116,5   1,219
11   121   11   1   **133** ≈ ***144***   1,198
(*7* x 19 = 133 = E7 Lie's group)
12   144   12   1   **157** ≈ 156   1,180 (156 + 132)/2 = ***144***)
13   169   13   1   **183**   1,165
14   196   14   1   **211**   1,153
15   225   15   1   **241**   1,142
(240 + ***8*** = **248** = **31***8* = E8 Lie's group)**

(Lie's Groups = L(n)* k with k = 2, 4, 6, 19, 8, for L(n) = **7,13, 13, 31**. Indeed, we have:

TABLE 1.1
Lie's Groups Factors: L(n)*k
G(n) = L(n)*k

14 = **7***2
52 = **13***4
78 = **13***6
133 = **7***19 = **1*133**
248 = **31***8 (Rif.2)

While the prime numbers smallest are **2, 3, 5, and 7**

We take the two series

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 21, 23, 24, 25, 27, 32, 36, 37, 40, 42, 45, 48.

63, 64, 65, 86, 90, 180, 189, 198, 270, 315, 378, 448, 567, 576, 720.

We note that 3 = 3, 7 = 7, 13 = 10 + 3; 21 = 21; 31 = 21 + 10; 43 = 36 + 7; 57 = 36 + 21; 73 = 63 + 10; 91 = 86 + 5; 111 = 90 + 21; 133 = 90 + 36 + 7; 157 = 90 + 64 + 3; 183 = 180 + 3; 211 = 198 + 10 + 3; 241 = 198 + 40 + 3.

Thence, Lie's numbers that are sums of numbers of the two series (or equal to them)

TABLE 2 (concerning 2T +1; in bleu, the numbers p(n))
(underlined and in black, the Witten's numbers)

T	2T-3	2T-2	2T-1	2T	2T+1	2T+2	2T+3	(2T +4)
1	-1	0	1	2	3	4	5	
3	3	4	5	6	7	8	9	
6	9	10	11	12	13	14	15	16
10	17	18	19	20	21	22	23	
15	27	28	29	30	31	32	33	
21	39	40	41	42	43	44	45	
28	53	54	55	56	57	58	59	

… … … … … … … …

Witten's numbers: 2, 4, 7, 8, 14, 16, 21, 32, 105, 154, 175, 256, 945, 4096, 8085, 10493, 74247, 363825 : up to 32, lying on the strip numbers from 2T-3 to 2T+3, with the exception of 16 (of the form 2T+4).

Thence, also their have a mathematical connection with the triangular numbers T and with the Lie's numbers 2T+1 (the Witten's numbers 7 and 21 are also Lie's numbers).

In the paper:" On the physical interpretation of the Riemann zeta function, the Rigid Surface Operators in Gauge Theory, the adeles and ideles groups applied to various formulae regarding the Riemann zeta function and the Selberg trace formula, padic strings, zeta strings and p-adic cosmology and mathematical connections with some sectors of String Theory and Number Theory" there are the Witten's numbers above mentioned:

2, 4, 7, 8, 14, 16, 21, 32, 105, 154, 175, 256, 945, 4096, 8085, 10493, 74247, 363825

with marked in red the powers of 2 (including for squares, underlined; in the Lie's numbers there aren't absolutely squares, since they are always halfway between a square and the next; and between the Fibonacci's numbers only 1 and 144 are squares).

The exponents of 2 are, in order: **1, 2, 3, 4, 5, 8 ,12** with ***1*, *2*, *3*, *5*, *8*** Fibonacci's numbers, and 12 ≈ ***13*** another Fibonacci's number. So we return to the Fibonacci's series and to the symmetries, which are also partially reflected in the partitions of numbers and in the Witten's numbers.

Further comparison between Witten's numbers and partitions

p(n)	1	2	3	5	7	11	15	22	30 …
n. Witten		**2**	**4**		**7**		**14**	**16**	**32…**

We note that, the partitions of numbers and the Witten's numbers are very near (2 e 7 coincide), at least in the beginning, the one that interested to the Nature. For the prime numbers n of the formula n^2+n+1 the Nature has chosen indeed 2, 3, 5, 7, and for the Fibonacci's numbers the Nature stop at 144. We don't know natural phenomena where are involved Fibonacci's numbers greater than 144, number of seeds in a sunflower.

We take the two series

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 21, 23, 24, 25, 27, 32, 36, 37, 40, 42, 45, 48.

63, 64, 65, 86, 90, 180, 189, 198, 270, 315, 378, 448, 567, 576, 720.

With regard the partitions, we note that: 2 = 2; 3 = 3; 5 = 5; 7 = 7; 11 = 11; 15 = 15; 22 = 15 + 7; 30 = 27 + 3.

With regard the following Witten's numbers, we note that

2, 4, 7, 8, 14, 16, 21, 32, 105, 154, 175, 256, 945

2 = 2; 4 = 4; 7 = 7; 8 = 8; 14 = 14; 16 = 16; 21 = 21; 32 = 32; 105 = 90 +15; 154 = 90 + 48 +16; 175 = 90 + 65 +20; 256 = 198 + 48 +10; 945 = 720 + 180 +45.

Thence, also here, numbers of partitions and Witten's numbers as sums of the numbers of the two series (or equal to them)

TABLE 3

Symmetries (Lie's numbers, Lie's Groups, Fibonacci's Numbers, Partitions of Numbers)

Lie's Numbers L(n) con n primo o potenza di primo	**Lie's Groups G(n)**	**Numeri di Fibonacci F(n)**	**Partizioni di Numeri p(n)**
Forma L(n) = n^2+n+1 = 2T+1 (T = numeri triangolari)	**Forma G(n) = k*L(n)**	**Forma F(n) = n^2+n±c con c piccolo numero**	**Forma p(n) =n^2+n±c' con c' ≈ c**
1^2+1+1 = 3		**(c=1) 3**	**(c'=1) 3**
2^2+2+1= 7	**7*2=7 + 7 = 14 = G2**	**(c=2) 8**	**(c'=1) 7**
3^2+3+1 = 13 **4^4+4+1= 21**	**13*4=13+13+13+13 =52=F4** **13*6=13+13+13+13+13+13= 78= E6**	**(c=1) 13** **(c=1) 21**	**(c'=-1)11** **(c'=2) 22**
5^5+5+1 = 31	**31*8 = 31+31+31+31+31+31+31+31= 248 =E8**	**(c=4) 34**	**(c'=1) 30**
6^6+6+1 = 43			**(c'=0)42**
7^2+7+1= 57		**(c=-1) 55**	**(c'=0)56**
…			
11^2+11+1=133	**133*1 = 133 = E7**	**(c=12) 144**	

… … … …

Now we remember that : 2T is the **sum** of the first n even numbers.

2
2+4=6
2+4+6=12
2+4+6+8= 20
2+4+6+8+10= 30
… … …

with the final values obtaining from the formula n^2+n = n(n+1).

To obtain the Lie's numbers just **add** them the number 1, and we have:

L(n) = n^2 + n + 1

from which then the two variants for F(n) and p(n) and their numbers, all present in the Nature:

a) the Lie's numbers by the Lie's groups of symmetry , G(n), in the Standard Model and in the String Theory;
b) the Fibonacci's numbers, in various natural phenomena, strings included;
c) the partitions of the numbers, in other phenomena.

Finally, similar to how the Lie's numbers are the **sum** of the first n even numbers +1, the Fibonacci's numbers are the **sum** of the two previous numbers, and the partitions of numbers are all the ways as a number p can be written as a **sum** of smaller numbers.
The prime numbers n are involved only in the Lie's numbers (which give rise to Lie's groups only if n is prime), and only some Fibonacci's numbers (for example 2,3,5,13, 89) and some partitions of numbers (for example 2,3,5,7,11) are prime numbers.
In red all the prime numbers of the form L(n) = n^2+n+1, including the Lie's prime numbers.

TABLE 4

Numeri primi p o loro potenze n	**Numeri primi di Lie: L(n) = n^2+n+1**	**Numeri di Fibonacci primi**	**Partizioni di numeri (p(n) primi**
1	**3**	**3**	**3**
2	**7**		**7**
3	**13**	**13**	
4=2^2	21	21	
5	**31**		
7	**57**	55	
9 = 3^2	91	**89**	
11	133		
…	…	**…**	**…**

These initial connections between prime numbers n, Lie's prime numbers, Fibonacci's prime numbers and prime numbers as partitions of numbers, may have their role, perhaps not yet known (but suspected by the fact that are all on the parable of the Lie's numbers), in the theories of physics – mathematics concerning the quantum physics and the string theory.

We remember that the numbers 7, 13, 31 and 133 = 7*19, with their respective multiples 2, 4 and 6, 8 and 1 , are the basis of the Lie's Groups
G2 = 14, F4 = 52, E6= 78, E8= 248 and E7 = 133. We note that 57 is the number of dimensions of the group E8 = 248, that is very important for the string theories.

Furthermore, we note that the numbers of the dimensions of the five Lie's groups of symmetry, are connected to the geometric solids:

a **tetrahedron** of measure **14**
an **octahedron** of measure **52**
three **dodecahedra** of measure **78**, **133** and **248**

(see TABLE 3)

The Lie's groups are related to the Fibonacci's numbers and to the Lie's numbers by the above geometric solids, and their respective sides F , i.e. 4, 8 and 12 become Fibonacci's numbers if we add 1 to the numbers 4 and 12, obtaining ***5*** and ***13*** that are Fibonacci's numbers (the middle ***8*** is unchanged); but also with their vertices 4, 6 and 20, subtracting 1 from 4 and 6 and adding 1 to 20, getting ***3***, ***5*** e ***21*** that are Fibonacci's numbers; while adding 1 at all their edges S 6, 12, and 30 (of the form S = 2T with T triangular numbers 3, 6 and 15) we obtain the Lie's numbers **7**, **13** and **31**, that are the basis of the Lie's groups:

14=2***7**, **52**=4***13**, **78**= 6***13**, **133** =**7***19 and **248** = 8***31**

We take the two series

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 21, 23, 24, 25, 27, 32, 36, 37, 40, 42, 45, 48.

63, 64, 65, 86, 90, 180, 189, 198, 270, 315, 378, 448, 567, 576, 720.

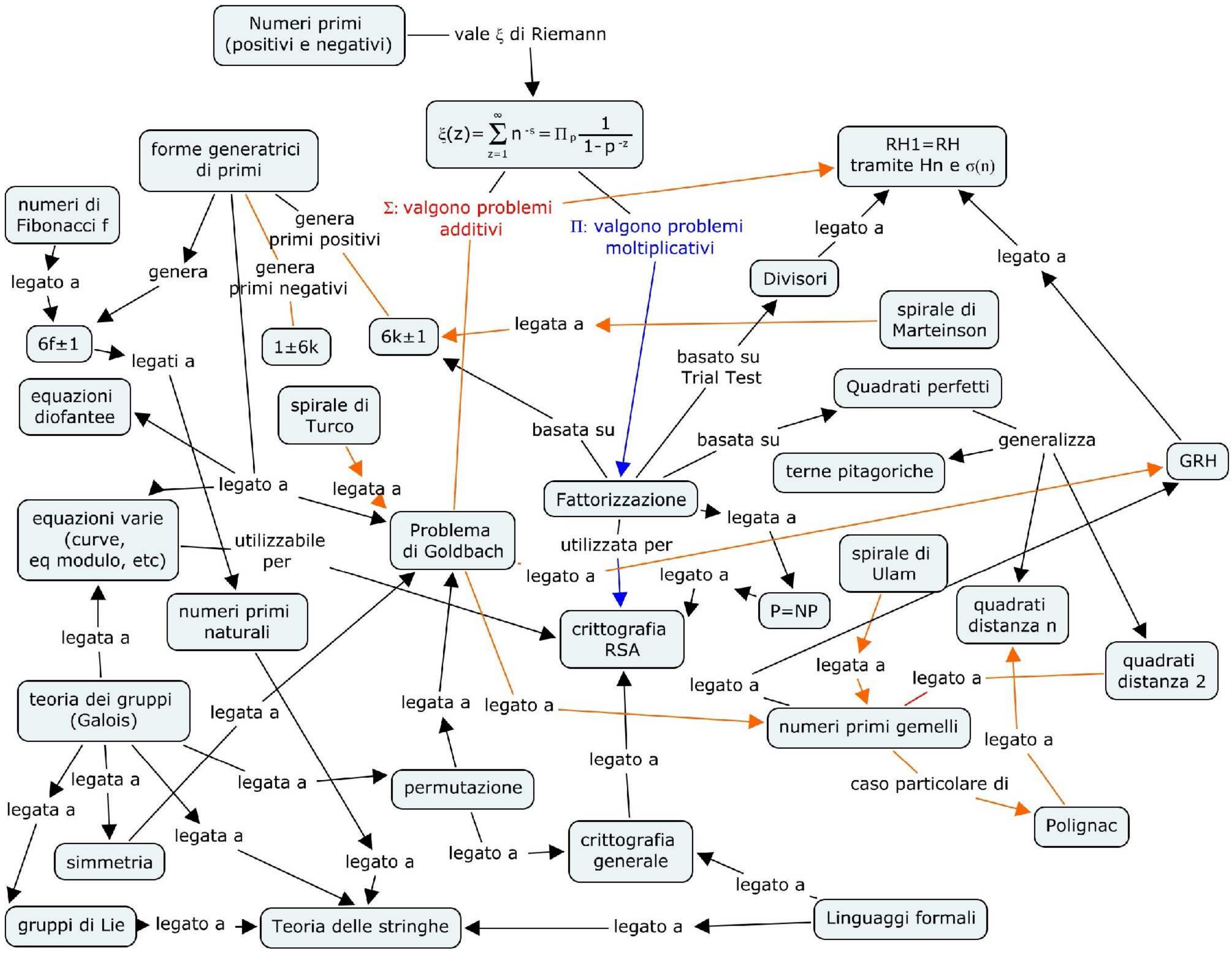
We note that 14 = 14; 52 = 45 + 7; 78 = 63 + 15; 133 = 90 + 36 + 7;

248 = 198 + 45 + 5.

Thence, also here mathematical connections between numbers of the two series (or their sums) and numbers of the Lie's Groups G(n).

*Now we have insert the scheme of the mathematical paths from the **prime numbers** at all the mathematical components connected to the **string theory** (and from here at the **TOE**, that are connected to the **Lie's groups**, specially **E8**, (from "Numeri primi in cerca d'Autore" on the site www.gruppoeratostene.com, section "Articoli sulla Teoria dei Numeri":*

"…In the follows we show a short scheme of the most interesting links that we have noted" (in Italian language)



By the prime numbers, through the darts, we will reach to the ***string theory****, connected to the* ***zeta function****, to the* ***Fibonacci's numbers*** *, to the* ***Lie's groups*** *(specially* ***E8,*** *linked to the* ***TOE*** *of Garrett Lisi).*

That is why this scheme is just very important: denote clearly the mathematical paths that connect the prime numbers with the components of the complex mathematical scenery of the strings.

Many of our works explores in the details these paths: Golden Ratio, Zeta Function, Symmetry, etc…

Appendix B (Christian Lange)

Here, we have showed the column "system" concerning the **universal music system based on Phi**

System
0,0131556174964
0,0135510312596
0,0142348954757
0,0143953404412
0,0150749962219
0,0155281000757
0,0162612375116
0,0167499958021
0,0172744085295
0,0177385302102
0,0186337200909
0,0191937872550
0,0200999949626
0,0203265468895
0,0212862362522
0,0219260291607
0,0230325447060
0,0232921501136
0,0243918562674
0,0251249937032
0,0263112349929
0,0271020625193
0,0279505801363
0,0287015447905
0,0301499924438
0,0310562001514
0,0325224750231
0,0328890437411
0,0344418537486
0,0354770604203
0,0372674401817
0,0376874905548
0,0394668524893

0,0406530937789
0,0425724725044
0,0438520583214
0,0452249886658
0,0464400750007
0,0487837125347
0,0502499874064
0,0526224699857
0,0532155906305
0,0557280900008
0,0574030895811
0,0602999848877
0,0609796406684
0,0638587087566
0,0657780874821
0,0688837074973
0,0709541208407
0,0731755688021
0,0751416197912
0,0789337049786
0,0813061875578
0,0851449450088
0,0861046343716
0,0901699437495
0,0928801500014
0,0975674250694
0,0986671312232
0,1033255612459
0,1064311812610
0,1114561800017
0,1148061791621
0,1184005574678
0,1215816947919
0,1277174175133
0,1315561749643
0,1377674149945
0,1393202250021
0,1458980337503
0,1502832395825
0,1578674099571
0,1596467718916
0,1671842700025
0,1722092687432
0,1803398874990
0,1857603000028
0,1915761262699
0,1967233145832
0,2066511224918

0,2128623625221
0,2229123600034
0,2254248593737
0,2360679774998
0,2431633895839
0,2554348350265
0,2583139031148
0,2705098312484
0,2786404500042
0,2917960675006
0,3005664791649
0,3099766837377
0,3183050093751
0,3343685400051
0,3444185374863
0,3606797749979
0,3647450843758
0,3819660112501
0,3934466291663
0,4133022449836
0,4179606750063
0,4376941012510
0,4508497187474
0,4721359549996
0,4863267791677
0,5015528100076
0,5150283239582
0,5410196624969
0,5572809000084
0,5835921350013
0,5901699437495
0,6180339887499
0,6366100187502
0,6687370800101
0,6762745781211
0,7082039324994
0,7294901687516
0,7639320225002
0,7868932583326
0,8115294937453
0,8333333333333
0,8753882025019
0,9016994374948
0,9442719099992
0,9549150281253
1,0000000000000
1,0300566479165
1,0820393249937

1,0942352531274
1,1458980337503
1,1803398874990
1,2360679774998
1,2732200375004
1,3130823037529
1,3483616572916
1,4164078649987
1,4589803375032
1,5278640450004
1,5450849718747
1,6180339887499
1,6666666666667
1,7507764050038
1,7705098312484
1,8541019662497
1,9098300562505
2,0000000000000
2,0601132958330
2,1246117974981
2,1816949906249
2,2917960675006
2,3606797749979
2,4721359549996
2,5000000000000
2,6180339887499
2,6967233145832
2,8328157299975
2,8647450843758
3,0000000000000
3,0901699437495
3,2360679774998
3,3333333333333
3,4376941012510
3,5300566479165
3,7082039324994
3,8196601125011
4,0000000000000
4,0450849718747
4,2360679774998
4,3633899812498
4,5835921350013
4,6352549156242
4,8541019662497
5,0000000000000
5,2360679774998
5,3934466291663
5,5623058987491

5,7117516385413
6,0000000000000
6,1803398874990
6,4721359549995
6,5450849718747
6,8541019662496
7,0601132958329
7,4164078649987
7,4999999999999
7,8541019662496
8,0901699437494
8,4721359549995
8,7267799624996
9,0000000000000
9,2418082864578
9,7082039324993
10,0000000000000
10,4721359549995
10,5901699437494
11,0901699437493
11,4235032770827
11,9999999999999
12,1352549156240
12,7082039324992
13,0901699437494
13,7082039324992
14,1202265916658
14,5623058987489
14,9535599249990
15,7082039324992
16,1803398874989
16,9442719099990
17,1352549156240
17,9442719099988
18,4836165729155
19,4164078649986
19,6352549156239
20,5623058987487
21,1803398874987
22,1803398874986
22,8470065541653
23,5623058987488
24,1953682114567
25,4164078649984
26,1803398874987
27,4164078649983
27,7254248593732
29,0344418537480

29,9071198499981
31,4164078649984
31,7705098312478
33,2705098312478
34,2705098312479
35,8885438199976
36,9672331458309
38,1246117974976
39,1489281364556
41,1246117974974
42,3606797749974
44,3606797749971
44,8606797749971
46,9787137637467
48,3907364229134
50,8328157299968
51,4057647468715
53,8328157299963
55,4508497187464
58,0688837074960
59,8142396999959
61,6869176962462
63,3442963479120
66,5410196624956
68,5410196624959
71,7770876399951
72,5861046343700
76,0131556174944
78,2978562729111
82,2492235949949
83,1762745781189
87,1033255612437
89,7213595499940
93,9574275274933
96,7814728458264
99,8115294937434
102,4932244843670
107,6656314599930
110,9016994374930
116,1377674149920
117,4467844093670
122,9918693812410
126,6885926958240
133,0820393249910
134,5820393249900
140,9361412912390
145,1722092687400
152,0263112349890

156,5957125458220
161,4984471899890
165,8375208322780
174,2066511224870
179,4427190999880
187,9148550549860
190,0328890437360
199,0050249987340
204,9864489687340
215,3312629199850
217,7583139031080
228,0394668524820
234,8935688187330
245,9837387624810
253,3771853916470
261,3099766837310
268,3307453166450
281,8722825824790
290,3444185374800
304,0526224699770
307,4796734531010
321,9968943799730
331,6750416645570
348,4133022449750
352,3403532280960
368,9756081437200
380,0657780874710
398,0100499974680
409,9728979374670
422,8084238737180
434,1682661489210
456,0789337049640
469,7871376374660
491,9674775249610
497,5125624968350
521,0019193787060
536,6614906332890
563,7445651649580
570,0986671312020
597,0150749962000
614,9593469062010
643,9937887599460
663,3500833291110
684,1184005574470
702,4990114655630
737,9512162874400
760,1315561749420
796,0200999949340

804,9922359499330

In the follow, the table where we have showed the difference between the values of Phi^(n/7) and the values of the column "system"

System	1,61803398875	n	Phi^(n/7)
	1,61803398875	-14,50	0,3690600455
0,3819660113	1,61803398875	-14,00	0,3819660113
0,3934466292	1,61803398875	-13,50	0,3953232964
0,4133022450	1,61803398875	-13,00	0,4091476835
0,4179606750	1,61803398875	-12,50	0,4234555071
0,4376941013	1,61803398875	-12,00	0,4382636727
0,4508497187	1,61803398875	-11,50	0,4535896773
0,4721359550	1,61803398875	-11,00	0,4694516297
0,4863267792	1,61803398875	-10,50	0,4858682718
0,5015528100	1,61803398875	-10,00	0,5028590010
0,5150283240	1,61803398875	-9,50	0,5204438930
0,5410196625	1,61803398875	-9,00	0,5386437257
0,5572809000	1,61803398875	-8,50	0,5574800034
0,5835921350	1,61803398875	-8,00	0,5769749824
0,5901699437	1,61803398875	-7,50	0,5971516975
0,6180339887	1,61803398875	-7,00	0,6180339887
0,6366100188	1,61803398875	-6,50	0,6396465301
0,6687370800	1,61803398875	-6,00	0,6620148584
0,6762745781	1,61803398875	-5,50	0,6851654032
0,7082039325	1,61803398875	-5,00	0,7091255185
0,7294901688	1,61803398875	-4,50	0,7339235149
0,7639320225	1,61803398875	-4,00	0,7595886929
0,7868932583	1,61803398875	-3,50	0,7861513778
0,8115294937	1,61803398875	-3,00	0,8136429551
0,8333333333	1,61803398875	-2,50	0,8420959081
0,8753882025	1,61803398875	-2,00	0,8715438560
0,9016994375	1,61803398875	-1,50	0,9020215935
0,9442719100	1,61803398875	-1,00	0,9335651322
0,9549150281	1,61803398875	-0,50	0,9662117429
1,0000000000	1,61803398875	0,00	1,0000000000
1,0300566479	1,61803398875	0,50	1,0349698266
1,0820393250	1,61803398875	1,00	1,0711625419
1,0942352531	1,61803398875	1,50	1,1086209102
1,1458980338	1,61803398875	2,00	1,1473891912
1,1803398875	1,61803398875	2,50	1,1875131922

1,2360679775	1,61803398875	3,00	1,2290403226
1,2732200375	1,61803398875	3,50	1,2720196495
1,3130823038	1,61803398875	4,00	1,3165019560
1,3483616573	1,61803398875	4,50	1,3625398011
1,4164078650	1,61803398875	5,00	1,4101875817
1,4589803375	1,61803398875	5,50	1,4595015968
1,5278640450	1,61803398875	6,00	1,5105401145
1,5450849719	1,61803398875	6,50	1,5633634404
1,6180339887	1,61803398875	7,00	1,6180339887
1,6666666667	1,61803398875	7,50	1,6746163567
1,7507764050	1,61803398875	8,00	1,7331774003
1,7705098312	1,61803398875	8,50	1,7937863134
1,8541019662	1,61803398875	9,00	1,8565147097
1,9098300563	1,61803398875	9,50	1,9214367071
2,0000000000	1,61803398875	10,00	1,9886290155
2,0601132958	1,61803398875	10,50	2,0581710273
2,1246117975	1,61803398875	11,00	2,1301449111
2,1816949906	1,61803398875	11,50	2,2046357093
2,2917960675	1,61803398875	12,00	2,2817314377
2,3606797750	1,61803398875	12,50	2,3615231903
2,4721359550	1,61803398875	13,00	2,4441052467
2,5000000000	1,61803398875	13,50	2,5295751833
2,6180339887	1,61803398875	14,00	2,6180339887
2,6967233146	1,61803398875	14,50	2,7095861833
2,8328157300	1,61803398875	15,00	2,8043399422
2,8647450844	1,61803398875	15,50	2,9024072236
3,0000000000	1,61803398875	16,00	3,0039039008
3,0901699437	1,61803398875	16,50	3,1089498993
3,2360679775	1,61803398875	17,00	3,2176693381
3,3333333333	1,61803398875	17,50	3,3301906768
3,4376941013	1,61803398875	18,00	3,4466468672
3,5300566479	1,61803398875	18,50	3,5671755104
3,7082039325	1,61803398875	19,00	3,6919190193
3,8196601125	1,61803398875	19,50	3,8210247871
4,0000000000	1,61803398875	20,00	3,9546453613
4,0450849719	1,61803398875	20,50	4,0929386237
4,2360679775	1,61803398875	21,00	4,2360679775
4,3633899812	1,61803398875	21,50	4,3842025400
4,5835921350	1,61803398875	22,00	4,5375173425
4,6352549156	1,61803398875	22,50	4,6961935370
4,8541019662	1,61803398875	23,00	4,8604186105
5,0000000000	1,61803398875	23,50	5,0303866064
5,2360679775	1,61803398875	24,00	5,2062983536
5,3934466292	1,61803398875	24,50	5,3883617041
5,5623058987	1,61803398875	25,00	5,5767917783
5,7117516385	1,61803398875	25,50	5,7718112196
6,0000000000	1,61803398875	26,00	5,9736504570
6,1803398875	1,61803398875	26,50	6,1825479774
6,4721359550	1,61803398875	27,00	6,3987506080
6,5450849719	1,61803398875	27,50	6,6225138070
6,8541019662	1,61803398875	28,00	6,8541019662
7,0601132958	1,61803398875	28,50	7,0937887233
7,4164078650	1,61803398875	29,00	7,3418572847
7,5000000000	1,61803398875	29,50	7,5986007606

7,8541019662	1,61803398875	30,00	7,8643225114
8,0901699437	1,61803398875	30,50	8,1393365057
8,4721359550	1,61803398875	31,00	8,4239676916
8,7267799625	1,61803398875	31,50	8,7185523808
9,0000000000	1,61803398875	32,00	9,0234386455
9,2418082865	1,61803398875	32,50	9,3389867300
9,7082039325	1,61803398875	33,00	9,6655694763
10,0000000000	1,61803398875	33,50	10,0035727646
10,4721359550	1,61803398875	34,00	10,3533959692
10,5901699437	1,61803398875	34,50	10,7154524306
11,0901699437	1,61803398875	35,00	11,0901699437
11,4235032771	1,61803398875	35,50	11,4779912633
12,0000000000	1,61803398875	36,00	11,8793746271
12,1352549156	1,61803398875	36,50	12,2947942976
12,7082039325	1,61803398875	37,00	12,7247411219
13,0901699437	1,61803398875	37,50	13,1697231120
13,7082039325	1,61803398875	38,00	13,6302660452
14,1202265917	1,61803398875	38,50	14,1069140849
14,5623058987	1,61803398875	39,00	14,6002304239
14,9535599250	1,61803398875	39,50	15,1107979497
15,7082039325	1,61803398875	40,00	15,6392199333
16,1803398875	1,61803398875	40,50	16,1861207420
16,9442719100	1,61803398875	41,00	16,7521465772
17,1352549156	1,61803398875	41,50	17,3379662376
17,9442719100	1,61803398875	42,00	17,9442719100
18,4836165729	1,61803398875	42,50	18,5717799866
19,4164078650	1,61803398875	43,00	19,2212319118
19,6352549156	1,61803398875	43,50	19,8933950582
20,5623058987	1,61803398875	44,00	20,5890636332
21,1803398875	1,61803398875	44,50	21,3090596177
22,1803398875	1,61803398875	45,00	22,0542337369
22,8470065542	1,61803398875	45,50	22,8254664657
23,5623058987	1,61803398875	46,00	23,6236690694
24,1953682115	1,61803398875	46,50	24,4497846797
25,4164078650	1,61803398875	47,00	25,3047894096
26,1803398875	1,61803398875	47,50	26,1896935066
27,4164078650	1,61803398875	48,00	27,1055425464
27,7254248594	1,61803398875	48,50	28,0534186683
29,0344418537	1,61803398875	49,00	29,0344418537
29,9071198500	1,61803398875	49,50	30,0497712499
31,4164078650	1,61803398875	50,00	31,1006065389
31,7705098312	1,61803398875	50,50	32,1881893558
33,2705098312	1,61803398875	51,00	33,3138047551
34,2705098312	1,61803398875	51,50	34,4787827297
35,8885438200	1,61803398875	52,00	35,6844997821
36,9672331458	1,61803398875	52,50	36,9323805506
38,1246117975	1,61803398875	53,00	38,2238994933
39,1489281365	1,61803398875	53,50	39,5605826293
41,1246117975	1,61803398875	54,00	40,9440093428
42,3606797750	1,61803398875	54,50	42,3758142486
44,3606797750	1,61803398875	55,00	43,8576891235
44,8606797750	1,61803398875	55,50	45,3913849059
46,9787137637	1,61803398875	56,00	46,9787137637

48,3907364229	1,61803398875	56,50	48,6215512365
50,8328157300	1,61803398875	57,00	50,3218384507
51,4057647469	1,61803398875	57,50	52,0815844139
53,8328157300	1,61803398875	58,00	53,9028683883
55,4508497187	1,61803398875	58,50	55,7878423474
58,0688837075	1,61803398875	59,00	57,7387335189

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