On some equations concerning Fivebranes and Knots, Wilson Loops in Chern-Simons Theory, cusp anomaly and integrability from String theory . Mathematical connections with some sectors of Number Theory

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Abstract

The present paper is a review, a thesis of some very important contributes of E. Witten, C. Beasley, R. Ricci, B. Basso et al. regarding various applications concerning the Jones polynomials, the Wilson loops and the cusp anomaly and integrability from string theory. In this work, in the **Section 1**, we have described some equations concerning the knot polynomials, the Chern-Simons from four dimensions, the D3-NS5 system with a theta-angle, the Wick rotation, the comparison to topological field theory, the Wilson loops, the localization and the boundary formula. We have described also some equations concerning electric-magnetic duality to N = 4 super Yang-Mills theory, the gravitational coupling and the framing anomaly for knots. Furthermore, we have described some equations concerning the gauge theory description, relation to Morse theory and the action.

In the **Section 2**, we have described some equations concerning the applications of non-abelian localization to analyze the Chern-Simons path integral including Wilson loop insertions.

In the **Section 3**, we have described some equations concerning the cusp anomaly and integrability from String theory and some equations concerning the cusp anomalous dimension in the transition regime from strong to weak coupling. In the **Section 4**, we have described also some equations concerning the "fractal" behaviour of the partition function.

Also here, we have described some mathematical connections between various equation described in the paper and (i) the Ramanujan's modular equations regarding the physical vibrations of the bosonic strings and the superstrings, thence the relationship with the Palumbo-Nardelli model, (ii) the mathematical connections with the Ramanujan's equations concerning π and, in conclusion, (iii) the mathematical connections with the aurea ratio ($\Phi = \frac{\sqrt{5}+1}{2} \cong 1{,}618033988$) and with 1,375 that is the mean real value for the number of partitions p(n).

1. On some equations concerning fivebranes and knots: an approach to Khovanov homology of knots via gauge theory. [1]

The Chern-Simons action for a gauge theory with gauge group G (here G is always a compact Lie group, and all representations considered are finite-dimensional and unitary) and gauge field A on a three-manifold W can be written

$$I = \frac{k}{4\pi}\int_W Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (1.1)$$

Here k is an integer for topological reasons; up to a choice of orientation, one may take k to be positive. In this theory, to an oriented embedded loop $K \subset W$ and a representation R of G, one can associate an observable, the trace of the holonomy or Wilson loop operator:

$$\mathcal{W}(K,R) = Tr_R P \exp\left(-\oint_K A\right). \quad (1.2)$$

Reversing the orientation of K has the same effect as replacing R by its complex conjugate. The Jones polynomial and its generalizations can be computed as expectation values of Wilson loop operators, if we express the argument q of the knot polynomials in terms of the Chern-Simons level k by

$$q = \exp(2\pi i/(k+h)), \quad (1.3)$$

where h is the dual Coxeter number of G. For example, if we take $G = SU(2)$, R to be the two-dimensional irreducible representation of $SU(2)$, and $W = S^3$, then the expectation value of $\mathcal{W}(K,R)$ is equal to the Jones polynomial:

$$\mathcal{J}(q;K) = \langle \mathcal{W}(K,R) \rangle. \quad (1.4)$$

Now we adopt a ten-dimensional notation in which $\mathcal{N} = 4$ super Yang-Mills theory comes by dimensional reduction from ten dimensions and the supersymmetries of the D3-brane transform under $SO(1,9)$ as a spinor **16** of definite chirality; thus a generator ε of supersymmetry obeys

$$\Gamma_{012\ldots9}\varepsilon = \varepsilon, \quad (1.5)$$

where Γ_I, $I = 0,\ldots 9$, are the $SO(1,9)$ gamma matrices. The supersymmetries transform as **16** = $V_8 \otimes V_2$, where V_2 is a two-dimensional real vector space. The natural operators that act on V_2 are the even elements of the $SO(1,9)$ Clifford algebra that commute with U, where $U = SO(1,2)\times SO(3)_X \times SO(3)_Y$. They are generated by

$$B_0 = \Gamma_{456789}; \quad B_1 = \Gamma_{3456}; \quad B_2 = \Gamma_{3789}, \quad (1.6)$$

and in view of the algebraic relations they obey, we can choose a basis for V_2 in which

$$B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.7)$$

We take also

$$\varepsilon_0 = \begin{pmatrix} -a \\ 1 \end{pmatrix}, \quad \bar{\varepsilon}_0 = \begin{pmatrix} 1 & a \end{pmatrix}. \quad (1.8)$$

The fermion fields λ of $\mathcal{N} = 4$ super Yang-Mills are adjoint-valued fields that transform as the **16** of $SO(1,9)$, like the supersymmetry generators. The boundary conditions they obey turn out to be

$$\lambda| \in \mathrm{V}_8 \otimes \vartheta, \quad (1.9)$$

where $\vartheta \in \mathrm{V}_2$ is

$$\vartheta = \begin{pmatrix} a \\ 1 \end{pmatrix}. \quad (1.10)$$

The boundary conditions on $\vec{X}$ at $x^3 = 0$ are

$$D_3 X_c - \frac{a}{1+a^2}\varepsilon_{cde}\left[X_d, X_e\right] = 0, \quad (1.11)$$

and the boundary conditions on the gauge fields at $x^3 = 0$ are

$$F_{3\mu} + \frac{a}{1-a^2}\varepsilon_{\mu\nu\lambda}F^{\nu\lambda} = 0. \quad (1.12)$$

At $a = 0$ and $a = \infty$, eqs. (1.11) and (1.12) reduce to the more obvious Neumann boundary conditions $D_3 X_a = F_{3\mu} = 0$. The additional terms in the boundary conditions for generic a reflect boundary corrections to the familiar $\mathcal{N} = 4$ super Yang-Mills action in bulk. Let us first consider $\vec{X}$. The usual bulk action for $\vec{X}$ is in Lorentz signature

$$I_{\vec{X}} = \frac{1}{g_{YM}^2}\int_{x^3 \geq 0} d^4x \sum_{\mu=0}^{3}\sum_{c=1}^{3} Tr D_\mu X_c D^\mu X_c. \quad (1.13)$$

If we place no restriction on the value of δX_c at $x^3 = 0$, we will learn that to make the boundary term in the variation of $I_{\vec{X}}$ vanish, the boundary condition must be $D_3 X_c = 0$. Suppose, however, that there is an additional boundary coupling

$$\tilde{I}_{\vec{X}} = \frac{2a}{3g_{YM}^2\left(1+a^2\right)}\int_{x^3=0} d^3x \varepsilon^{cde} Tr X_c\left[X_d, X_e\right]. \quad (1.14)$$

If we now vary $\hat{I}_{\vec{X}} = I_{\vec{X}} + \tilde{I}_{\vec{X}}$ with respect to $\vec{X}$, placing again no restriction on $\delta X_c|$, we find that setting the boundary variation of $\hat{I}_{\vec{X}}$ to zero gives the boundary condition (1.11). So the boundary coupling (1.14) underlies the boundary condition (1.11). The boundary coupling $\tilde{I}_{\vec{X}}$ is unfamiliar, but it has a more familiar analog for gauge fields. The analog of (1.13) for the gauge field A, whose field strength we denote as $F_{\mu\nu}$, is

$$I_A = \frac{1}{2g_{YM}^2}\int_{x^3>0} d^4x \sum_{\mu,\nu=0}^{3} TrF_{\mu\nu}F^{\mu\nu} . \quad (1.15)$$

Thence, we obtain also that:

$$I_{\vec{X}} = \frac{1}{g_{YM}^2}\int_{x^3\geq 0} d^4x \sum_{\mu=0}^{3}\sum_{c=1}^{3} TrD_\mu X_c D^\mu X_c = \frac{1}{2g_{YM}^2}\int_{x^3>0} d^4x \sum_{\mu,\nu=0}^{3} TrF_{\mu\nu}F^{\mu\nu} . \quad (1.15b)$$

If we work just with this action, then setting its boundary variation to zero, we learn that the boundary condition on the gauge field must be $F_{3\mu}\Big| = 0$.

To arrive at (1.12), we need an additional term in the action. This extra term is the usual topological term of four-dimensional gauge theory

$$\tilde{I}_A = -\frac{\theta}{32\pi^2}\int_{x^3\geq 0} d^4x \varepsilon^{\mu\nu\alpha\beta} TrF_{\mu\nu}F_{\alpha\beta} , \quad (1.16)$$

with

$$\frac{\theta}{2\pi} = \frac{2a}{1-a^2}\frac{4\pi}{g_{YM}^2} . \quad (1.16b)$$

Viewed as an equation for a with θ, g_{YM} fixed, (1.16b) has two roots. The two roots correspond to half-BPS boundary conditions of the D3-NS5 and $D3-\overline{NS}\,\overline{5}$ systems, respectively.

Although written as a bulk integral, $\tilde{I}_A$ has only a boundary variation, simply because on a manifold V without boundary, $\int_V TrF \wedge F$ is a topological invariant. In fact, we can almost write $\tilde{I}_A$ as a boundary integral, the integral over the surface $x^3 = 0$ of the Chern-Simons form:

$$\tilde{I}_A = -\frac{\theta}{8\pi^2}\int_{x^3=0} d^3x \varepsilon^{\mu\nu\lambda} Tr\left(A_\mu \partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda \right). \quad (1.17)$$

Now we want to show an interesting equation concerning the gauge fields as described in the **Jormakka's paper** "*Solutions to Yang-Mills equations*" and connected with an Ramanujan's identity concerning π in the my recent paper: "*On some equations concerning quantum electrodynamics coupled to quantum gravity, the gravitational contributions to the gauge couplings and quantum effects in the theory of gravitation: mathematical connections with some sector of String Theory and Number Theory*".

$$\int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\int dy_1 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_1^2} = \int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dx_2 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_2^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dy_2 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_2^2} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$= \left[\int d^2 x e^{-\beta^2 \left(y_2^2 + y_3^2\right)} \frac{1}{\sqrt{3}} \sqrt{2\pi} \left(\sqrt{2}\beta\right)^{-1} \right]^2 = 8 \frac{\pi^3}{4} \times \frac{1}{\left(\sqrt{2}\beta\right)^6} \Rightarrow$$

$$\Rightarrow \pi^3 \left(\int_0^\infty x^3 \frac{\cos \pi x^2}{\sinh \pi x} dx + \int_0^\infty x^3 \frac{\sin \pi x^2}{\sinh \pi x} dx \right) = \frac{1}{16} \left(\frac{\pi^3}{4} - 3\pi + \pi^2 \right); \quad (1.18)$$

This equation can be connected with the eq. (1.17) that we have multiplied for $\frac{\pi^2}{16}$ as follows:

$$\frac{\theta}{2\pi} = \frac{2a}{1-a^2} \frac{4\pi}{g_{YM}^2} \Rightarrow \frac{\theta}{2\pi} \times \frac{\pi^2}{16} = \frac{2a}{1-a^2} \frac{4\pi}{g_{YM}^2} \times \frac{\pi^2}{16} = \frac{2a}{1-a^2} \frac{\pi^3}{4g_{YM}^2}, \quad (1.19)$$

thence,

$$\tilde{I}_A = -\frac{\theta}{2\pi} \times \frac{1}{4\pi} \int_{x^3=0} d^3 x \varepsilon^{\mu\nu\lambda} Tr \left(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right) \Rightarrow$$

$$\frac{\pi^2}{16} \cdot \tilde{I}_A = -\frac{\theta}{2\pi} \times \frac{1}{4\pi} \times \frac{\pi^2}{16} \int_{x^3=0} d^3 x \varepsilon^{\mu\nu\lambda} Tr \left(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right) \Rightarrow$$

$$\frac{\pi^2}{16} \cdot \tilde{I}_A = -\frac{1}{4\pi} \times \frac{2a}{1-a^2} \frac{\pi^3}{4g_{YM}^2} \int_{x^3=0} d^3 x \varepsilon^{\mu\nu\lambda} Tr \left(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right). \quad (1.20)$$

We obtain the following mathematical connections:

$$\frac{\pi^2}{16} \cdot \tilde{I}_A = -\frac{1}{4\pi} \times \frac{2a}{1-a^2} \frac{\pi^3}{4g_{YM}^2} \int_{x^3=0} d^3 x \varepsilon^{\mu\nu\lambda} Tr \left(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right) \Rightarrow$$

$$\Rightarrow \int d^2 x e^{-\beta^2 \left(y_2^2 + y_3^2\right)} \frac{1}{\sqrt{3}} \int dy_1 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_1^2} = \int d^2 x e^{-\beta^2 \left(y_2^2 + y_3^2\right)} \frac{1}{\sqrt{3}} \sqrt{2\pi} \left(\sqrt{2}\beta\right)^{-1} =$$

$$= \frac{1}{\sqrt{3}} \sqrt{2\pi} \left(\sqrt{2}\beta\right)^{-1} \int dx_3 e^{-\beta^2 y_3^2} \int dx_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} = \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \sqrt{2\pi} \left(\sqrt{2}\beta\right)^{-1} \int dx_3 e^{-\beta^2 y_3^2} \int dy_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} =$$

$$= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} (2\pi) \left(\sqrt{2}\beta\right)^{-2} \int dx_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \frac{1}{2} (2\pi) \left(\sqrt{2}\beta\right)^{-2} \int dy_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \frac{1}{2} (2\pi)^{\frac{3}{2}} \left(\sqrt{2}\beta\right)^{-3}$$

$$= \left[\int d^2 x e^{-\beta^2 \left(y_2^2 + y_3^2\right)} \frac{1}{\sqrt{3}} \sqrt{2\pi} \left(\sqrt{2}\beta\right)^{-1} \right]^2 = 8 \frac{\pi^3}{4} \times \frac{1}{\left(\sqrt{2}\beta\right)^6} \Rightarrow$$

$$\Rightarrow \pi^3 \left(\int_0^\infty x^3 \frac{\cos \pi x^2}{\sinh \pi x} dx + \int_0^\infty x^3 \frac{\sin \pi x^2}{\sinh \pi x} dx \right) = \frac{1}{16} \left(\frac{\pi^3}{4} - 3\pi + \pi^2 \right). \quad (1.21)$$

A Wick rotation $x^0 \to -ix^0$ reverse the sign of $\tilde{I}_X$, and multiplies $\tilde{I}_A$ by $-i$. So in Euclidean signature, combining the terms involving X and A, the boundary interactions of the D3-NS5 system are

$$I^* = \frac{1}{g_{YM}^2}\int_{x^3=0} d^3x\left(-\frac{2a}{3(1+a^2)}\varepsilon^{abc}TrX_a[X_b,X_c]+i\frac{2a}{1-a^2}\varepsilon^{\mu\nu\lambda}Tr\left(A_\mu\partial_\nu A_\lambda+\frac{2}{3}A_\mu A_\nu A_\lambda\right)\right). \quad (1.22)$$

In a convenient notation in which $\mathcal{N}=4$ super Yang-Mills is obtained by dimensional reduction from ten dimensions, with the ten dimensions labelled by $x^0,...,x^9$, the Euclidean signature version of the chirality condition for supersymmetry generators and fermions is

$$\Gamma_0\Gamma_1...\Gamma_9\varepsilon=-i\varepsilon\,,\quad \Gamma_0\Gamma_1...\Gamma_9\lambda=-i\lambda\,. \quad (1.23)$$

To define a topological field theory, one defines a group $SO'(4)$ that acts by rotating $x^0,...,x^3$ in the usual way, while simultaneously rotating four normal coordinates $x^4,...x^7$. We pick a supersymmetry generator ε that is $SO'(4)$-invariant, meaning that it obeys

$$(\Gamma_{\mu\nu}+\Gamma_{4+\mu,4+\nu})\varepsilon=0\,,\quad \mu,\nu=0,...,3\,. \quad (1.24)$$

From the point of view of $SO'(4)$ symmetry, four of the adjoint-valued scalar fields of $\mathcal{N}=4$ super Yang-Mills theory are reinterpreted as an adjoint-valued one-form $\phi=\sum_{\mu=0}^{3}\phi_\mu dx^\mu$, while the other two combine two an adjoint-valued complex scalar field σ. $SO'(4)$ commutes with a group $SO(2)\cong U(1)$ of R-symmetries that rotates x^8 and x^9. We normalize its generator F so that σ has charge 2.
We identify that tangential part of ϕ, that is $\vec{\phi}=\sum_{\mu=0}^{2}\phi_\mu dx^\mu$, with $\vec{X}$, and we identify the normal part ϕ_3 with a component of $\vec{Y}$, say Y_1. The boundary couplings (1.22) become in this notation

$$I^* = \frac{1}{g_{YM}^2}\int_{x^3=0} d^3x\varepsilon^{\mu\nu\lambda}Tr\left(-\frac{4a}{3(1+a^2)}\phi_\mu\phi_\nu\phi_\lambda+i\frac{2a}{1-a^2}\left(A_\mu\partial_\nu A_\lambda+\frac{2}{3}A_\mu A_\nu A_\lambda\right)\right). \quad (1.25)$$

The condition (1.24) for $SO'(4)$-invariance of the supersymmetry generator actually has a two-dimensional space of solutions. It is possible to pick a basis of solutions $\varepsilon_\ell,\varepsilon_r$ that are chiral in the four-dimensional sense,

$$\Gamma_{0123}\varepsilon_\ell=-\varepsilon_\ell\,,\quad \Gamma_{0123}\varepsilon_r=\varepsilon_r\,. \quad (1.26)$$

It is possible to normalize ε_ℓ and ε_r so that, for $\mu=0,1,2,$ or 3,

$$\Gamma_{\mu,4+\mu}\varepsilon_\ell=-\varepsilon_r\,,\quad \Gamma_{\mu,4+\mu}\varepsilon_r=\varepsilon_\ell\,. \quad (1.27)$$

In constructing a topological field theory, we may take the supersymmetry generator ε to be an arbitrary linear combination of ε_ℓ and ε_r. Up to an inessential scaling, we take

$$\varepsilon=\varepsilon_\ell+t\varepsilon_r\,. \quad (1.28)$$

Now we can make contact with the D3-NS5 system. From (1.23), (1.26), and (1.6), we have

$$B_0\varepsilon_\ell=i\varepsilon_\ell\,,\quad B_0\varepsilon_r=-i\varepsilon_r\,. \quad (1.29)$$

Using also (1.27) and (1.24), one can show, with some gamma matrix algebra, that

$$B_1\varepsilon_\ell = -\varepsilon_r\,, \quad B_1\varepsilon_r = -\varepsilon_\ell\,. \quad (1.30)$$

It follows that

$$\left(1 + i\frac{1-t^2}{1+t^2}B_0 + \frac{2t}{1+t^2}B_1\right)(\varepsilon_\ell + t\varepsilon_r) = 0\,. \quad (1.31)$$

On the other hand, with the help of (1.7), we see that the object ε_0 defined in (1.8) obeys the same equation

$$\left(1 + i\frac{1-t^2}{1+t^2}B_0 + \frac{2t}{1+t^2}B_1\right)\varepsilon_0 = 0 \quad (1.32)$$

if and only if the parameter a used in describing the D3-NS5 system is related to the parameter t of the topological field theory

$$a = i\frac{1-it}{1+it}\,. \quad (1.33)$$

Substituting (1.33) in (1.16b) and solving for t^2, we get the surprisingly simple result

$$t^2 = \frac{\bar{\tau}}{\tau}\,. \quad (1.34)$$

The operation $t \to -t$ corresponds to $a \to -1/a$ and to exchange of the D3-NS5 and D3-$\overline{NS}\,\overline{5}$ systems. With the aid of (1.33), the boundary couplings (1.25) can be rewritten

$$I^* = \frac{1}{g_{YM}^2}\int_{x^3=0} d^3x\varepsilon^{\mu\nu\lambda}Tr\left(-\frac{t+t^{-1}}{3}\phi_\mu\phi_\nu\phi_\lambda + \frac{t+t^{-1}}{t-t^{-1}}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right)\right). \quad (1.35)$$

Thence, we can obtain the following mathematical connection:

$$I^* = \frac{1}{g_{YM}^2}\int_{x^3=0} d^3x\varepsilon^{\mu\nu\lambda}Tr\left(-\frac{4a}{3(1+a^2)}\phi_\mu\phi_\nu\phi_\lambda + i\frac{2a}{1-a^2}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right)\right) \Rightarrow$$
$$\Rightarrow \frac{1}{g_{YM}^2}\int_{x^3=0} d^3x\varepsilon^{\mu\nu\lambda}Tr\left(-\frac{t+t^{-1}}{3}\phi_\mu\phi_\nu\phi_\lambda + \frac{t+t^{-1}}{t-t^{-1}}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right)\right). \quad (1.36)$$

$\mathcal{N} = 4$ super Yang-Mills theory in four dimensions admits 1/16-BPS Wilson loop operators. They are constructed as follows. The supersymmetry transformation law for the bosonic fields of this theory is

$$\delta A_I = i\bar{\varepsilon}\Gamma_I\lambda = -i\bar{\lambda}\Gamma_I\varepsilon\,, \quad I = 0,...,9\,. \quad (1.37)$$

Here we use a ten-dimensional notation: for $I \leq 3$, A_I is a component of a gauge field, and for $I \geq 4$, it is a scalar field. By twisting, we have converted four of the scalar fields to a one-form ϕ.

Usually, we use Greek letters $\mu,\nu\ldots$ for four-dimensional indices, so we write $A=\sum_{\mu=0}^{3}A_\mu dx^\mu$, $\phi=\sum_{\mu=0}^{3}\phi_\mu dx^\mu=\sum_{\mu=0}^{3}A_{4+\mu}dx^\mu$. Suppose that ε is such that

$$(\Gamma_\mu + i\Gamma_{4+\mu})\varepsilon = 0, \quad \mu = 0,\ldots,3. \quad (1.38)$$

Clearly, in this case, Wilson operators of the form

$$Tr_R P\exp\left(-\oint_K (A+i\phi)\right) \quad (1.39)$$

are invariant, for an arbitrary embedded loop K in spacetime and any representation R of the gauge group. Similarly, if

$$(\Gamma_\mu - i\Gamma_{4+\mu})\varepsilon = 0, \quad \mu = 0,\ldots,3. \quad (1.40)$$

then there are supersymmetric Wilson operators of the form

$$Tr_R P\exp\left(-\oint_K (A-i\phi)\right). \quad (1.41)$$

For a Wilson operator supported entirely at the boundary of V, we can use the boundary conditions obeyed by λ, as well as the conditions obeyed by ε, to establish supersymmetry. We will describe the conditions that on the boundary of V

$$0=\delta(A_\mu + w\phi_\mu) = -i\bar{\lambda}(\Gamma_\mu + w\Gamma_{4+\mu})\varepsilon, \quad \mu = 0,1,2. \quad (1.42)$$

In (1.42) w is a complex number, to be determined. If (1.42) holds, then upon setting

$$\mathrm{A}_w = A + w\phi, \quad (1.43)$$

we can construct supersymmetric Wilson operators

$$Tr_R P\exp\left(-\oint_K \mathrm{A}_w\right). \quad (1.44)$$

for any knot K in the boundary of V. The action I of $\mathcal{N}=4$ super Yang-Mills theory on a four-manifold V is the sum of a term proportional to $1/g_{YM}^2$, which contains the kinetic energy for all fields, and a term proportional to θ:

$$I=\frac{1}{g_{YM}^2}\int_V d^4x\sqrt{g}\,\mathcal{L}_{kin} + i\frac{\theta}{32\pi^2}\int_V d^4x\varepsilon^{\mu\nu\alpha\beta}TrF_{\mu\nu}F_{\alpha\beta}. \quad (1.45)$$

Also this equation can be related with the Jormakka's equation (1.18) multiplying both the sides for $8\pi^5$ and obtaining:

$$I=\frac{8\pi^5}{g_{YM}^2}\int_V d^4x\sqrt{g}\,\mathcal{L}_{kin} + i\frac{\theta\pi^3}{4}\int_V d^4x\varepsilon^{\mu\nu\alpha\beta}TrF_{\mu\nu}F_{\alpha\beta} \Rightarrow$$

$$\Rightarrow \int d^2 x e^{-\beta^2\left(y_2^2+y_3^2\right)} \frac{1}{\sqrt{3}} \int dy_1 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_1^2} = \int d^2 x e^{-\beta^2\left(y_2^2+y_3^2\right)} \frac{1}{\sqrt{3}} \sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} =$$

$$= \frac{1}{\sqrt{3}} \sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} \int dx_3 e^{-\beta^2 y_3^2} \int dx_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} = \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} \int dx_3 e^{-\beta^2 y_3^2} \int dy_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} =$$

$$= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} (2\pi)\left(\sqrt{2}\beta\right)^{-2} \int dx_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \frac{1}{2} (2\pi)\left(\sqrt{2}\beta\right)^{-2} \int dy_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \frac{1}{2} (2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$= \left[\int d^2 x e^{-\beta^2\left(y_2^2+y_3^2\right)} \frac{1}{\sqrt{3}} \sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2 = 8\frac{\pi^3}{4} \times \frac{1}{\left(\sqrt{2}\beta\right)^6} \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3 \frac{\cos \pi x^2}{\sinh \pi x} dx + \int_0^\infty x^3 \frac{\sin \pi x^2}{\sinh \pi x} dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right). \quad (1.45b)$$

Here, the part of $\mathcal{L}_{kin}$ that involves A,ϕ only is (in Euclidean signature)

$$\mathcal{L}_{kin}^{A,\phi} = -Tr\left(\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi_\nu D^\mu \phi^\nu + R_{\mu\nu} \phi^\mu \phi^\nu + \frac{1}{2}\left[\phi_\mu, \phi_\nu\right]^2\right). \quad (1.46)$$

Both terms on the right hand side of (1.45) are Q-invariant. The θ term is Q-invariant because, more generally, it is a topological invariant, unchanged in any continuous deformations. It represents a nonzero class in the cohomology of Q. One might suspect that the integral of $\mathcal{L}_{kin}$ would vanish in the cohomology of Q, as happens in many twisted topological field theories, but this is actually not the case. Instead, the first term on the right of (1.45) is equivalent $\bmod\{Q,...\}$ to a multiple of the second term. The precise relation is

$$I = \{Q,...\} + \frac{2\pi i \Psi}{32\pi^2} \int_V d^4 x \varepsilon^{\mu\nu\alpha\beta} Tr F_{\mu\nu} F_{\alpha\beta}, \quad (1.47)$$

where

$$\Psi = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} \frac{t - t^{-1}}{t + t^{-1}} \quad (1.48)$$

is the canonical parameter.
Thence, the eq. (1.45) can be rewritten also as follows:

$$I = \frac{1}{g_{YM}^2} \int_V d^4 x \sqrt{g} \mathcal{L}_{kin} + i\frac{\theta}{32\pi^2} \int_V d^4 x \varepsilon^{\mu\nu\alpha\beta} Tr F_{\mu\nu} F_{\alpha\beta} = \{Q,...\} + \frac{2\pi i \Psi}{32\pi^2} \int_V d^4 x \varepsilon^{\mu\nu\alpha\beta} Tr F_{\mu\nu} F_{\alpha\beta}. \quad (1.48b)$$

Under a general S-duality transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad (1.49)$$

t transforms by

$$t \to \frac{c\tau + d}{|c\tau + d|} t \quad (1.50)$$

and that Ψ transforms just as τ does:

$$\Psi \to \frac{a\Psi + b}{c\Psi + d}. \quad (1.51)$$

The formula (1.48) for Ψ holds for all τ, t. Imposing the relations (1.16b), (1.33) that are natural in studying the D3-NS5 system, we can derive several interesting alternative formulas. Eliminating t in favour of g_{YM} and θ, we find

$$\Psi = \frac{|\tau|^2}{\mathrm{Re}\,\tau}, \quad (1.52)$$

showing that Ψ is always real for the D3-NS5 system with physical values of the parameters (real g_{YM} and θ). Alternatively, eliminating θ in favour of g_{YM} and t, we get

$$\Psi = \frac{4\pi i}{g_{YM}^2}\left(\frac{t - t^{-1}}{t + t^{-1}} - \frac{t + t^{-1}}{t - t^{-1}}\right). \quad (1.53)$$

The integral $\int_V d^4x \varepsilon^{\mu\nu\alpha\beta} Tr F_{\mu\nu} F_{\alpha\beta}$ is no longer Q-invariant, but varies by a boundary term. It is convenient to replace this integral by a multiple of the Chern-Simons function. We define the Chern-Simons function $CS(\mathrm{A})$, for any connection A, possibly complex-valued, by

$$CS(\mathrm{A}) = \frac{1}{4\pi}\int_{\partial V} d^3x \varepsilon^{\mu\nu\lambda} Tr\left(\mathrm{A}_\mu \partial_\nu \mathrm{A}_\lambda + \frac{2}{3}\mathrm{A}_\mu \mathrm{A}_\nu \mathrm{A}_\lambda\right). \quad (1.54)$$

In terms of this function, we can make the following substitution on the right hand side of eq. (1.47):

$$\frac{2\pi i \Psi}{32\pi^2}\int_V d^4x \varepsilon^{\mu\nu\alpha\beta} Tr F_{\mu\nu} F_{\alpha\beta} \to i\Psi CS(A). \quad (1.55)$$

Writing h for the dual Coxeter number of G, we can write a formula equivalent to (1.54) in terms of a trace Tr_{ad} in the adjoint representation of G:

$$CS(\mathrm{A}) = \frac{1}{8\pi h}\int_{\partial V} d^3x \varepsilon^{\mu\nu\lambda} Tr_{ad}\left(\mathrm{A}_\mu \partial_\nu \mathrm{A}_\lambda + \frac{2}{3}\mathrm{A}_\mu \mathrm{A}_\nu \mathrm{A}_\lambda\right). \quad (1.56)$$

Thence, we have the following connection:

$$CS(\mathrm{A}) = \frac{1}{4\pi}\int_{\partial V} d^3x \varepsilon^{\mu\nu\lambda} Tr\left(\mathrm{A}_\mu \partial_\nu \mathrm{A}_\lambda + \frac{2}{3}\mathrm{A}_\mu \mathrm{A}_\nu \mathrm{A}_\lambda\right) = \frac{1}{8\pi h}\int_{\partial V} d^3x \varepsilon^{\mu\nu\lambda} Tr_{ad}\left(\mathrm{A}_\mu \partial_\nu \mathrm{A}_\lambda + \frac{2}{3}\mathrm{A}_\mu \mathrm{A}_\nu \mathrm{A}_\lambda\right). \quad (1.56b)$$

Also this expression, can be related with the Jormakka's equation (1.18), multiplying both the sides for π^4 and obtaining:

$$CS(\mathrm{A})=\frac{\pi^3}{4}\int_{\partial V}d^3x\varepsilon^{\mu\nu\lambda}Tr\left(\mathrm{A}_\mu\partial_\nu\mathrm{A}_\lambda+\frac{2}{3}\mathrm{A}_\mu\mathrm{A}_\nu\mathrm{A}_\lambda\right)=\frac{\pi^3}{8h}\int_{\partial V}d^3x\varepsilon^{\mu\nu\lambda}Tr_{ad}\left(\mathrm{A}_\mu\partial_\nu\mathrm{A}_\lambda+\frac{2}{3}\mathrm{A}_\mu\mathrm{A}_\nu\mathrm{A}_\lambda\right)\Rightarrow$$

$$\Rightarrow\int d^2xe^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\int dy_1e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2y_1^2}=\int d^2xe^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}=$$

$$=\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3e^{-\beta^2y_3^2}\int dx_2e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2y_2^2}=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3e^{-\beta^2y_3^2}\int dy_2e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2y_2^2}=$$

$$=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3e^{-\beta^2y_3^2}=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3e^{-\beta^2y_3^2}=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$=\left[\int d^2xe^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2=8\frac{\pi^3}{4}\times\frac{1}{\left(\sqrt{2}\beta\right)^6}\Rightarrow$$

$$\Rightarrow\pi^3\left(\int_0^\infty x^3\frac{\cos\pi x^2}{\sinh\pi x}dx+\int_0^\infty x^3\frac{\sin\pi x^2}{\sinh\pi x}dx\right)=\frac{1}{16}\left(\frac{\pi^3}{4}-3\pi+\pi^2\right).\quad(1.56c)$$

The boundary couplings must be a function of A_w only (modulo Q-exact terms), since this is the only non-trivial Q-invariant combination of boundary fields.

The coefficient of $CS(\mathrm{A}_w)$ is precisely $i\Psi$. The generalization of (1.47) in the presence of a boundary is

$$I=\{Q,...\}+i\Psi CS(\mathrm{A}_w).\quad(1.57)$$

Where $CS(\mathrm{A}_w)$ is written explicitly as a function of A and ϕ, the ϕ-dependent terms are given by local, gauge-invariant integrals, since

$$CS(\mathrm{A}_w)=CS(A)+\frac{1}{4\pi}\int_{\partial V}d^3x\varepsilon^{\mu\nu\lambda}Tr\left(w\phi_\mu F_{\nu\lambda}+w^2\phi_\mu D_\nu\phi_\lambda+\frac{2w^3}{3}\phi_\mu\phi_\nu\phi_\lambda\right).\quad(1.58)$$

The coefficient of $CS(A)$ in the boundary interaction is $i\Psi$, and in view of (1.58), the coefficient of $CS(\mathrm{A}_w)$ must be the same.

Under favourable conditions, computations in topological field theory can be localized on configurations that obey $\{Q,\zeta\}=0$, for all fermion fields ζ. Among the fermions of F = – 1 in the present model are a selfdual two-form χ^+, and anti-selfdual two-form χ^-, and a scalar η. They have the property that $\upsilon^+=\{Q,\chi^+\}$, $\upsilon^-=\{Q,\chi^-\}$, and $\upsilon^0=\{Q,\eta\}$ depend on A,ϕ only:

$$\upsilon^+=\left(F-\phi\wedge\phi+td_A\phi\right)^+,\quad\upsilon^-=\left(F-\phi\wedge\phi-t^{-1}d_A\phi\right)^-,\quad\upsilon^0=D_\mu\phi^\mu.\quad(1.59)$$

Here for any two-form α, we write α^+ and α^- for its selfdual and anti-selfdual projections. Localization of A and ϕ can be achieved for real t by adding a suitable term to the action I:

$$I\to I-\frac{1}{\varepsilon}\left\{Q,\int_V Tr\left(\chi^+\upsilon^++\chi^-\upsilon^-+\chi^0\upsilon^0\right)\right\}=I-\frac{1}{\varepsilon}\int_V Tr\left(\left(\upsilon^+\right)^2+\left(\upsilon^-\right)^2+\left(\upsilon^2\right)^2+...\right),\quad(1.60)$$

where ε is a small parameter and the omitted terms are fermion bilinears. For t real, $\mathcal{V}^+$, $\mathcal{V}^-$, and $\mathcal{V}^0$ are real, and the modified action diverges as $1/\varepsilon$ unless the localization equations

$$\left(F-\phi\wedge\phi+td_A\phi\right)^+=\left(F-\phi\wedge\phi-t^{-1}d_A\phi\right)^-=D_\mu\phi^\mu=0 \quad (1.61)$$

are satisfied. So the path integral is supported, for $\varepsilon\to 0$, on the space of solutions of those equations.

To understand explicitly the origin of the ϕ-dependent boundary terms in (1.58), we have to make more explicit the relation of the localization procedure of eq. (1.60) to the physical action of $\mathcal{N}=4$ Yang-Mills theory. The identity we need is the following:

$$-\int_V d^4x Tr\left(\frac{t^{-1}}{t+t^{-1}}\mathcal{V}^+_{\mu\nu}\mathcal{V}^{+\mu\nu}+\frac{t}{t+t^{-1}}\mathcal{V}^-_{\mu\nu}\mathcal{V}^{-\mu\nu}+\left(\mathcal{V}^0\right)^2\right)=\int_V d^4x\sqrt{g}\mathcal{L}^{A,\phi}_{kin}+\frac{t-t^{-1}}{4\left(t+t^{-1}\right)}\int_V d^4x\varepsilon^{\mu\nu\alpha\beta}TrF_{\mu\nu}F_{\alpha\beta}+$$
$$+\int_{\partial V} d^3x\varepsilon^{\mu\nu\lambda}Tr\left(-\frac{2}{t+t^{-1}}\phi_\mu F_{\nu\lambda}-\frac{t-t^{-1}}{t+t^{-1}}\phi_\mu D_\nu\phi_\lambda+\frac{4}{3}\frac{1}{t+t^{-1}}\phi_\mu\phi_\nu\phi_\lambda\right). \quad (1.62)$$

After multiplying by $1/g^2_{YM}$ and making the substitution (1.55) in one term, we can rewrite (1.62) as follows:

$$\frac{1}{g^2_{YM}}\int_V d^4x\sqrt{g}\mathcal{L}_{kin}=\{Q,\ldots\}+\frac{1}{g^2_{YM}}\int_{\partial V} d^3x\varepsilon^{\mu\nu\lambda}Tr\left(-\frac{t-t^{-1}}{t+t^{-1}}\left(A_\mu\partial_\nu A_\lambda+\frac{2}{3}A_\mu A_\nu A_\lambda\right)+\frac{2}{t+t^{-1}}\phi_\mu F_{\nu\lambda}+\right.$$
$$\left.+\frac{t-t^{-1}}{t+t^{-1}}\phi_\mu D_\nu\phi_\lambda-\frac{4}{3}\frac{1}{t+t^{-1}}\phi_\mu\phi_\nu\phi_\lambda\right). \quad (1.63)$$

When we add the boundary terms that have appeared in (1.63) to the boundary terms (1.35) that are already present in the physical theory, before twisting, we find that the action has the expected form

$$\{Q,\ldots\}+i\Psi CS(\mathrm{A}_w), \quad (1.64)$$

with the expected value $w=\left(t-t^{-1}\right)/2$.

The transformation law (1.51) for the canonical parameter Ψ tells us that the parameter $\Psi^\vee$ of the dual theory is related to Ψ by

$$\Psi^\vee=-\frac{1}{n_g\Psi}. \quad (1.65)$$

On the other hand, since $t^\vee=1$, the formula (1.48) for $\Psi^\vee$ reduces to

$$\Psi^\vee=\frac{\theta^\vee}{2\pi}. \quad (1.66)$$

Combining these formulas,

$$\theta^\vee=2\pi\Psi^\vee=-\frac{2\pi}{n_g\Psi}. \quad (1.67)$$

For $G^\vee = SU(N)$, we define the instanton number of the $G^\vee$ gauge theory by

$$\mathrm{P} = \frac{1}{32\pi^2}\int_V \varepsilon^{\mu\nu\alpha\beta} Tr F_{\mu\nu} F_{\alpha\beta}\,, \quad (1.68)$$

where Tr is the trace in the N-dimensional representation. For any $G^\vee$, we can take

$$\mathrm{P} = \frac{1}{2h^\vee}\frac{1}{32\pi^2}\int_V \varepsilon^{\mu\nu\alpha\beta} Tr_{adj} F_{\mu\nu} F_{\alpha\beta}\,. \quad (1.69)$$

where $h^\vee$ is the dual Coxeter number of $G^\vee$, and Tr_{adj} is the trace in the adjoint representation of $G^\vee$.

The instanton number of a $G^\vee$-bundle $E \to V$ is a topological invariant if V is a four-manifold without boundary. It remains a topological invariant if V has a non-empty boundary and we are given a trivialization of E on $W = \partial V$. We have just discovered that instead of being trivialized on W, E is identified on W with the tangent bundle TW to W; the gauge field A restricted to W is similarly identified with the Riemannian connection ω on TW, or more precisely with its $G^\vee$-valued image $\xi(\omega)$, where $\xi : \mathfrak{su}(2) \to \mathfrak{g}^\vee$ is a principal embedding. This means that the instanton number P is not invariant under a change of metric of V. In general, under any change in the gauge field A, the change in P is given by the change in the Chern-Simons invariant of the restriction of A to the boundary W:

$$\delta\,\mathrm{P} = \frac{1}{2\pi}\delta\,\mathrm{CS}(A). \quad (1.70)$$

Since when restricted to W we have $A = \xi(\omega)$, we can equivalently write

$$\delta\,\mathrm{P} = \frac{1}{2\pi}\delta\,\mathrm{CS}(\xi(\omega)). \quad (1.71)$$

In turn, $\mathrm{CS}(\xi(\omega))$ is the same as $\mathfrak{k}\,\mathrm{CS}(\omega)$ where $\mathrm{CS}(\omega)$ is the Chern-Simons invariant of ω as an $SU(2)$ connection, and $\mathfrak{k}$ is an integer, analyzed presently, that results from the embedding. So we can slightly simplify (1.71) to

$$\delta\,\mathrm{P} = \frac{\mathfrak{k}}{2\pi}\delta\,\mathrm{CS}(\omega). \quad (1.72)$$

If V is a compact manifold with boundary, there is a simple cure for this. We simply modify the definition (1.69) of P by subtracting the integral over V of a suitable curvature integral. The curvature integral is a multiple of $\int_V TrR \wedge R$, with R the Riemann tensor of V. This integral is a topological invariant if $\partial V =$ Ø, and in general its variation is a multiple of $\delta\,\mathrm{CS}(\omega)$. We pick the coefficient to cancel the boundary term in the variation of P. Thus, we replace the definition (1.69) with

$$\hat{P} = \frac{1}{2h^\vee}\frac{1}{32\pi^2}\int_V \varepsilon^{\mu\nu\alpha\beta} Tr_{adj} F_{\mu\nu} F_{\alpha\beta} - \frac{\mathfrak{k}}{4}\frac{1}{32\pi^2}\int_V \varepsilon^{\mu\nu\alpha\beta} Tr_{TV} R_{\mu\nu} R_{\alpha\beta}\,, \quad (1.73)$$

where we view the Riemann tensor as a two-form with values in endomorphisms of the tangent bundle TV of V and take the trace accordingly. $\hat{P}$ is an integer-valued topological invariant.

We note that $32\pi^2 = 315{,}8273408 \cong 315{,}83$ that is possible to connect at the following value: $315{,}7587390 \cong 315{,}76$ (see 1/1,375, i.e. $\sigma =$ partition numbers) and also to the value $315{,}6949704 \cong 315{,}69$ (see * 2,71828 = "e"), i.e. the Table concerning the "**universal music system based on Phi**".

We use eq. (1.56), in which $\mathrm{CS}(A)$ is defined for any connection A using a trace in the adjoint representation (and we set $h = 2$). It is convenient to evaluate the right hand side of (1.56) as the sum of an integral over W_1 with the connection ω, an integral over W_0 with the connection $\hat{\omega}$, and a correction term on the common boundary Ξ of W_0 and W_1 that involves the gauge transformation between ω and $\hat{\omega}$:

$$\mathrm{CS}\left(\omega_{(j)}\right) = \frac{1}{16\pi}\int_{W_1} Tr_{ad}\left(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right) + \frac{1}{16\pi}\int_{W_0} Tr_{ad}\hat{\omega} \wedge d\hat{\omega} - \frac{1}{16\pi}\int_{\Xi} Tr_{ad} ds \wedge \hat{\omega}. \quad (1.74)$$

We note that the following equation can be related with the Ramanujan' modular equation concerning the superstrings, multiplying both the sides for $\frac{2}{31}$. Indeed , we obtain:

$$\frac{2}{31}\mathrm{CS}\left(\omega_{(j)}\right) = \frac{1}{248\pi}\int_{W_1} Tr_{ad}\left(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right) + \frac{1}{248\pi}\int_{W_0} Tr_{ad}\hat{\omega} \wedge d\hat{\omega} - \frac{1}{248\pi}\int_{\Xi} Tr_{ad} ds \wedge \hat{\omega} \Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.74b)$$

Furthermore, we note that $\frac{2}{31} = 0.064516129 \cong 0.0645 \cong 0.0638$, value that is inserted in the columns (π) and (* 1,375) concerning the **universal music system based on Phi**.

The terms in (1.74) that depend on the framing of K are the integrals over W_0 and Ξ. A straightforward evaluation gives

$$\mathrm{CS}\left(\omega_{(j)}\right) = -\tau j(j+1) + \ldots \quad (1.75)$$

where the ellipses come from the integral over W_1 and do not depend on the framing of K. Using the following equation

$$q^{-\nu CS(\omega)/2\pi} = q^{-\nu CS_{grav}/8\pi}, \quad (1.76)$$

(with $v = 1$ for $G^{\vee} = SO(3)$), the dependence of the partition function on $\mathrm{CS}\left(\omega_{(j)}\right)$ is a factor of $q^{-CS\left(\omega_{(j)}\right)/2\pi}$. So finally, under a unit change in framing, $\tau \to \tau + 2\pi$, the partition function is multiplied by $q^{j(j+1)}$, just as in Chern-Simons theory.
According to the following equations:

$$F^+ - \frac{1}{4}B\times B - \frac{1}{2}D_y B = 0; \quad (1.77) \qquad F_{y\mu} + \sum_{\nu=0}^{3} D^\nu B_{\nu\mu} = 0, \quad \mu = 0,\ldots,3; \quad (1.78)$$

the equations for a supersymmetric field configuration in this theory (i.e. the gauge theory description) read

$$F^+ - \frac{1}{4}B\times B - \frac{1}{2}D_y B = 0; \qquad F_{y\mu} + D^\nu B_{\nu\mu} = 0. \quad (1.79)$$

On a manifold Z, with local coordinates u^i, a metric tensor γ_{ij}, and a Morse function Γ, the flow equations of Morse theory read

$$\frac{du^i}{dt} = -\gamma^{ij}\frac{\partial\Gamma}{\partial u^j}. \quad (1.80)$$

We endow $W_3 \times R_+$ with a metric $g_{ij}dx^i dx^j + dy^2$. On the space of fields on $W_3 \times R_+$, we define the metric

$$ds^2 = -\int_{W_3\times R_+} d^3x dy \sqrt{g}\, Tr\left(g^{ij}\delta A_i \delta A_j + \delta A_y \delta A_y + g^{ij}\delta B_{0i}\delta B_{0j}\right). \quad (1.81)$$

And then we define the Morse function

$$\Gamma = -\int_{W_3\times R_+} d^3x dy Tr\left(\sqrt{g}\, g^{ij}F_{yi}B_{0j} + \frac{1}{2}\varepsilon^{ijk}\left(A_i\partial_j A_k + \frac{2}{3}A_i A_j A_k - B_{0i}D_j B_{0k}\right) + \sqrt{g}\, w\right), \quad (1.82)$$

with w a constant chosen so that the integral converges for $y \to \infty$. A straightforward computation shows that the supersymmetric equations (1.79), in the gauge $A_0 = 0$, are indeed the flow equations with Γ as a Morse function.
The first-order supersymmetric equations (1.79) imply the second order Euler-Lagrange equations of supersymmetric Yang-Mills theory. Setting

$$Y_{\mu\nu} = \left(F^+ - \frac{1}{4}B\times B - \frac{1}{2}D_y B\right)_{\mu\nu}, \qquad Z_\mu = F_{y\mu} + D^\sigma B_{\sigma\mu}, \quad (1.83)$$

so that the supersymmetric equations are $Y = Z = 0$, we find the following identity

$$-\int_{M_4\times R_+} d^4x dy\sqrt{g}\, Tr\left(Y_{\mu\nu}Y^{\mu\nu} + Z_\mu Z^\mu\right) = -\int_{M_4\times R_+} d^4x dy \sqrt{g}\, Tr\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + F_{y\mu}F^{y\mu} + \frac{1}{4}\left(D_y B_{\mu\nu}\right)^2 + \right.$$
$$\left. + \frac{1}{4}\left(D_\alpha B_{\mu\nu}\right)^2 + \frac{1}{16}(B\times B)_{\mu\nu}(B\times B)^{\mu\nu} + \frac{R}{8}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}R_{\lambda\nu\mu\tau}B^{\lambda\nu}B^{\mu\tau}\right) + \ldots \quad (1.84)$$

Here $R_{\lambda\nu\mu\tau}$ and R are the Riemann tensor and Ricci scalar of M_4; these curvature couplings are dictated by supersymmetry when M_4 becomes curved. In (1.84), the ellipses represent the omission of certain terms whose local variations vanish – both surface terms and a multiple of the instanton number evaluated on M_4. In fact, with our boundary conditions, both the volume integral on the right hand side of (1.84) and the omitted terms are divergent. The right hand side of (1.84) is

essentially the bosonic part of the action of maximally supersymmetric Yang-Mills theory in five dimensions.
Indeed, we can to connect this equation with the Ramanujan's modular equation concerning the superstrings:

$$-\int_{M_4\times R_+} d^4xdy\sqrt{g}Tr\left(Y_{\mu\nu}Y^{\mu\nu}+Z_\mu Z^\mu\right)=-\int_{M_4\times R_+} d^4xdy\sqrt{g}Tr\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+F_{y\mu}F^{y\mu}+\frac{1}{4}\left(D_yB_{\mu\nu}\right)^2+\right.$$

$$\left.+\frac{1}{4}\left(D_\alpha B_{\mu\nu}\right)^2+\frac{1}{16}(B\times B)_{\mu\nu}(B\times B)^{\mu\nu}+\frac{R}{8}B_{\mu\nu}B^{\mu\nu}-\frac{1}{4}R_{\lambda\nu\mu\tau}B^{\lambda\nu}B^{\mu\tau}\right)+\ldots\Rightarrow$$

$$\Rightarrow\frac{1}{3}\frac{4\left[anti\log\dfrac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\dfrac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.85)$$

2. On some equations concerning the applications of non-abelian localization to analyze the Chern-Simons path integral including Wilson loop insertions [2]

We recall that a Wilson loop operator $W_R(C)$ in any gauge theory on a manifold M is described by the data of an oriented, closed curve C which is smoothly embedded in M and which is decorated by an irreducible representation R of the gauge group G. As a classical functional of the connection A, the Wilson loop operator is then given simply by the trace in R of the holonomy of A around C,

$$W_R(C)=Tr_RP\exp\left(-\oint_C A\right). \quad (2.1)$$

To describe the expectation value of $W_R(C)$ in the Lagrangian formulation of Chern-Simons theory, we introduce the absolutely-normalized Wilson loop path integral

$$Z(k;C,R)=\int\mathcal{D}AW_R(C)\exp\left[i\frac{k}{4\pi}\int_M Tr\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)\right], \quad (2.2)$$

in terms of which the Wilson loop expectation value is given by the ratio

$$\langle W_R(C)\rangle=\frac{Z(k;C,R)}{Z(k)}. \quad (2.3)$$

With regard $\frac{1}{4\pi}$ in the eq. (2.2), we have that $\frac{1}{4\pi}=0{,}07957\cong 0{,}07893$, value inserted in the column (*Pigreco) concerning the Table of the **universal music system based on Phi**.

Let us consider the simplest Wilson loop – namely, the unknot Wilson loop – in Chern-Simons theory on S^3 with gauge group $SU(2)$. For the unknot, the absolutely-normalized Wilson loop path integral in (2.2) is given exactly by

$$Z(k;\mathrm{O},j)=\sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi j}{k+2}\right), \qquad j=1,...,k+1. \quad (2.4)$$

We have that j runs without loss over the finite set of irreducible representations which are integrable in the $SU(2)$ current algebra at level k. This simple result was first obtained by Witten using the Hamiltonian formulation of Chern-Simons theory, and as a special case, when $j=1$ is trivial, the general formula for $Z(k;\mathrm{O},j)$ reduces to the standard expression for the $SU(2)$ partition function $Z(k)$ of Chern-Simons theory on S^3. From the semi-classical perspective, we can gain greater insight into the exact formula for $Z(k;\mathrm{O},j)$ by rewriting (2.4) as a contour integral over the real axis,

$$Z(k;\mathrm{O},j)=\frac{1}{2\pi i}e^{-\frac{i\pi(1+j^2)}{2(k+2)}}\int_{-\infty}^{+\infty}dx\,ch_j\left(e^{\frac{i\pi}{4}}\frac{x}{2}\right)\sinh^2\left(e^{\frac{i\pi}{4}}\frac{x}{2}\right)\exp\left(-\frac{(k+2)}{8\pi}x^2\right). \quad (2.5)$$

Here ch_j is the character of $SU(2)$ associated to the representation j,

$$ch_j(y)=\frac{\sinh(jy)}{\sinh(y)}=e^{(j-1)y}+e^{(j-3)y}+...+e^{-(j-3)y}+e^{-(j-1)y}, \quad (2.6)$$

and the equality between the expressions in (2.4) and (2.5) follows by evaluating (2.5) as a sum of elementary Gaussian integrals. Thence, we obtain the following expression:

$$Z(k;\mathrm{O},j)=\sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi j}{k+2}\right)=\frac{1}{2\pi i}e^{-\frac{i\pi(1+j^2)}{2(k+2)}}\int_{-\infty}^{+\infty}dx\,ch_j\left(e^{\frac{i\pi}{4}}\frac{x}{2}\right)\sinh^2\left(e^{\frac{i\pi}{4}}\frac{x}{2}\right)\exp\left(-\frac{(k+2)}{8\pi}x^2\right). \quad (2.6b)$$

The non-degenerate inner product on $R\oplus\tilde{L}_{\mathfrak{g}}$ is given by

$$((p,\phi,a),(q,\psi,b))=-\int_M d\tau Tr(\phi\psi)-pb-qa. \quad (2.7)$$

A non-degenerate, invariant inner product on the Lie algebra of $U(1)_R\times\tilde{\mathcal{G}}_0$ is given by

$$((p,\phi,a),(q,\psi,b))=-\int_M \kappa\wedge d\kappa Tr(\phi\psi)-pb-qa, \quad (2.8)$$

in direct correspondence with (2.7). Furthermore, the action of $U(1)_R\times\tilde{\mathcal{G}}_0$ on $\overline{\mathrm{A}}$ is Hamiltonian, with moment map

$$\langle\mu,(p,\phi,a)\rangle=-\frac{1}{2}p\int_M\kappa\wedge Tr(\mathcal{L}_R A\wedge A)-\int_M\kappa\wedge Tr(\phi F_A)+\int_M d\kappa\wedge Tr(\phi A)+a. \quad (2.9)$$

From (2.8) and (2.9), we see immediately that

$$(\mu,\mu)=\int_M \kappa\wedge Tr(\mathcal{L}_R A\wedge A)-\int_M \kappa\wedge d\kappa Tr\left[\left(\frac{\kappa\wedge F_A-d\kappa\wedge A}{\kappa\wedge d\kappa}\right)^2\right]. \quad (2.10)$$

Using the identity

$$\iota_R A=\frac{d\kappa\wedge A}{d\kappa\wedge\kappa}, \quad (2.11)$$

let us rewrite (2.10) as

$$(\mu,\mu)=\int_M \kappa\wedge Tr(\mathcal{L}_R A\wedge A)+2\int_M \kappa\wedge Tr[(\iota_R A)F_A]-\int_M \kappa\wedge d\kappa Tr\left[(\iota_R A)^2\right]-\int_M \frac{1}{\kappa\wedge d\kappa}Tr\left[(\kappa\wedge F_A)^2\right]. \quad (2.12)$$

We also require the following identity

$$\mathrm{CS}(A)=\int_M Tr\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)=\int_M \kappa\wedge Tr(\mathcal{L}_R A\wedge A)+2\int_M \kappa\wedge Tr[(\iota_R A)F_A]+$$
$$-\int_M \kappa\wedge d\kappa Tr\left[(\iota_R A)^2\right]. \quad (2.13)$$

With regard $\frac{2}{3}$ and 2 in the eq. (2.13), we have: $\frac{2}{3}=0{,}66666\cong 0{,}66666667$, i.e. $0{,}8333333\cdot\frac{4}{5}$, where 0,8333333 is the value inserted in the column “System” in the Table concerning the **universal music system based on Phi**.

We consider a general Wilson loop operator

$$W_R(C)=Tr_R P\exp\left(-\oint_C A\right), \quad (2.14)$$

where C is an oriented closed curve smoothly embedded in M, and R is an irreducible representation of the simply-connected gauge group G. The basic Wilson loop path integral is:

$$Z(\varepsilon;C,R)=\frac{1}{Vol(\mathcal{G})}\left(\frac{1}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}}\int \mathcal{D}A W_R(C)\exp\left[\frac{i}{2\varepsilon}\int_M Tr\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)\right],$$
$$\varepsilon=\frac{2\pi}{k}, \quad \Delta_{\mathcal{G}}=\dim\mathcal{G} \quad (2.15)$$

Thence, the eq. (2.15) can be rewritten also as follows:

$$Z(\varepsilon;C,R)=\frac{1}{Vol(\mathcal{G})}\left(\frac{1}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}}\int \mathcal{D}A W_R(C)\exp\left[\frac{ik}{4\pi}\int_M Tr\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)\right]. \quad (2.15b)$$

The Wilson loop path integral in a shift-invariant form is:

$$Z(\varepsilon;C,R)=\frac{1}{Vol(\mathcal{G})}\frac{1}{Vol(\mathcal{S})}\left(\frac{1}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}}\int \mathcal{D}A\mathcal{D}\Phi\mathcal{W}_R(C)\exp\left[\frac{i}{2\varepsilon}\mathcal{CS}(A-\kappa\Phi)\right]. \quad (2.16)$$

Here $\mathcal{W}_R(C)$ denotes the generalized Wilson loop operator defined not using A but using the shift-invariant combination $A-\kappa\Phi$, so that

$$\mathcal{W}_R(C)=Tr_R P\exp\left[-\oint_C (A-\kappa\Phi)\right]. \quad (2.17)$$

The semi-classical description for the Wilson loop operator is:

$$W_R(C)=\varepsilon^{\Delta_\alpha/2}\int_{\mathrm{LO}_\alpha}\mathcal{D}U\exp\left[i\mathcal{cs}_\alpha\left(U;A|_C\right)\right], \quad \varepsilon=\frac{2\pi}{k}, \quad \Delta_\alpha=\dim L\mathrm{O}_\alpha. \quad (2.18)$$

We write $\varepsilon L\mathrm{O}_\alpha$ to indicate the loopspace $L\mathrm{O}_\alpha$ equipped with the sigma model metric induced from the invariant Kahler metric on $\varepsilon\mathrm{O}_\alpha\equiv\mathrm{O}_{\varepsilon\alpha}$. We have that

$$W_R(C)=\int_{\varepsilon\mathrm{LO}_\alpha}\mathcal{D}U\exp\left[i\mathcal{cs}_\alpha\left(U;A|_C\right)\right]. \quad (2.19)$$

We can formulate the basic Wilson loop path integral in (2.15) using (2.19), as a path integral over the product $\mathrm{A}\times\varepsilon L\mathrm{O}_\alpha$,

$$Z(\varepsilon;C,R)=\frac{1}{Vol(\mathcal{G})}\left(\frac{1}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}}\int_{\mathrm{A}\times\varepsilon\mathrm{LO}_\alpha}\mathcal{D}A\mathcal{D}U\exp\left[\frac{i}{2\varepsilon}\mathcal{CS}(A)+i\mathcal{cs}_\alpha\left(U;A|_C\right)\right]. \quad (2.20)$$

The topological sigma model action for U in terms of a bulk integral over M is:

$$\mathcal{cs}_\alpha\left(U;A|_C\right)=\oint_C Tr\left(\alpha\cdot g^{-1}d_A g\right)=\int_M \delta_C\wedge Tr\left(\alpha\cdot g^{-1}d_A g\right). \quad (2.21)$$

Thence, we note that the eq. (2.20) can be rewritten also as follows:

$$Z(\varepsilon;C,R)=\frac{1}{Vol(\mathcal{G})}\left(\frac{1}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}}\int_{\mathrm{A}\times\varepsilon\mathrm{LO}_\alpha}\mathcal{D}A\mathcal{D}U\exp\left[\frac{i}{2\varepsilon}\mathcal{CS}(A)+i\int_M \delta_C\wedge Tr\left(\alpha\cdot g^{-1}d_A g\right)\right]. \quad (2.21b)$$

As in (2.16), we first consider the generalization of the following equation

$$Z(\varepsilon;C,R)=\frac{1}{Vol(\mathcal{G})}\left(\frac{1}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}}\int_{\mathrm{A}\times\varepsilon\mathrm{LO}_\alpha}\mathcal{D}A\mathcal{D}U\exp\left[\frac{i}{2\varepsilon}\mathcal{CS}(A)+i\mathcal{cs}_\alpha\left(U;A|_C\right)\right], \quad (2.22)$$

obtained by replacing A with the shift-invariant combination $A-\kappa\Phi$,

$$Z(\varepsilon;C,R)=\frac{1}{Vol(\mathcal{G})}\frac{1}{Vol(\mathcal{S})}\left(\frac{1}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}}\int \mathcal{D}A\mathcal{D}U\mathcal{D}\Phi\exp\left[\frac{i}{2\varepsilon}S(A,\Phi,U)\right], \quad (2.23)$$

where

$$S(A,\Phi,U)=\mathcal{CS}(A-\kappa\Phi)+2\varepsilon\mathcal{cs}_\alpha(U;A-\kappa\Phi). \quad (2.24)$$

We assume that the shift symmetry S acts on A and Φ just as before, and S acts trivially on U. Upon setting $\Phi=0$ with the shift symmetry, we reproduce (2.22) as before. On the other hand, to underscore the significance of (2.23), let us expand the shift-invariant sigma model action $\mathcal{cs}_\alpha(U;A-\kappa\Phi)$ in terms of Φ. From (2.21), we immediately find

$$\mathcal{cs}_\alpha(U;A-\kappa\Phi)=\mathcal{cs}_\alpha(U;A)-\int_M \kappa\wedge\delta_C Tr\left[\left(g\alpha g^{-1}\right)\Phi\right]. \quad (2.25)$$

The essential observation to make about (2.25) is simply that Φ appears linearly. Thus Φ still enters the total shift-invariant action $S(A,\Phi,U)$ quadratically. To be explicit, we expand $S(A,\Phi,U)$ in terms of Φ to obtain

$$S(A,\Phi,U)=\mathcal{CS}(A)+2\varepsilon\mathcal{cs}_\alpha(U;A)-\int_M\left[2\kappa\wedge Tr(\Phi\mathcal{F}_A)-\kappa\wedge d\kappa Tr\left(\Phi^2\right)\right]. \quad (2.26)$$

Here as a convenient shorthand, we introduce a "generalized" curvature $\mathcal{F}_A$ which includes the delta-function contribution from (2.25), so that

$$\mathcal{F}_A=F_A+\varepsilon\left(g\alpha g^{-1}\right)\delta_C. \quad (2.27)$$

By virtue of the shift symmetry, the remaining integral over the affine space A then reduces to an integral over the quotient $\overline{\mathrm{A}}=\mathrm{A}/S$, and we obtain the shift-invariant reformulation of the general Wilson loop path integral in Chern-Simons theory. Thus,

$$Z(\varepsilon;C,R)=\frac{1}{Vol(\mathcal{G})}\left(\frac{-i}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}/2}\int_{\overline{\mathrm{A}}\times\varepsilon\mathrm{LO}_\alpha}\overline{\mathcal{D}A}\,\mathcal{D}U\exp\left[\frac{i}{2\varepsilon}S(A,U)\right], \quad (2.28)$$

where

$$S(A,U)=\mathcal{CS}(A)+2\varepsilon\mathcal{cs}_\alpha\left(U;A|_C\right)-\int_M\frac{1}{\kappa\wedge d\kappa}Tr\left[(\kappa\wedge\mathcal{F}_A)^2\right]. \quad (2.29)$$

We consider a product of Wilson loop operators associated to oriented curves C_ℓ which are linked in M and decorated by irreducible representations R_ℓ with highest weights α_ℓ for $\ell=1,\ldots\mathrm{L}$. On each curve we introduce a corresponding sigma model field U_ℓ, and we apply the semi-classical description of $W_R(C)$ in (2.19) to write the obvious generalization of (2.20),

$$Z(\varepsilon;(C_1,R_1),\ldots,(C_\mathrm{L},R_\mathrm{L}))=\frac{1}{Vol(\mathcal{G})}\left(\frac{1}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}}\times$$

$$\times\int_{\mathrm{A}\times\varepsilon\mathrm{LO}_{\alpha_1}\times\ldots\times\varepsilon\mathrm{LO}_{\alpha_\mathrm{L}}}\mathcal{D}A\mathcal{D}U_1\ldots\mathcal{D}U_\mathrm{L}\exp\left[\frac{i}{2\varepsilon}\mathcal{CS}(A)+i\sum_{\ell=1}^{\mathrm{L}}\mathcal{cs}_{\alpha_\ell}\left(U_\ell;A|_C\right)\right]. \quad (2.30)$$

Through some manipulations, we find that the shift invariant version of the Wilson link path integral in (2.30) is given by

$$Z\left(\varepsilon;(C_1,R_1),\ldots,(C_{\mathrm{L}},R_{\mathrm{L}})\right)=\frac{1}{Vol(\mathcal{G})}\left(\frac{-i}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}/2}\times$$

$$\times\int_{\overline{\mathrm{A}}\times\varepsilon\mathrm{LO}_{\alpha_1}\times\ldots\times\varepsilon\mathrm{LO}_{\alpha_{\mathrm{L}}}}\overline{\mathfrak{D}A}\,\mathfrak{D}U_1\ldots\mathfrak{D}U_{\mathrm{L}}\exp\left[\frac{i}{2\varepsilon}S(A,U_1,\ldots,U_{\mathrm{L}})\right],\quad (2.31)$$

where

$$S(A,U_1,\ldots,U_{\mathrm{L}})=\mathcal{CS}(A)+2\varepsilon\sum_{\ell=1}^{\mathrm{L}}cs_{\alpha_\ell}\left(U_\ell;A|_C\right)-\int_M\frac{1}{\kappa\wedge d\kappa}Tr\left[(\kappa\wedge\mathcal{F}_A)^2\right],\quad (2.32)$$

with

$$\mathcal{F}_A=F_A+\varepsilon\sum_{\ell=1}^{\mathrm{L}}\left[\left(g\alpha g^{-1}\right)\delta_C\right]_\ell.\quad (2.33)$$

Thence, we can rewrite the eq. (2.32) also as follows:

$$S(A,U_1,\ldots,U_{\mathrm{L}})=\mathcal{CS}(A)+2\varepsilon\sum_{\ell=1}^{\mathrm{L}}cs_{\alpha_\ell}\left(U_\ell;A|_C\right)-\int_M\frac{1}{\kappa\wedge d\kappa}Tr\left[\left(\kappa\wedge\left(F_A+\varepsilon\sum_{\ell=1}^{\mathrm{L}}\left[\left(g\alpha g^{-1}\right)\delta_C\right]_\ell\right)\right)^2\right].\ (2.33\mathrm{b})$$

The moment map μ for the action of H on $L\mathrm{O}_\alpha$ is given up to a constant by

$$\langle\mu,(p,\phi,a)\rangle=-\oint_C\kappa Tr\left[\alpha\cdot\left(pg^{-1}\mathcal{L}_R g+g^{-1}\phi g\right)\right].\quad (2.34)$$

The shift-invariant path integral describing $Z(\varepsilon;C,R)$ in (2.28) becomes a symplectic integral over $\overline{\mathrm{A}}_\alpha=\overline{\mathrm{A}}\times\varepsilon L\mathrm{O}_\alpha$,

$$Z(\varepsilon;C,R)=\frac{1}{Vol(\mathcal{G})}\left(\frac{-i}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}/2}\int_{\overline{\mathrm{A}}_\alpha}\exp\left[\Omega_\alpha+\frac{i}{2\varepsilon}S(A,U)\right].\quad (2.35)$$

The moment map which describes the Hamiltonian action of H on the product $\overline{\mathrm{A}}_\alpha=\overline{\mathrm{A}}\times\varepsilon L\mathrm{O}_\alpha$, is the sum of the moment map for $\overline{\mathrm{A}}$ in (2.9) with ε times the moment map for $L\mathrm{O}_\alpha$ in (2.34), so that the total moment map on $\overline{\mathrm{A}}_\alpha$ is given by

$$\langle\mu,(p,\phi,a)\rangle=a-p\int_M\kappa\wedge Tr\left[\frac{1}{2}\mathcal{L}_R A\wedge A+\varepsilon\alpha\left(g^{-1}\mathcal{L}_R g\right)\delta_C\right]-\int_M\kappa\wedge Tr(\phi\mathcal{F}_A)+\int_M d\kappa\wedge Tr(\phi A),\quad (2.36)$$

where

$$\mathcal{F}_A=F_A+\varepsilon\left(g\alpha g^{-1}\right)\delta_C.\quad (2.37)$$

Again, $\mathcal{F}_A$ is the generalized curvature (2.27) appearing already in the shift-invariant action $S(A,U)$. From the description of the invariant form on the Lie algebra of H in (2.8), we see that

$$(\mu,\mu)=\int_M \kappa\wedge Tr\left[\mathcal{L}_R A\wedge A+2\varepsilon\alpha\left(g^{-1}\mathcal{L}_R g\right)\delta_C\right]-\int_M \kappa\wedge d\kappa Tr\left[\left(\frac{\kappa\wedge\mathcal{F}_A-d\kappa\wedge A}{\kappa\wedge d\kappa}\right)^2\right]. \quad (2.38)$$

To simplify (2.38), let us expand the last term therein as

$$\int_M \kappa\wedge d\kappa Tr\left[\left(\frac{\kappa\wedge\mathcal{F}_A-d\kappa\wedge A}{\kappa\wedge d\kappa}\right)^2\right]=\int_M \frac{1}{\kappa\wedge d\kappa}Tr\left[(\kappa\wedge\mathcal{F}_A)^2-2(\kappa\wedge\mathcal{F}_A)(d\kappa\wedge A)+(d\kappa\wedge A)^2\right]. \quad (2.39)$$

The term in (2.39) which is quadratic in $\mathcal{F}_A$ appears explicitly in $S(A,U)$, and as for the term linear in $\mathcal{F}_A$, we need only extract the new contribution from the Seifert loop operator,

$$-2\varepsilon\int_M \kappa\wedge\delta_C Tr\left[\left(g\alpha g^{-1}\right)\left(\frac{d\kappa\wedge A}{d\kappa\wedge\kappa}\right)\right]=-2\varepsilon\oint_C \kappa Tr\left[\alpha\cdot\left(g^{-1}\iota_R Ag\right)\right]. \quad (2.40)$$

Here we have applied the identity in (2.11). After a little bit of algebra, we thus rewrite (μ,μ) using (2.40) as

$$(\mu,\mu)=\int_M \kappa\wedge Tr\left(\mathcal{L}_R A\wedge A\right)+2\int_M \kappa\wedge Tr\left[(\iota_R A)F_A\right]-\int_M \kappa\wedge d\kappa Tr\left[(\iota_R A)^2\right]+$$
$$+2\varepsilon\oint_C \kappa Tr\left[\alpha\cdot\left(g^{-1}\mathcal{L}_R g+g^{-1}\iota_R Ag\right)\right]-\int_M \frac{1}{\kappa\wedge d\kappa}Tr\left[(\kappa\wedge\mathcal{F}_A)^2\right]. \quad (2.41)$$

At this stage, we apply our identity in (2.13) to recognize the first line in (2.41) as the Chern-Simons action $\mathcal{CS}(A)$. We also have the much more transparent identity

$$\mathcal{cs}_\alpha\left(U;A|_C\right)=\oint_C Tr\left(\alpha\cdot g^{-1}d_A g\right)=\oint_C \kappa Tr\left[\alpha\cdot\left(g^{-1}\mathcal{L}_R g+g^{-1}\iota_R Ag\right)\right]. \quad (2.42)$$

The identity in (2.42) follows immediately if we recall that the vector field R is tangent to C and satisfies $\langle\kappa,R\rangle=1$. So from (2.13), (2.41), and (2.42), we finally obtain the following result

$$(\mu,\mu)=\mathcal{CS}(A)+2\varepsilon\mathcal{cs}_\alpha\left(U;A|_C\right)-\int_M \frac{1}{\kappa\wedge d\kappa}Tr\left[(\kappa\wedge\mathcal{F}_A)^2\right]=S(A,U). \quad (2.43)$$

Consequently the Seifert loop path integral in (2.35) assumes the canonical symplectic form required for non-abelian localization,

$$Z(\varepsilon;C,R)=\frac{1}{Vol(\mathcal{G})}\left(\frac{-i}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}/2}\int_{\bar{A}_\alpha}\exp\left[\Omega_\alpha+\frac{i}{2\varepsilon}(\mu,\mu)\right]. \quad (2.44)$$

With regard $\frac{1}{2\pi}$, we have that: $\frac{1}{2\pi}=0{,}159154943\cong 0{,}360674\cdot\frac{4}{9}=0{,}160299$; or $0{,}159154943\cong 0{,}159649217$, and this values are in the columns (*Pigreco) and (*1/Pigreco) of the Table of **universal music system based on Phi**.

We let C_ℓ for $\ell = 1,\dots,\mathrm{L}$ be a set of disjoint Seifert fibers of M, each fiber labelled by an irreducible representation R_ℓ with highest weight α_ℓ. We then consider the symplectic space

$$\overline{\mathrm{A}}_\alpha = \overline{\mathrm{A}} \times \varepsilon L\mathrm{O}_{\alpha_1} \times \dots \times \varepsilon L\mathrm{O}_{\alpha_\mathrm{L}}\,, \quad (2.45)$$

with symplectic form

$$\Omega_\alpha = \Omega + \varepsilon \sum_{\ell=1}^{\mathrm{L}} \Upsilon_{\alpha_\ell}\,, \quad (2.46)$$

where $\alpha = (\alpha_1,\dots,\alpha_\mathrm{L})$ serves as a multi-index. The group $\mathrm{H} = U(1)_R \times \tilde{\mathcal{G}}_0$ now acts on $\overline{\mathrm{A}}_\alpha$ in a Hamiltonian fashion with moment map

$$\langle \mu, (p,\phi,a) \rangle = a - p\int_M \kappa \wedge Tr\left(\frac{1}{2} \mathcal{L}_R A \wedge A + \varepsilon \sum_{\ell=1}^{\mathrm{L}} \left[\alpha\left(g^{-1}\mathcal{L}_R g\right)\delta_C \right]_\ell \right) - \int_M \kappa \wedge Tr(\phi \mathcal{F}_A) + \int_M d\kappa \wedge Tr(\phi A)\,, \quad (2.47)$$

where

$$\mathcal{F}_A = F_A + \varepsilon \sum_{\ell=1}^{\mathrm{L}} \left[\left(g\alpha g^{-1} \right)\delta_C \right]_\ell\,. \quad (2.48)$$

By the same calculations leading to (2.43), the shift-invariant action $S(A,U_1,\dots,U_\mathrm{L})$ in (2.32) is precisely the square of the moment map (2.47) for the Hamiltonian action of H on $\overline{\mathrm{A}}_\alpha$. So when applied to multiple Seifert loop operators, the shift-invariant path integral in (2.31) can also be rewritten in the canonical symplectic form,

$$Z(\varepsilon;(C_1,R_1),\dots,(C_\mathrm{L},R_\mathrm{L})) = \frac{1}{Vol(\mathcal{G})}\left(\frac{-i}{2\pi\varepsilon} \right)^{\Delta_\mathcal{G}/2} \int_{\overline{\mathrm{A}}_\alpha} \exp\left[\Omega_\alpha + \frac{i}{2\varepsilon}(\mu,\mu) \right]. \quad (2.49)$$

Non-abelian localization provides a general means to study a symplectic integral of the canonical form

$$Z(\varepsilon) = \frac{1}{Vol(H)}\left(\frac{1}{2\pi\varepsilon} \right)^{\Delta_H/2} \int_X \exp\left[\Omega - \frac{1}{2\varepsilon}(\mu,\mu) \right], \quad \Delta_H = \dim H\,. \quad (2.50)$$

Here X is a symplectic manifold with symplectic form Ω, and H is a Lie group which acts on X in a Hamiltonian fashion with moment map μ. Finally, (. , .) is an invariant, positive-definite quadratic form on the Lie algebra $\mathfrak{h}$ of H and dually on $\mathfrak{h}^*$ which we use to define the "action" $S = \frac{1}{2}(\mu,\mu)$ and the volume $Vol(H)$ of H that appear in (2.50). To apply non-abelian localization to an integral of the form (2.50), we first observe that $Z(\varepsilon)$ can be rewritten as

$$Z(\varepsilon) = \frac{1}{Vol(H)} \int_{\mathfrak{h}\times X} \left[\frac{d\phi}{2\pi} \right] \exp\left[\Omega - i\langle \mu, \phi \rangle - \frac{\varepsilon}{2}(\phi,\phi) \right]. \quad (2.51)$$

Here ϕ is an element of the Lie algebra $\mathfrak{h}$ of H, and $[d\phi]$ is the Euclidean measure on $\mathfrak{h}$ that is determined by the same invariant form (. , .) which we use to define the volume $Vol(H)$ of H. The measure $[d\phi/2\pi]$ includes a factor of $1/2\pi$ for each real component of ϕ. The Gaussian integral over ϕ in (2.51) then leads immediately to the expression for Z in (2.50).
We define the local contribution to Z from the component $\mathcal{C} \subset X$ by the following symplectic integral over N,

$$Z(\varepsilon)\big|_{\mathcal{M}} = \frac{1}{Vol(H)} \int_{\mathfrak{h}\times N} \left[\frac{d\phi}{2\pi}\right] \exp\left[\Omega - i\langle \mu, \phi\rangle - \frac{\varepsilon}{2}(\phi,\phi) + sD\Psi\right]. \quad (2.52)$$

So long as s is non-zero and Ψ is given by the following equation

$$\Psi = JdS = (\mu, Jd\mu); \quad \text{in components} \quad \Psi = dx^m J_m^n \partial_n S = dx^m \mu^a J_m^n \partial_n \mu_a, \quad (2.53)$$

the integral (2.52) over the non-compact space N is both convergent and independent of s, so that $Z(\varepsilon)\big|_{\mathcal{M}}$ is well-defined. $Z(\varepsilon)\big|_{\mathcal{M}}$ in (2.52) is given by the following integral over $\mathfrak{h}_0 \times \mathcal{M}$,

$$Z(\varepsilon)\big|_{\mathcal{M}} = \frac{1}{Vol(H_0)} \int_{\mathfrak{h}_0\times\mathcal{M}} \left[\frac{d\psi}{2\pi}\right] \frac{e_{H_0}(\mathcal{M}, E_0)}{e_{H_0}(\mathcal{M}, E_1)} \exp\left[\Omega + \varepsilon\Theta - i(\gamma_0,\psi) - \frac{\varepsilon}{2}(\psi,\psi)\right]. \quad (2.54)$$

With regard the non-abelian localization formula in (2.54), let us mention two particularly simple special cases. At one extreme, we suppose that H acts freely on a neighbourhood of the vanishing locus $\mathcal{C} = \mu^{-1}(0) \subset X$ of the moment map μ. Thus H_0 is trivial, and $\gamma_0 = E_0 = E_1 = 0$. The non-abelian localization formula in this case reduces to the following integral over $\mathcal{M} = \mu^{-1}(0)/H$,

$$Z(\varepsilon)\big|_{\mathcal{M}} = \int_{\mathcal{M}} \exp[\Omega + \varepsilon\Theta]. \quad (2.55)$$

Here Θ is now the degree-four characteristic class associated to $\mu^{-1}(0)$, regarded as a principal H-bundle over $\mathcal{M}$, and determined under the Chern-Weil homomorphism by $-\frac{1}{2}(\phi,\phi)$. At the opposite extreme, we allow the stabilizer $H_0 \subset H$ to be non-trivial, but we assume that $\mathcal{M}$ is simply a point. The non-abelian localization formula for $Z\big|_{\mathcal{M}}$ in (2.54) then reduces to an integral over the Lie algebra $\mathfrak{h}_0$,

$$Z(\varepsilon)\big|_{\mathcal{M}} = \frac{1}{Vol(H_0)} \int_{\mathfrak{h}_0} \left[\frac{d\psi}{2\pi}\right] \det\left(\frac{\psi}{2\pi}\Big|_{E_0}\right) \det\left(\frac{\psi}{2\pi}\Big|_{E_1}\right)^{-1} \exp\left[-i(\gamma_0,\psi) - \frac{\varepsilon}{2}(\psi,\psi)\right]. \quad (2.56)$$

Here we have written the H_0-equivariant Euler classes in (2.54) more explicitly as determinants of $\psi \in \mathfrak{h}_0$ acting on the respective vector spaces E_0 and E_1.
Now we apply non-abelian localization to the Seifert loop path integral, which takes the canonical form (see eq. (2.44))

$$Z(\varepsilon;C,R)=\frac{1}{Vol(\mathcal{G})}\left(\frac{-i}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}}/2}\int_{\overline{\mathrm{A}}_\alpha}\exp\left[\Omega_\alpha+\frac{i}{2\varepsilon}(\mu,\mu)\right]. \quad (2.57)$$

By the general properties of the canonical symplectic integral, $Z(\varepsilon;C,R)$ localizes onto the critical points in $\overline{\mathrm{A}}_\alpha=\overline{\mathrm{A}}\times\varepsilon L\mathrm{O}_\alpha$ of the shift-invariant action

$$S(A,U)=CS(A)+2\varepsilon\oint_C Tr\left(\alpha\cdot g^{-1}d_A g\right)-\int_M\frac{1}{\kappa\wedge d\kappa}Tr\left[(\kappa\wedge\mathcal{F}_A)^2\right], \quad (2.58)$$

where

$$\mathcal{F}_A=F_A+\varepsilon\left(g\alpha g^{-1}\right)\delta_C. \quad (2.59)$$

Varying $S(A,U)$ in (2.58) with respect to A, we immediately find one classical equation of motion,

$$F_A+\varepsilon\left(g\alpha g^{-1}\right)\delta_C-\left(\frac{\kappa\wedge\mathcal{F}_A}{\kappa\wedge d\kappa}\right)d\kappa-\kappa\wedge d_A\left(\frac{\kappa\wedge\mathcal{F}_A}{\kappa\wedge d\kappa}\right)=0. \quad (2.60)$$

Varying with respect to g, we find the other equation of motion,

$$\left[\alpha, g^{-1}d_A g-\kappa g^{-1}\left(\frac{\kappa\wedge\mathcal{F}_A}{\kappa\wedge d\kappa}\right)g\right]=0. \quad (2.61)$$

To express $Z(\varepsilon;C,R)$ in a manner which makes the semi-classical interpretation of the Seifert loop operator manifest, we find it useful to introduce the quantities

$$\varepsilon_r=\frac{2\pi}{k+2};\qquad \mathrm{P}=\prod_{j=1}^{N}a_j \ \text{ if } \ N\geq 1,\ \ \mathrm{P}=1 \text{ otherwise};\qquad \theta_0=3-\frac{d}{P}+12\sum_{j=1}^{N}s\left(b_j,a_j\right). \quad (2.62)$$

Here ε_r is the renormalized coupling incorporating the shift $k\to k+2$ in the Chern-Simons level in the case $G=SU(2)$, and $s(b,a)$ is the Dedekind sum,

$$s(b,a)=\frac{1}{4a}\sum_{l=1}^{a-1}\cot\left(\frac{\pi l}{a}\right)\cot\left(\frac{\pi l b}{a}\right). \quad (2.63)$$

We also introduce the analytic functions

$$F(z)=\left(2\sinh\left(\frac{z}{2}\right)\right)^{2-N}\cdot\prod_{j=1}^{N}\left(2\sinh\left(\frac{z}{2a_j}\right)\right);\quad G^{(l)}(z)=\frac{i}{4\varepsilon_r}\left(\frac{d}{P}\right)z^2-\frac{2\pi l}{\varepsilon_r}z. \quad (2.64)$$

We introduce the character $ch_j(z)$ for the irreducible representation j of $SU(2)$ with dimension j,

$$ch_j(z)=\frac{\sinh(jz)}{\sinh(z)}=e^{(j-1)z}+e^{(j-3)z}+\ldots+e^{-(j-3)z}+e^{-(j-1)z}. \quad (2.65)$$

The Seifert loop path integral on M can then be written exactly as

$$Z(\varepsilon;C,j)=(-1)\frac{\exp\left[\frac{3\pi i}{4}-\frac{i}{4}\left(\theta_0+\left(j^2-1\right)P\right)\varepsilon_r\right]}{4\sqrt{P}}\times\left\{\sum_{l=0}^{d-1}\frac{1}{2\pi i}\int_{\mathcal{C}^{(l)}}dz\,ch_j\left(\frac{z}{2}\right)F(z)\exp\left[G^{(l)}(z)\right]-\right.$$
$$\left.-\sum_{t=1}^{2P-1}\mathrm{Res}\left(\frac{ch_j\left(\frac{z}{2}\right)F(z)\exp\left[G^{(0)}(z)\right]}{1-\exp\left(-\frac{2\pi}{\varepsilon_r}z\right)}\right)\Bigg|_{z=2\pi it}-\sum_{l=1}^{d-1}\sum_{t=1}^{\left[\frac{2Pl}{d}\right]}\mathrm{Res}\left(ch_j\left(\frac{z}{2}\right)F(z)\exp\left[G^{(l)}(z)\right]\right)\Big|_{z=-2\pi it}\right\}. \quad (2.66)$$

Here $\mathcal{C}^{(l)}$ for $l=0,...,d-1$ denote a set of contours in the complex plane over which we evaluate the integrals in the first line of (2.66). In particular, $\mathcal{C}^{(0)}$ is the diagonal line contour through the origin,

$$\mathcal{C}^{(0)}=e^{\frac{i\pi}{4}}\times\mathrm{R}, \quad (2.67)$$

and the other contours $\mathcal{C}^{(l)}$ for $l>0$ are diagonal line contours parallel to $\mathcal{C}^{(0)}$ running through the stationary phase point of the integrand, given by $z=-4\pi il(P/d)$. Also, "Res" denotes the residue of the given analytic function evaluated at the given point.

We see from the formula (2.66) that $Z(\varepsilon;C,j)$ has exactly the same structure as $Z(\varepsilon)$ even when j is non-trivial. Again, $Z(\varepsilon;C,j)$ appears as a sum of terms associated to each component in the moduli space $\mathcal{M}$, and the Seifert loop operator is universally described on each component by the character ch_j. Of the terms in (2.66), the contour integral for $l=0$ represents the contribution from the trivial connection, which is given explicitly by

$$Z(\varepsilon;C,j)\big|_{\{0\}}=(-1)\frac{\exp\left[\frac{3\pi i}{4}-\frac{i}{4}\left(\theta_0+\left(j^2-1\right)P\right)\varepsilon_r\right]}{4\sqrt{P}}\times\frac{1}{2\pi i}\int_{\mathcal{C}^{(0)}}dz\,ch_j\left(\frac{z}{2}\right)\exp\left[\frac{i}{4\varepsilon_r}\left(\frac{d}{P}\right)z^2\right]$$
$$\left(2\sinh\left(\frac{z}{2}\right)\right)^{2-N}\cdot\prod_{j=1}^{N}\left(2\sinh\left(\frac{z}{2a_j}\right)\right). \quad (2.68)$$

To gain a bit more insight into the empirical formula (2.66) for $Z(\varepsilon;C,j)$, let us again specialize to the case of torus knots $\mathcal{K}_{p,q}$ in S^3. With the Seifert invariants given from the following expressions:

$$h=0, \qquad a_1=\mathbf{p}, \qquad a_2=\mathbf{q}. \quad (2.69);$$

$$n=-1, \qquad b_1=\mathbf{p}-\mathbf{r}, \qquad b_2=\mathbf{s}. \quad (2.70),$$

the formula for $Z(\varepsilon;C,j)$ becomes

$$Z(\varepsilon;\mathcal{K}_{p,q},j)=(-1)\frac{\exp\left[\frac{3\pi i}{4}-\frac{i}{4}\left(\frac{p}{q}+\frac{q}{p}+pq(j^2-1)\right)\varepsilon_r\right]}{\sqrt{pq}}\times\left\{\frac{1}{2\pi i}\int_{\mathcal{C}^{(0)}}dz ch_j\left(\frac{z}{2}\right)\sinh\left(\frac{z}{2p}\right)\sinh\left(\frac{z}{2q}\right)\right.$$

$$\left.\exp\left[\frac{i}{4\varepsilon_r}\left(\frac{1}{pq}\right)z^2\right]+\left(\frac{j}{k+2}\right)\sum_{t=1}^{2pq-1}(-1)^{t(j+1)}\sin\left(\frac{\pi t}{p}\right)\sin\left(\frac{\pi t}{q}\right)\exp\left(\frac{-i\pi(k+2)}{2pq}t^2\right)\right\}. \quad (2.71)$$

In passing from (2.66) to (2.71), we have explicitly evaluated the phase θ_0 in (2.62) for the Seifert presentation of S^3 with fiber $\mathcal{K}_{p,q}$. Here we use two arithmetic properties of the Dedekind sum $s(.,.)$ that enters θ_0. First, as follows more or less directly from the definition (2.63),

$$s(\mathbf{p}-\mathbf{r},\mathbf{p}) = s(\mathbf{q},\mathbf{p}), \qquad s(\mathbf{s},\mathbf{q}) = s(\mathbf{p},\mathbf{q}), \qquad \mathbf{ps}-\mathbf{qr}=1. \quad (2.72)$$

Much more non-trivially, we also use Dedekind reciprocity, which states that

$$12\mathbf{pq}[s(\mathbf{p},\mathbf{q}) + s(\mathbf{q},\mathbf{p})] = \mathbf{p}^2+\mathbf{q}^2-3\mathbf{pq}+1, \quad \gcd(\mathbf{p},\mathbf{q})=1. \quad (2.73)$$

Together, we apply (2.72) and (2.73) to compute θ_0 as

$$\theta_0 = 3-\frac{1}{pq}+12\,[s(\mathbf{p}-\mathbf{r},\mathbf{p}) + s(\mathbf{s},\mathbf{q})] = \frac{p}{q}+\frac{q}{p}. \quad (2.74)$$

We have also evaluated the residues appearing in the empirical formula for $Z(\varepsilon;C,R)$. These residues appear in the sum over t in (2.71), in terms of which we decompose $Z(\varepsilon;\mathcal{K}_{p,q},j)$ as

$$Z(\varepsilon;\mathcal{K}_{p,q},j)=Z(\varepsilon;\mathcal{K}_{p,q},j)\big|_{\{0\}}+Z(\varepsilon;\mathcal{K}_{p,q},j)_{res}, \quad (2.75)$$

where

$$Z(\varepsilon;\mathcal{K}_{p,q},j)\big|_{\{0\}}=(-1)\frac{\exp\left[\frac{3\pi i}{4}-\frac{i}{4}\left(\frac{p}{q}+\frac{q}{p}+pq(j^2-1)\right)\varepsilon_r\right]}{\sqrt{pq}}\times\frac{1}{2\pi i}\int_{\mathcal{C}^{(0)}}dz ch_j\left(\frac{z}{2}\right)\sinh\left(\frac{z}{2p}\right)$$

$$\sinh\left(\frac{z}{2q}\right)\exp\left[\frac{i}{4\varepsilon_r}\left(\frac{1}{pq}\right)z^2\right], \quad (2.76)$$

and

$$Z(\varepsilon;\mathcal{K}_{p,q},j)_{res}=(-1)\frac{\exp\left[\frac{3\pi i}{4}-\frac{i}{4}\left(\frac{p}{q}+\frac{q}{p}+pq(j^2-1)\right)\varepsilon_r\right]}{\sqrt{pq}}\times$$

$$\times\left\{\left(\frac{j}{k+2}\right)\sum_{t=1}^{2pq-1}(-1)^{t(j+1)}\sin\left(\frac{\pi t}{p}\right)\sin\left(\frac{\pi t}{q}\right)\exp\left(\frac{-i\pi(k+2)}{2pq}t^2\right)\right\}. \quad (2.77)$$

We have that $\frac{3\pi}{4} = 2{,}35619 \cong 2{,}35350$, value inserted in the column (*1/1,375) concerning the Table of the **universal music system based on Phi**.

As we have already mentioned, $Z(\varepsilon; \mathcal{K}_{p,q}, j)|_{\{0\}}$ can be naturally interpreted as the contribution to the Seifert loop path integral from the reducible point $\{\rho_{ab}\}$ in the extended moduli space $\mathcal{M}(\mathcal{K}_{p,q}, j)$. Equivalently $Z(\varepsilon; \mathcal{K}_{p,q}, j)|_{\{0\}}$ is the contribution from the trivial connection $\{0\}$ on S^3. But in the relevant semi-classical limit, for which $\varepsilon \to 0$ with **j** fixed, $\{\rho_{ab}\}$ is indeed the only point in $\mathcal{M}(\mathcal{K}_{p,q}, j)$. Hence our localization result for the Seifert loop path integral implies that the additional, oscillatory Gaussian sum in (2.77) must actually vanish,

$$Z(\varepsilon; \mathcal{K}_{p,q}, j)_{res} = 0, \quad \gcd(\mathbf{p}, \mathbf{q}) = 1. \quad (2.78)$$

After applying the vanishing result (2.78), we obtain a compact formula for the expectation value of an arbitrary Wilson loop operator wrapping the torus knot $\mathcal{K}_{p,q}$ in S^3 and decorated with the irreducible $SU(2)$ representation **j**,

$$Z(\varepsilon; \mathcal{K}_{p,q}, j) = Z(\varepsilon; \mathcal{K}_{p,q}, j)|_{\{0\}} = \frac{1}{2\pi i}\frac{1}{\sqrt{pq}} \exp\left[-\frac{i\pi}{2(k+2)}\left(\frac{p}{q} + \frac{q}{p} + pq(j^2 - 1)\right)\right] \times$$
$$\times \int_R dx\, ch_j\left(e^{\frac{i\pi}{4}}\frac{x}{2}\right) \sinh\left(e^{\frac{i\pi}{4}}\frac{x}{2p}\right) \sinh\left(e^{\frac{i\pi}{4}}\frac{x}{2q}\right) \exp\left[-\frac{(k+2)}{8\pi}\left(\frac{x^2}{pq}\right)\right]. \quad (2.79)$$

With regard $\frac{\pi}{2(k+2)}$ and $\frac{k+2}{8\pi}$ for $k = 1$, we have that $\frac{\pi}{2(1+2)} = \frac{\pi}{6} = 0{,}52359 \cong 0{,}52520$; $\frac{3}{8\pi} = 0{,}119366 \cong 0{,}118393$. Bothe the values are inserted in the column (*1,375) of the Table regarding the **universal music system based on Phi**. Furthermore, we have that the eq. (2.79) can be related with the Ramanujan modular equation concerning the superstrings and with the Palumbo-Nardelli model equation, thence:

$$Z(\varepsilon; \mathcal{K}_{p,q}, j) = Z(\varepsilon; \mathcal{K}_{p,q}, j)|_{\{0\}} = \frac{1}{2\pi i}\frac{1}{\sqrt{pq}} \exp\left[-\frac{i\pi}{2(k+2)}\left(\frac{p}{q} + \frac{q}{p} + pq(j^2 - 1)\right)\right] \times$$
$$\times \int_R dx\, ch_j\left(e^{\frac{i\pi}{4}}\frac{x}{2}\right) \sinh\left(e^{\frac{i\pi}{4}}\frac{x}{2p}\right) \sinh\left(e^{\frac{i\pi}{4}}\frac{x}{2q}\right) \exp\left[-\frac{(k+2)}{8\pi}\left(\frac{x^2}{pq}\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] \quad (2.79b)$$

In writing (2.79), we have rotated the contour $\mathcal{C}^{(0)}=e^{\frac{i\pi}{4}}\times R$ to the real axis and substituted $\varepsilon_r=2\pi/(k+2)$, so that (2.79) appears as a simple generalization of the corresponding formula (2.5) for the unknot $\mathrm{O}=\mathcal{K}_{1,1}$.

We now apply the non-abelian localization formula in (2.54) to the Seifert loop path integral. Because $\{\rho_{ab}\}\cong \mathrm{O}_\alpha/G$ is a point, the path integral immediately reduces via (2.56) to an integral over the finite-dimensional Lie algebra $\mathfrak{h}_0^\alpha=R\oplus\mathfrak{g}_\alpha\oplus R$ of the stabilizer H_0^α,

$$Z(\varepsilon;C,R)\Big|_{\mathrm{O}_\alpha/G}=\frac{(2\pi\varepsilon)}{Vol(G_\alpha)}\int_{\mathfrak{h}_0^\alpha}\left[\frac{d\psi}{2\pi}\right]\det\left(\frac{\psi}{2\pi}\Big|_{\varsigma_0^\alpha}\right)\det\left(\frac{\psi}{2\pi}\Big|_{\varsigma_1^\alpha}\right)^{-1}\times\exp\left[-i(\gamma_0,\psi)-\frac{i\varepsilon}{2}(\psi,\psi)\right]. \quad (2.80)$$

Here ψ is an element in the algebra $\mathfrak{h}_0^\alpha$. Because the group $H_0^\alpha=U(1)_R\times G_\alpha\times U(1)_Z$ decomposes as a product, we frequently write ψ in terms of components

$$\psi=(p,\phi,a)\in R\oplus\mathfrak{g}_\alpha\oplus R, \quad (2.81)$$

where p and a generate $U(1)_R$ and $U(1)_Z$ respectively, and ϕ is an element of $\mathfrak{g}_\alpha$.

In arriving at the expression for $Z(\varepsilon;C,R)\big|_{\mathrm{O}_\alpha/G}$ in (2.80), we have multiplied the result obtained directly from (2.56) by

$$Vol[U(1)_R]\cdot Vol[U(1)_Z]\cdot 2\pi\varepsilon, \quad (2.82)$$

which accounts for the prefactor involving ε in (2.80). By definition, $\gamma_0\in\mathfrak{h}_0^\alpha$ is the dual of the value of the moment map μ evaluated at the point $\alpha\in\mathrm{O}_\alpha$. According to (2.36), μ is generally given on $\overline{\mathrm{A}}_\alpha$ by

$$\langle\mu,(p,\phi,a)\rangle=a-p\int_M\kappa\wedge Tr\left[\frac{1}{2}\mathcal{L}_R A\wedge A+\varepsilon\alpha\left(g^{-1}\mathcal{L}_R g\right)\delta_C\right]-\int_M\kappa\wedge Tr(\phi\mathcal{F}_A)+\int_M d\kappa\wedge Tr(\phi A). \quad (2.83)$$

Points in O_α correspond to classical configurations of (A,U) which are annihilated by $\mathcal{L}_R$ and satisfy $\mathcal{F}_A=0$, so only the first and last terms in (2.83) contribute when μ is evaluated at points in O_α. We compute directly the last term in (2.83) to be

$$\int_M d\kappa \wedge Tr(\phi A) = \int_M \kappa \wedge Tr(\phi F_A) = -\varepsilon \int_M \kappa \wedge \delta_C Tr(\alpha\phi) = -\varepsilon Tr(\alpha\phi) = \varepsilon\langle \alpha, \phi \rangle . \quad (2.84)$$

From (2.83) and (2.84) we thereby obtain

$$(\gamma_0, \psi) = \langle \mu, (p, \phi, a) \rangle \big|_{\alpha \in \mathrm{O}_\alpha} = a + \varepsilon \langle \alpha, \phi \rangle . \quad (2.85)$$

We also recall from (2.8) that the norm of ψ is given by

$$(\psi, \psi) = -\int_M \kappa \wedge d\kappa Tr(\phi^2) - 2pa = -\frac{d}{P} Tr(\phi^2) - 2pa . \quad (2.86)$$

In passing to the second line of (2.86), we use the description of $d\kappa$ in the following expression

$$d\kappa = \left(n + \sum_{j=1}^{N} \frac{b_j}{a_j} \right) \pi^* \hat{\omega} , \quad (2.86b)$$

along with the identity in the following expression

$$n + \sum_{j=1}^{N} \frac{b_j}{a_j} = \pm \prod_{j=1}^{N} \frac{1}{a_j} , \quad (2.86c)$$

to compute the integral $\int_M \kappa \wedge d\kappa = d / P$, where d is defined as

$$d = |H_1(M)| \quad (2.86d)$$

and P is defined in (2.62). Via (2.85) and (2.86), the integral over $\mathfrak{h}_0^\alpha$ then takes the more explicit form

$$Z(\varepsilon; C, R)\big|_{\mathrm{O}_\alpha / G} = \frac{(2\pi\varepsilon)}{Vol(G_\alpha)} \int_{R \times \mathfrak{g}_\alpha \times R} \left[\frac{dp}{2\pi} \right] \left[\frac{d\phi}{2\pi} \right] \left[\frac{da}{2\pi} \right] \det\left(\frac{\psi}{2\pi}\Big|_{\xi_0^\alpha} \right) \det\left(\frac{\psi}{2\pi}\Big|_{\xi_1^\alpha} \right)^{-1} \times$$
$$\times \exp\left[-ia - i\varepsilon\langle \alpha, \phi \rangle + \frac{i\varepsilon}{2}\left(\frac{d}{P} \right) Tr(\phi^2) + i\varepsilon pa \right] . \quad (2.87)$$

The vector bundles ξ_0^α and ξ_1^α both decompose into summands associated to the respective factors in the product $\overline{\mathrm{A}}_\alpha = \overline{\mathrm{A}} \times \varepsilon L\mathrm{O}_\alpha$, so that

$$\xi_0^\alpha = \xi_0 \oplus \mathfrak{g}^{(1,0)} , \quad \xi_1^\alpha = \xi_1 \oplus \overline{\mathcal{N}}_\alpha . \quad (2.88)$$

Consequently, in any regularization, we can factorize the ratio of determinants appearing in (2.87) as

$$\det\left(\frac{\psi}{2\pi}\Big|_{\xi_0^\alpha} \right) \cdot \det\left(\frac{\psi}{2\pi}\Big|_{\xi_1^\alpha} \right)^{-1} = e(\overline{\mathrm{A}}) \cdot e(L\mathrm{O}_\alpha) , \quad (2.89)$$

where we introduce the separate ratios

$$e(\overline{\mathrm{A}}) = \det\left(\frac{\psi}{2\pi}\Big|_{\xi_0}\right)\det\left(\frac{\psi}{2\pi}\Big|_{\xi_1}\right)^{-1}, \quad e(L\mathrm{O}_\alpha) = \det\left(\frac{\psi}{2\pi}\Big|_{\mathfrak{g}^{(1,0)}}\right)\det\left(\frac{\psi}{2\pi}\Big|_{\overline{\mathcal{N}}_\alpha}\right)^{-1}. \quad (2.90)$$

The integral in (2.87) immediately becomes

$$Z(\varepsilon;C,R)\big|_{\mathrm{O}_\alpha/G} = \frac{(2\pi\varepsilon)}{Vol(G_\alpha)}\int_{R\times\mathfrak{g}_\alpha\times R}\left[\frac{dp}{2\pi}\right]\left[\frac{d\phi}{2\pi}\right]\left[\frac{da}{2\pi}\right] e(\overline{\mathrm{A}})\cdot e(L\mathrm{O}_\alpha)\times$$
$$\times\exp\left[-ia - i\varepsilon\langle\alpha,\phi\rangle + \frac{i\varepsilon}{2}\left(\frac{d}{P}\right)Tr(\phi^2) + i\varepsilon pa\right]. \quad (2.91)$$

The essence of localization on O_α/G now lies in evaluating $e(\overline{\mathrm{A}})$ and $e(L\mathrm{O}_\alpha)$. Because the central generator a of $U(1)_Z$ acts trivially, $e(\overline{\mathrm{A}})$ depends only on the generators (p,ϕ) of $U(1)_R \times G_\alpha$ and is given by

$$e(\overline{\mathrm{A}}) = \exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\cdot\frac{(2\pi)^{\Delta_G}}{\left(p\sqrt{P}\right)^{\Delta_T}}\times\exp\left[\frac{i\breve{c}_{\mathfrak{g}}}{4\pi p^2}\left(\frac{d}{P}\right)Tr(\phi^2)\right]\prod_{\beta>0}\langle\beta,\phi\rangle^{-2}\left[2\sin\left(\frac{\langle\beta,\phi\rangle}{2p}\right)\right]^{2-N}$$
$$\prod_{j=1}^{N}\left[2\sin\left(\frac{\langle\beta,\phi\rangle}{2a_j p}\right)\right], \qquad \Delta_G = \dim G, \quad \Delta_T = \dim T. \quad (2.92)$$

Thence, we can rewrite the eq. (2.91) also as follows:

$$Z(\varepsilon;C,R)\big|_{\mathrm{O}_\alpha/G} = \frac{(2\pi\varepsilon)}{Vol(G_\alpha)}\int_{R\times\mathfrak{g}_\alpha\times R}\left[\frac{dp}{2\pi}\right]\left[\frac{d\phi}{2\pi}\right]\left[\frac{da}{2\pi}\right]\cdot$$
$$\exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\cdot\frac{(2\pi)^{\Delta_G}}{\left(p\sqrt{P}\right)^{\Delta_T}}\times\exp\left[\frac{i\breve{c}_{\mathfrak{g}}}{4\pi p^2}\left(\frac{d}{P}\right)Tr(\phi^2)\right]\prod_{\beta>0}\langle\beta,\phi\rangle^{-2}\left[2\sin\left(\frac{\langle\beta,\phi\rangle}{2p}\right)\right]^{2-N}$$
$$\prod_{j=1}^{N}\left[2\sin\left(\frac{\langle\beta,\phi\rangle}{2a_j p}\right)\right]\cdot e(L\mathrm{O}_\alpha)\times\exp\left[-ia - i\varepsilon\langle\alpha,\phi\rangle + \frac{i\varepsilon}{2}\left(\frac{d}{P}\right)Tr(\phi^2) + i\varepsilon pa\right]. \quad (2.92b)$$

We have, with regard the eq. (2.92), that $\frac{1}{4\pi} = 0{,}079577 \cong 0{,}078929$, and $\frac{\pi}{2} = 1{,}5707963 \cong 1{,}57560$ and both the values are inserted in the column (*1,375) and 0,079577 is very near to value 0,07801 that is in the column (*2,71828…) of the Table regarding the **universal music system based on Phi**.

Here we recall that $T \subset G$ is a maximal torus, and in writing this formula for $e(\overline{\mathrm{A}})$, we assume without loss than ϕ lies in the associated Cartan subalgebra $\mathfrak{t}$. Each $\beta > 0$ is then a positive root of G, and $\langle\cdot,\cdot\rangle$ is the canonical dual paring.

The product of determinants associated to the free loopspace $L\mathrm{O}_\alpha$, is

$$e(L\mathrm{O}_\alpha) = \det\left(\frac{\psi}{2\pi}\Big|_{\mathfrak{g}^{(1,0)}}\right)\det\left(\frac{\psi}{2\pi}\Big|_{\mathcal{N}_\alpha}\right)^{-1}. \quad (2.93)$$

Eventually these determinants, along with the moment map on O_α which enters the argument of the exponential in (2.91), will determine the invariant function of ϕ which represents the Seifert loop operator under localization at the trivial connection on M.
Taking the ratio between the determinants in the following expressions

$$\det\left(\frac{[\phi,\cdot]}{2\pi}\Big|_{\mathfrak{g}^{(1,0)}}\right) = \prod_{(\beta_+,\alpha)>0}\left(\frac{i}{2\pi}\langle\beta_+,\phi\rangle\right) = \left(\frac{i}{2\pi}\right)^{(\Delta_G-\Delta_{G_\alpha})/2} \cdot \prod_{(\beta_+,\alpha)>0}\langle\beta_+,\phi\rangle, \quad \Delta_G = \dim G \quad \Delta_{G_\alpha} = \dim G_\alpha, \quad (2.94)$$

$$\det\left(\frac{\psi}{2\pi}\Big|_{\mathcal{N}_\alpha}\right) = (2\pi)^{(\Delta_G-\Delta_{G_\alpha})/2}\exp\left(i\frac{\langle\rho^{[\alpha]},\phi\rangle}{p}\right)\prod_{(\beta_+,\alpha)>0}\frac{2}{\langle\beta_+,\phi\rangle}\sin\left(\frac{\langle\beta_+,\phi\rangle}{2p}\right), \quad (2.95)$$

we see that $e(L\mathrm{O}_\alpha)$ is given by

$$e(L\mathrm{O}_\alpha) = \left(\frac{1}{2\pi}\right)^{(\Delta_G-\Delta_{G_\alpha})}\exp\left[\frac{i\pi}{4}(\Delta_G-\Delta_{G_\alpha}) - i\frac{\langle\rho^{[\alpha]},\phi\rangle}{p}\right] \times \prod_{(\beta_+,\alpha)>0}(\beta_+,\phi)^2\cdot\left[2\sin\left(\frac{\langle\beta_+,\phi\rangle}{2p}\right)\right]^{-1}, \quad (2.96)$$

$$\Delta_G = \dim G, \quad \Delta_{G_\alpha} = \dim G_\alpha$$

As manifest in (2.92) and (2.96), neither $e(\overline{\mathrm{A}})$ nor $e(L\mathrm{O}_\alpha)$ depends upon the variable a which parametrizes the Lie algebra of $U(1)_Z$. Because $U(1)_Z$ acts in a completely trivial fashion on $\overline{\mathrm{A}}_\alpha$, the result could hardly have been otherwise. Yet this observation does have an important consequence. We recall from (2.91) that the local contribution from $\{\rho_{ab}\} \cong \mathrm{O}_\alpha / G$ to the Seifert loop path integral is given by

$$Z(\varepsilon;C,R)\Big|_{\mathrm{O}_\alpha/G} = \frac{(2\pi\varepsilon)}{Vol(G_\alpha)}\int_{R\times\mathfrak{g}_\alpha\times R}\left[\frac{dp}{2\pi}\right]\left[\frac{d\phi}{2\pi}\right]\left[\frac{da}{2\pi}\right]e(\overline{\mathrm{A}})\cdot e(L\mathrm{O}_\alpha)\times$$
$$\times\exp\left[-ia - i\varepsilon\langle\alpha,\phi\rangle + \frac{i\varepsilon}{2}\left(\frac{d}{P}\right)Tr(\phi^2) + i\varepsilon pa\right]. \quad (2.97)$$

Since a enters the integrand of (2.97) only linearly in the argument of the exponential, we can immediately integrate over a using the elementary identity

$$\int_{-\infty}^{+\infty} dy\exp(-ixy) = 2\pi\delta(x). \quad (2.98)$$

Hence the integral over a yields a delta-function $2\pi\delta(1-\varepsilon p)$. Next, we use the delta-function to perform the integral over p, thereby setting $p = 1/\varepsilon$. In the process, the prefactor of $2\pi\varepsilon$ which appears in the normalization of (2.97) is cancelled, and the integral over $R\oplus\mathfrak{g}_\alpha\oplus R$ reduces to an integral over $\mathfrak{g}_\alpha$ alone,

$$Z(\varepsilon;C,R)\Big|_{\mathrm{O}_\alpha/G}=\exp\left[-\frac{i\pi}{2}\left(\eta_0(0)-\frac{1}{2}\left(\Delta_G-\Delta_{G_\alpha}\right)\right)\right]\frac{1}{Vol(G_\alpha)}\left(\frac{\varepsilon}{\sqrt{P}}\right)^{\Delta_T}\times\int_{\mathfrak{g}_\alpha}[d\phi]\exp\left[-i\varepsilon\left\langle\alpha+\rho^{[\alpha]},\phi\right\rangle+\frac{i\varepsilon}{2}\right.$$

$$\left.\left(\frac{d}{P}\right)\left(1+\frac{\varepsilon\breve{c}_{\mathfrak{g}}}{2\pi}\right)Tr\left(\phi^2\right)\right]\times\prod_{\beta>0}\langle\beta,\phi\rangle^{-2}\left[2\sin\left(\frac{\varepsilon\langle\beta,\phi\rangle}{2}\right)\right]^{2-N}\prod_{j=1}^{N}\left[2\sin\left(\frac{\varepsilon\langle\beta,\phi\rangle}{2a_j}\right)\right]\times$$

$$\times\prod_{(\beta_+,\alpha)>0}\langle\beta_+,\phi\rangle^2\left[2\sin\left(\frac{\varepsilon\langle\beta_+,\phi\rangle}{2}\right)\right]^{-1}. \quad (2.99)$$

Here we have substituted the expressions for $e(\overline{\mathrm{A}})$ and $e(L\mathrm{O}_\alpha)$ in (2.92) and (2.96). Also we emphasize that the products over β and β_+ in (2.99) run over distinct sets of roots whenever α is not regular.

Because the integrand of (2.99) is invariant under the adjoint action of G_α, we can apply the Weyl integral formula to reduce the integral from $\mathfrak{g}_\alpha$ to $\mathfrak{t}$. In its infinitesimal version, the Weyl integral formula generally states that if f is a function on a Lie algebra $\mathfrak{g}$ invariant under the adjoint action of a group G, then

$$\int_{\mathfrak{g}}[d\phi]f(\phi)=\frac{1}{|\mathrm{W}|}\frac{Vol(G)}{Vol(T)}\int_{\mathfrak{t}}[d\phi]\prod_{\beta>0}\langle\beta,\phi\rangle^2 f(\phi). \quad (2.100)$$

Here $|\mathrm{W}|$ is the order of the Weyl group of G, and the product over positive roots β of G appearing on the right in (2.100) is a Jacobian factor generalizing the classical van der Monde determinant.

We want to apply the Weyl integral formula (2.100) not for G but for G_α. The roots of G_α are precisely those roots $\beta_\perp$ of G orthogonal to α in the invariant metric on $\mathfrak{t}^*$, such that

$$(\beta_\perp,\alpha)=0. \quad (2.101)$$

Consequently, when we apply the Weyl integral formula to reduce the integral in (2.99) from $\mathfrak{g}_\alpha$ to $\mathfrak{t}$, the Weyl Jacobian for G_α conspires to cancel against the following product of factors in (2.99),

$$\prod_{\beta>0}\langle\beta,\phi\rangle^{-2}\cdot\prod_{(\beta_+,\alpha)>0}\langle\beta_+,\phi\rangle^2=\prod_{\beta_\perp>0}\langle\beta_\perp,\phi\rangle^{-2}, \quad (2.102)$$

implying

$$Z(\varepsilon;C,R)\Big|_{\mathrm{O}_\alpha/G}=\exp\left[-\frac{i\pi}{2}\left(\eta_0(0)-\frac{1}{2}\left(\Delta_G-\Delta_{G_\alpha}\right)\right)\right]\frac{1}{|\mathrm{W}_\alpha|}\frac{1}{Vol(T)}\left(\frac{1}{\sqrt{P}}\right)^{\Delta_T}\times\int_{\mathfrak{t}}[d\phi]\exp\left[-i\left\langle\alpha+\rho^{[\alpha]},\phi\right\rangle+\frac{i}{2\varepsilon_r}\right.$$

$$\left.\left(\frac{d}{P}\right)Tr\left(\phi^2\right)\right]\times\prod_{\beta>0}\left[2\sin\left(\frac{\langle\beta,\phi\rangle}{2}\right)\right]^{2-N}\prod_{j=1}^{N}\left[2\sin\left(\frac{\varepsilon\langle\beta,\phi\rangle}{2a_j}\right)\right]\times\prod_{(\beta_+,\alpha)>0}\left[2\sin\left(\frac{\langle\beta_+,\phi\rangle}{2}\right)\right]^{-1}. \quad (2.103)$$

In passing to (2.103), we have performed a change of variables $\phi\mapsto\varepsilon\phi$ to remove extraneous factors of ε. In the process, we introduce the renormalized coupling ε_r,

$$\varepsilon_r = \frac{2\pi}{k + \breve{c}_{\mathfrak{g}}}, \quad (2.104)$$

to absorb the explicit shift in the coefficient of $Tr(\phi^2)$ that arises from $e(\overline{\mathrm{A}})$ in (2.92). Also, as hopefully clear, $|\mathsf{M}_\alpha|$ denotes the order of the Weyl group of G_α. If $G_\alpha = T$ is abelian, then M_α is trivial and $|\mathsf{M}_\alpha| = 1$. We now make two further substitutions to relate the formula in (2.103) to the empirical result in (2.68). First, we rotate the contour of integration from $\boldsymbol{t} = \boldsymbol{t} \times R$ to $\boldsymbol{t} \times e^{-\frac{i\pi}{4}}$. Second, we make a change of variables $\phi \mapsto i\phi$. Hence

$$Z(\varepsilon; C, R)\Big|_{\mathrm{O}_\alpha / G} = \exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\frac{1}{|\mathsf{M}_\alpha|}\frac{(-1)^{(\Delta_{G_\alpha} - \Delta_T)/2}}{Vol(T)}\left(\frac{1}{i\sqrt{P}}\right)^{\Delta_T} \times \int_{\boldsymbol{t}\times \mathfrak{e}^{(0)}} [d\phi] \exp\left[-\langle \alpha + \rho^{[\alpha]}, \phi\rangle - \frac{i}{2\varepsilon_r}\right.$$
$$\left.\left(\frac{d}{P}\right) Tr(\phi^2)\right] \times \prod_{\beta > 0}\left[2\sinh\left(\frac{\langle \beta, \phi\rangle}{2}\right)\right]^{2-N} \prod_{j=1}^{N}\left[2\sinh\left(\frac{\langle \beta, \phi\rangle}{2a_j}\right)\right] \times \prod_{(\beta_+, \alpha) > 0}\left[2\sinh\left(\frac{\langle \beta_+, \phi\rangle}{2}\right)\right]^{-1}. \quad (2.105)$$

Let us now interpret our result (2.105) for $Z(\varepsilon; C, R)\big|_{\mathrm{O}_\alpha / G}$ in light of the character formula. This interpretation is slightly more straightforward when α is a regular weight of G, so we specialize to the regular case first. When α is regular, $G_\alpha = T$, $|\mathsf{M}_\alpha| = 1$, and $\rho^{[\alpha]}$ reduces to the Weyl vector ρ itself. Also, the product over roots β_+ satisfying $(\beta_+, \alpha) > 0$ in (2.105) is simply the product over all positive roots $\beta > 0$ of G. As a result, the final factor in the integrand of (2.105) reduces to the Weyl denominator A_ρ,

$$A_\rho(\phi) = \prod_{\beta > 0} 2\sinh\left(\frac{\langle \beta, \phi\rangle}{2}\right). \quad (2.106)$$

Thus for regular weights,

$$Z(\varepsilon; C, R)\Big|_{\mathrm{O}_\alpha / G} = \exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\frac{1}{Vol(T)}\left(\frac{1}{i\sqrt{P}}\right)^{\Delta_T} \times \int_{\boldsymbol{t}\times \mathfrak{e}^{(0)}} [d\phi]\frac{1}{A_\rho(\phi)} \exp\left[-\langle \alpha + \rho, \phi\rangle - \frac{i}{2\varepsilon_r}\right.$$
$$\left.\left(\frac{d}{P}\right) Tr(\phi^2)\right] \times \prod_{\beta > 0}\left[2\sinh\left(\frac{\langle \beta, \phi\rangle}{2}\right)\right]^{2-N} \prod_{j=1}^{N}\left[2\sinh\left(\frac{\langle \beta, \phi\rangle}{2a_j}\right)\right], \quad \alpha \text{ regular}. \quad (2.107)$$

The measure $[d\phi]$ in the contour integral is invariant under the Weyl group M of G. Moreover, the integrand of (2.107) can generally be decomposed as a sum of terms, each of which transforms in a one-dimensional representation of M. Then since $[d\phi]$ is Weyl invariant, only the Weyl invariant piece of the integrand actually contributes to the integral over ϕ.
Since M is generated by reflections in the root lattice of G, the expression in the last line of (2.107) is also Weyl invariant, as it arises from a product over all positive roots $\beta > 0$ of the even function

$$F_\beta(\phi) = \left[2\sinh\left(\frac{\langle \beta, \phi\rangle}{2}\right)\right]^{2-N} \prod_{j=1}^{N}\left[2\sinh\left(\frac{\langle \beta, \phi\rangle}{2a_j}\right)\right], \quad F_\beta(\phi) = F_\beta(-\phi) = F_{-\beta}(\phi). \quad (2.108)$$

So in the integrand of (2.107), we are left to consider the factor

$$S_\alpha(\phi) = \frac{e^{-\langle \alpha+\rho,\phi\rangle}}{A_\rho(\phi)}. \quad (2.109)$$

By construction, the Weyl denominator $A_\rho(\phi)$ is alternating under M. Therefore, only the alternating piece of the numerator $\exp[-\langle \alpha+\rho,\phi\rangle]$ in $S_\alpha(\phi)$ actually contributes to the contour integral over ϕ in (2.107). We immediately recognize that alternating piece to be

$$A\left[e^{-\langle \alpha+\rho,\phi\rangle}\right] \equiv \frac{1}{|\mathsf{M}|}\sum_{w\in\mathsf{M}}(-1)^w e^{\langle w\cdot(\alpha+\rho),-\phi\rangle} = \frac{1}{|\mathsf{M}|}A_{\alpha+\rho}(-\phi) = (-1)^{(\Delta_G-\Delta_T)/2}\cdot\frac{1}{|\mathsf{M}|}A_{\alpha+\rho}(\phi). \quad (2.110)$$

Without loss, we replace $S_\alpha(\phi)$ in the integrand of (2.107) with the Weyl-invariant function

$$S_\alpha(\phi) \overset{[\cdot]^{\mathsf{M}}}{\mapsto} \frac{(-1)^{(\Delta_G-\Delta_T)/2}}{|\mathsf{M}|}\cdot\frac{A_{\alpha+\rho}(\phi)}{A_\rho(\phi)}. \quad (2.111)$$

Via the following character formula

$$ch_R\left(e^\phi\right) = \frac{A_{\alpha+\rho}(\phi)}{A_\rho(\phi)}, \quad e^\phi\in T, \quad (2.112)$$

we finally obtain the following result for the contribution of $\{\rho_{\alpha\beta}\}\cong \mathrm{O}_\alpha/G$ to the Seifert loop path integral,

$$Z(\varepsilon,C,R)\Big|_{\mathrm{O}_\alpha/G} = \exp\left[-\frac{i\pi}{2}\eta_0(0)\right]\frac{1}{|\mathsf{M}|}\frac{(-1)^{(\Delta_G-\Delta_T)/2}}{Vol(T)}\left(\frac{1}{i\sqrt{P}}\right)^{\Delta_T}\times$$
$$\times\int_{t\times\mathfrak{e}^{(0)}}[d\phi]ch_R\left(e^\phi\right)\exp\left[-\frac{i}{2\varepsilon_r}\left(\frac{d}{P}\right)Tr\left(\phi^2\right)\right]\times\prod_{\beta>0}\left[2\sinh\left(\frac{\langle\beta,\phi\rangle}{2}\right)\right]^{2-N}\prod_{j=1}^{N}\left[2\sinh\left(\frac{\langle\beta,\phi\rangle}{2a_j}\right)\right], \quad \alpha \text{ regular.}$$
(2.113)

As claimed, all dependence on the weight α has been subsumed into the character ch_R, which represents the Seifert loop operator under localization on O_α/G.

Now we decompose the roots β of G into two sets, consisting of roots β_+ for which $(\beta_+,\alpha)\neq 0$ and roots $\beta_\perp$ for which $(\beta_\perp,\alpha)=0$, just as in (2.102). The set of roots $\beta_\perp$ is empty when α is regular, and the set of roots $\beta_\perp$ runs over all roots when α vanishes. The Weyl denominator A_ρ in (2.106) then factorizes as a product over each set,

$$A_\rho(\phi) = \left[\prod_{(\beta_+,\alpha)>0}2\sinh\left(\frac{\langle\beta_+,\phi\rangle}{2}\right)\right]\cdot\left[\prod_{\beta_\perp>0}2\sinh\left(\frac{\langle\beta_\perp,\phi\rangle}{2}\right)\right]. \quad (2.114)$$

Using (2.114), we rewrite the general contour integral in (2.105) as

$$Z(\varepsilon, C, R)\Big|_{\mathrm{O}_\alpha/G} = \exp\left[-\frac{i\pi}{2}\eta_0(0)\right]\frac{1}{|\mathsf{M}_\alpha|}\frac{(-1)^{(\Delta_G-\Delta_T)/2}}{Vol(T)}\left(\frac{1}{i\sqrt{P}}\right)^{\Delta_T}\times$$
$$\times\int_{\mathfrak{t}\times\mathfrak{e}^{(0)}}[d\phi]\frac{1}{A_\rho(\phi)}\left[\prod_{\beta_\perp>0}2\sinh\left(\frac{\langle\beta_\perp,\phi\rangle}{2}\right)\right]\exp\left[-\langle\alpha+\rho^{[\alpha]},\phi\rangle\right]\times$$
$$\exp\left[-\frac{i}{2\varepsilon_r}\left(\frac{d}{P}\right)Tr(\phi^2)\right]\times\prod_{\beta>0}\left[2\sinh\left(\frac{\langle\beta,\phi\rangle}{2}\right)\right]^{2-N}\prod_{j=1}^{N}\left[2\sinh\left(\frac{\langle\beta,\phi\rangle}{2a_j}\right)\right], \quad (2.115)$$

Once again, we wish to tease the character ch_R out of the integrand in (2.115). To do so, let us introduce the following function of ϕ,

$$B_\alpha(\phi) = e^{\langle\alpha+\rho^{[\alpha]},\phi\rangle}\cdot\prod_{\beta_\perp>0}\left[2\sinh\left(\frac{\langle\beta_\perp,\phi\rangle}{2}\right)\right], \quad (2.116)$$

in terms of which we write the factor in the second line of (2.115) as

$$S_\alpha(\phi) = \frac{e^{-\langle\alpha+\rho^{[\alpha]},\phi\rangle}}{A_\rho(\phi)}\cdot\left[\prod_{\beta_\perp>0}2\sinh\left(\frac{\langle\beta_\perp,\phi\rangle}{2}\right)\right] = (-1)^{(\Delta_{G_\alpha}-\Delta_T)/2}\cdot\frac{B_\alpha(-\phi)}{A_\rho(\phi)}. \quad (2.117)$$

Only the Weyl-invariant component of $S_\alpha(\phi)$, or equivalently the alternating component of $B_\alpha(\phi)$, contributes to the contour integral over ϕ. We have that $B_\alpha(\phi)$ satisfies an identity which extends the denominator formula in (2.106). According to this extended denominator formula, $B_\alpha(\phi)$ can be rewritten as an alternating sum over elements w' of the Weyl group M_α of the stabilizer G_α, so that

$$B_\alpha(\phi) = \sum_{w'\in\mathsf{M}_\alpha}(-1)^{w'}e^{\langle w'\cdot(\alpha+\rho),\phi\rangle}. \quad (2.118)$$

Given the identity in (2.118), the alternating component of $B_\alpha(\phi)$ is easy to evaluate. Clearly,

$$A[B_\alpha(\phi)] = \frac{1}{|\mathsf{M}|}\sum_{w\in\mathsf{M}}(-1)^w B_\alpha(w\cdot\phi) = \frac{1}{|\mathsf{M}|}\sum_{w\in\mathsf{M}}\sum_{w'\in\mathsf{M}_\alpha}(-1)^{(ww')}e^{\langle(ww')\cdot(\alpha+\rho),\phi\rangle} = \frac{1}{|\mathsf{M}|}\sum_{w'\in\mathsf{M}_\alpha}A_{\alpha+\rho}(\phi) = \frac{|\mathsf{M}_\alpha|}{|\mathsf{M}|}A_{\alpha+\rho}(\phi). \quad (2.119)$$

In complete analogy to (2.111), we apply the identity in (2.119) to symmetrize $S_\alpha(\phi)$ under M,

$$S_\alpha(\phi)\overset{[\cdot]^{\mathsf{M}}}{\longmapsto}(-1)^{(\Delta_G-\Delta_{G_\alpha})/2}\cdot\frac{|\mathsf{M}_\alpha|}{|\mathsf{M}|}\cdot\frac{A_{\alpha+\rho}(\phi)}{A_\rho(\phi)}. \quad (2.120)$$

The sign on the right in (2.120) again arises after a reflection from $-\phi$ to ϕ in the argument of $A_{\alpha+\rho}$. Via the character formula (2.112), the contour integral in (2.115) then becomes

$$Z(\varepsilon, C, R)\Big|_{\mathrm{O}_\alpha/G} = \exp\left[-\frac{i\pi}{2}\eta_0(0)\right]\frac{1}{|\mathrm{M}|}\frac{(-1)^{(\Delta_G-\Delta_T)/2}}{Vol(T)}\left(\frac{1}{i\sqrt{P}}\right)^{\Delta_T}\times$$

$$\times\int_{\mathfrak{t}\times e^{(0)}}[d\phi]ch_R(e^\phi)\exp\left[-\frac{i}{2\varepsilon_r}\left(\frac{d}{P}\right)Tr(\phi^2)\right]\times\prod_{\beta>0}\left[2\sinh\left(\frac{\langle\beta,\phi\rangle}{2}\right)\right]^{2-N}\prod_{j=1}^{N}\left[2\sinh\left(\frac{\langle\beta,\phi\rangle}{2a_j}\right)\right], \quad (2.121)$$

exactly as in the regular case (2.113). So regardless of whether α is regular or irregular, the Seifert loop operator reduces to the character ch_R under localization on $\{\rho_{\alpha\beta}\}\cong \mathrm{O}_\alpha/G$ and subsequent pushdown to the trivial connection $\{0\}$ on M.

We use the localization formula in (2.54) to reduce the Seifert loop path integral over the infinite-dimensional space $\overline{\mathrm{A}}_\alpha = \overline{\mathrm{A}}\times\varepsilon L\mathrm{O}_\alpha$ to the integral of an appropriate de Rham cohomology class $[d\mu]$ on each smooth component $\mathcal{M}_0(C,\alpha)$ of the moduli space $\mathcal{M}(C,\alpha)$. Schematically,

$$Z(\varepsilon;C,R)\big|_{\mathcal{M}_0(C,\alpha)} = \int_{\mathcal{M}_0(C,\alpha)}[d\mu], \qquad [d\mu]\in H^*(\mathcal{M}_0(C,\alpha)), \quad (2.122)$$

where the class $[d\mu]$ generally depends upon the discrete parameters (n,k,α) which specify respectively the degree of the S^1-bundle, the Chern-Simons level, and the highest weight of the irreducible representation R.

We now possess all the ingredients required to apply the non-abelian localization formula in (2.54) to compute the cohomology class $[d\mu]$ in (2.122). Immediately,

$$Z(\varepsilon;C,R)\Big|_{\mathcal{M}_0(C,\alpha)} = \frac{(2\pi\varepsilon)}{|\mathrm{Z}(G)|}\int_{\mathfrak{h}_0\times\mathcal{M}_0(C,\alpha)}\left[\frac{dp}{2\pi}\right]\left[\frac{da}{2\pi}\right]\frac{e_{H_0}(\mathcal{M}_0(C,\alpha),\xi_0^\alpha)}{e_{H_0}(\mathcal{M}_0(C,\alpha),\xi_1^\alpha)}\times\exp[\Omega_\lambda + i\varepsilon n\Theta - ia + i\varepsilon pa].$$

(2.123)

Here the prefactor involving ε arises for the same reason as the corresponding prefactor in (2.80). Otherwise, the semiclassical contribution to $Z(\varepsilon;C,R)$ from $\mathcal{M}_0(C,\alpha)$ reduces to an integral over the abelian Lie algebra $\mathfrak{h}_0\cong R\oplus R$ of the stabilizer H_0, as well as an integral over $\mathcal{M}_0(C,\alpha)$ itself. Our main task here is to evaluate the ratio of equivariant Euler classes associated to the bundles $(\xi_0^\alpha,\xi_1^\alpha)$ over $\mathcal{M}_0(C,\alpha)$. Using the multiplicative property of the Euler class and the identification in the following expression

$$\xi_1^\alpha = \xi_1\oplus\overline{\mathcal{N}}_\alpha, \quad (2.124)$$

we immediately factor the ratio in (2.123) as

$$\frac{e_{H_0}(\mathcal{M}_0(C,\alpha),\xi_0^\alpha)}{e_{H_0}(\mathcal{M}_0(C,\alpha),\xi_1^\alpha)} = q^*\left[\frac{e_{H_0}(\mathcal{M}_0,\xi_0)}{e_{H_0}(\mathcal{M}_0,\xi_1)}\right]\cdot\frac{1}{e_{H_0}(\mathcal{M}_0(C,\alpha),\overline{\mathcal{N}}_\alpha)}. \quad (2.125)$$

In obtaining (2.125), we observe that ξ_0 and ξ_1 are defined in the following expression

$$\xi_0 = \bigoplus_{t\ge1}H^0_{\bar\partial}(\Sigma, ad(P)\otimes(\mathcal{L}^t\oplus\mathcal{L}^{-t})), \qquad \xi_1 = \bigoplus_{t\ge1}H^1_{\bar\partial}(\Sigma, ad(P)\otimes(\mathcal{L}^t\oplus\mathcal{L}^{-t})), \quad (2.125b)$$

as equivariant bundles on $\mathcal{M}_0$ which pull back to $\mathcal{M}_0(C,\alpha)$, implying that the ratio of Euler classes pulls back as well.

Evaluating the ratio of equivariant Euler classes associated to ξ_0 and ξ_1 on $\mathcal{M}_0$ turns out to be fairly tricky. We have that this ratio is

$$\frac{e_{H_0}(\mathcal{M}_0,\xi_0)}{e_{H_0}(\mathcal{M}_0,\xi_1)}=\exp\left[-\frac{i\pi}{2}\eta_0(0)+\frac{\pi}{p}c_1(\mathcal{M}_0)+\frac{in\breve{c}_{\mathfrak{g}}}{2\pi p^2}\Theta\right]\cdot\prod_{j=1}^{\dim_C \mathcal{M}_0}\frac{\varpi_j}{2\sinh(\pi\varpi_j/p)}\ ,\quad \eta_0(0)=-\frac{n\Delta_G}{6}\,. \quad (2.126)$$

Here ϖ_j for $j=1,\ldots,\dim_C \mathcal{M}_0$ are the Chern roots of the complex tangent bundle of $\mathcal{M}_0$, so that

$$c(\mathcal{M}_0)=\prod_{j=1}^{\dim_C \mathcal{M}_0}(1+\varpi_j),\qquad c_1(\mathcal{M}_0)=\prod_{j=1}^{\dim_C \mathcal{M}_0}\varpi_j\,. \quad (2.127)$$

According to the general description in the following expression

$$e_{H_0}(\mathcal{M},E)=\prod_{j=1}^{\dim E}\left(\frac{i\langle\alpha_j,\psi\rangle}{2\pi}+e_j\right),\quad (2.127b)$$

the equivariant Euler class of $\overline{\mathcal{N}_\alpha}$ is given by the formal product

$$e_{H_0}(\mathcal{M}_0(C,\alpha),\overline{\mathcal{N}_\alpha})=\prod_{t\neq 0}\prod_{\beta>0}(-itp+\langle\beta,\boldsymbol{u}\rangle).\quad (2.128)$$

With regard the computation of the equivariant Euler class of the bundle $\overline{\mathcal{N}_\alpha}$, we obtain

$$e_{H_0}(\mathcal{M}_0(C,\alpha),\overline{\mathcal{N}_\alpha})=\exp\left(-\frac{2\pi\langle\rho,\boldsymbol{u}\rangle}{p}\right)\cdot\prod_{\beta>0}\frac{2}{\langle\beta,\boldsymbol{u}\rangle}\sinh\left(\frac{\pi\langle\beta,\boldsymbol{u}\rangle}{p}\right),\quad (2.129)$$

where ρ is the usual Weyl vector. Combining the formulae in (2.126) and (2.129), we see that the ratio of equivariant Euler classes in (2.125) becomes

$$\frac{e_{H_0}(\mathcal{M}_0(C,\alpha),\xi_0^\alpha)}{e_{H_0}(\mathcal{M}_0(C,\alpha),\xi_1^\alpha)}=\exp\left(-\frac{i\pi}{2}\eta_0(0)+\frac{2\pi\langle\rho,\boldsymbol{u}\rangle}{p}\right)\cdot\prod_{\beta>0}\frac{\langle\beta,\boldsymbol{u}\rangle}{2\sinh(\pi\langle\beta,\boldsymbol{u}\rangle/p)}\times$$
$$\times q^*\left[\exp\left(\frac{\pi}{p}c_1(\mathcal{M}_0)+\frac{in\breve{c}_{\mathfrak{g}}}{2\pi p^2}\Theta\right)\cdot\prod_{j=1}^{\dim_C \mathcal{M}_0}\frac{\varpi_j}{2\sinh(\pi\varpi_j/p)}\right].\quad (2.130)$$

The ratio in (2.130) depends only on the coordinate p, not a, in the Lie algebra $\mathfrak{h}_0\cong R\oplus R$. Thus we express $Z(\varepsilon;C,R)\big|_{\mathcal{M}_0(C,\alpha)}$ solely as an integral over the classical Seifert loop moduli space $\mathcal{M}_0(C,\alpha)$,

$$Z(\varepsilon;C,R)\big|_{\mathcal{M}_0(C,\alpha)}=\frac{1}{|Z(G)|}\exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\times\int_{\mathcal{M}_0(C,\alpha)}\exp(2\pi\varepsilon\langle\alpha+\rho,\boldsymbol{u}\rangle)\cdot\prod_{\beta>0}\frac{\langle\beta,\boldsymbol{u}\rangle}{2\sinh(\pi\varepsilon\langle\beta,\boldsymbol{u}\rangle)}\times$$
$$\times q^*\left[\exp\left(\Omega+\pi\varepsilon c_1(\mathcal{M}_0)+i\varepsilon n(1+\frac{\varepsilon\breve{c}_{\mathfrak{g}}}{2\pi})\Theta\right)\cdot\prod_{j=1}^{\dim_C \mathcal{M}_0}\frac{\varpi_j}{2\sinh(\pi\varepsilon\varpi_j)}\right].\quad (2.131)$$

When obtaining (2.131), we have used the symplectic decomposition

$$\Omega_\lambda = q^*\Omega + 2\pi\varepsilon\langle\alpha,\boldsymbol{u}\rangle, \qquad \lambda = \alpha/k. \quad (2.132)$$

To make the cohomological interpretation of (2.131) more transparent, let us rescale each element in the cohomology ring of $\mathcal{M}_0(C,\alpha)$ by a factor $(2\pi\varepsilon)^{-q/2}$, where q is the degree of the given class. For instance, the Chern roots ϖ_j and $\boldsymbol{u}$, each of degree two, scale by

$$\varpi_j \mapsto \frac{1}{2\pi\varepsilon}\varpi_j, \qquad \boldsymbol{u} \mapsto \frac{1}{2\pi\varepsilon}\boldsymbol{u}. \quad (2.133)$$

To preserve the value of the integral over $\mathcal{M}_0(C,\alpha)$, we simultaneously scale the integral itself by an overall factor $(2\pi\varepsilon)^d$, where $d = \dim_C \mathcal{M}_0(C,\alpha)$. After this change of variables to clear away extraneous factors of ε, $Z(\varepsilon;C,R)\big|_{\mathcal{M}_0(C,\alpha)}$ becomes

$$Z(\varepsilon;C,R)\big|_{\mathcal{M}_0(C,\alpha)} = \frac{1}{|Z(G)|}\exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\times\int_{\mathcal{M}_0(C,\alpha)}\frac{e^{\langle\alpha+\rho,\boldsymbol{u}\rangle}}{A_\rho(\boldsymbol{u})}\prod_{\beta>0}\langle\beta,\boldsymbol{u}\rangle\times$$
$$\times q^*\left[\exp\left(\frac{1}{2\pi\varepsilon}\Omega + \frac{1}{2}c_1(\mathcal{M}_0) + i\frac{n}{4\pi^2\varepsilon_r}\Theta\right)\hat{A}(\mathcal{M}_0)\right]. \quad (2.134)$$

Also here we have that $\frac{\pi}{2} = 1{,}570796 \cong 1{,}5756097$; $\frac{1}{2\pi} = 0{,}1591549 \cong 0{,}159649217$; $\frac{1}{4\pi^2} = 0{,}025330295 \cong 0{,}02562136$, values that are inserted in the columns (*1,375), (*1/Pigreco) and (*1,375) of the Table regarding the **universal music system based on Phi**.

The appearance of the $\hat{A}$-genus of the orbit $O_{-\lambda}$ in (2.134) is no accident. We have a holomorphic fibration of complex manifolds

$$2\pi O_{-\lambda} \to \underset{\substack{\downarrow_q \\ \mathcal{M}_0}}{\mathcal{M}_0(C,\alpha)}, \qquad \lambda = \frac{\alpha}{k}. \quad (2.135)$$

The fibration of $\mathcal{M}_0(C,\alpha)$ over $\mathcal{M}_0$ in (2.135) implies the relation

$$\hat{A}(\mathcal{M}_0(C,\alpha)) = \hat{A}(O_{-\lambda})\cdot\hat{A}(\mathcal{M}_0). \quad (2.136)$$

We note that the first Chern class of $\mathcal{M}_0(C,\alpha)$ is given by the sum

$$c_1(\mathcal{M}_0(C,\alpha)) = q^*c_1(\mathcal{M}_0) + c_1(O_{-\lambda}) = q^*c_1(\mathcal{M}_0) + \sum_{\beta>0}\langle\beta,\boldsymbol{u}\rangle = q^*c_1(\mathcal{M}_0) + 2\langle\rho,\boldsymbol{u}\rangle. \quad (2.137)$$

Using (2.132), (2.136), and (2.137), we then rewrite $Z(\varepsilon; C, R)\big|_{\mathcal{M}_0(C,\alpha)}$ in terms of classes defined intrinsically on $\mathcal{M}_0(C,\alpha)$,

$$Z(\varepsilon; C, R)\big|_{\mathcal{M}_0(C,\alpha)} = \frac{1}{|Z(G)|}\exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\times\int_{\mathcal{M}_0(C,\alpha)}\hat{A}(\mathcal{M}_0(C,\alpha))\cdot\exp$$
$$\left[\left(\frac{1}{2\pi\varepsilon}\Omega_\lambda + \frac{1}{2}c_1(\mathcal{M}_0(C,\alpha)) + i\frac{n}{4\pi^2\varepsilon_r}q^*\Theta\right)\right]. \quad (2.138)$$

The integral over $\mathcal{M}_0(C,\alpha)$ in (2.138) should be compared to the following expression for the partition function $Z(\varepsilon)\big|_{\mathcal{M}_0}$:

$$Z(\varepsilon)\big|_{\mathcal{M}_0} = \frac{1}{|Z(G)|}\exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\int_{\mathcal{M}_0}\hat{A}(\mathcal{M}_0)\cdot\exp\left[\frac{1}{2\pi\varepsilon}\Omega + \frac{1}{2}c_1(\mathcal{M}_0) + i\frac{n}{4\pi^2\varepsilon_r}\Theta\right]. \quad (2.139)$$

To recast the result (2.134) entirely in terms of ε_r we apply a theorem of Drezet and Narasimhan which determines $c_1(\mathcal{M}_0)$ in the case $G = SU(r+1)$ to be

$$c_1(\mathcal{M}_0) = 2(r+1)\Omega_0, \qquad \Omega_0 = \frac{1}{4\pi^2}\Omega, \quad \text{thence} \quad c_1(\mathcal{M}_0) = 2(r+1)\frac{1}{4\pi^2}\Omega. \quad (2.140)$$

Since $\breve{c}_{\mathfrak{g}} = r+1$ as well, the local contribution from $\mathcal{M}_0(C,\alpha)$ to $Z(\varepsilon; C, R)$ becomes

$$Z(\varepsilon; C, R)\big|_{\mathcal{M}_0(C,\alpha)} = \frac{1}{|Z(G)|}\exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\times\int_{\mathcal{M}_0(C,\alpha)}\frac{e^{\langle\alpha+\rho,\boldsymbol{u}\rangle}}{A_\rho(\boldsymbol{u})}\prod_{\beta>0}\langle\beta,\boldsymbol{u}\rangle$$
$$\cdot q^*\left[\exp\left(\frac{1}{2\pi\varepsilon_r}(\Omega + i\frac{n}{2\pi}\Theta)\right)\cdot\hat{A}(\mathcal{M}_0)\right], \quad (2.141)$$

and all dependence on k has been absorbed into the renormalized coupling ε_r.

Also here we have that $\frac{\pi}{2} = 1{,}570796 \cong 1{,}5756097$; $\frac{1}{2\pi} = 0{,}1591549 \cong 0{,}159649217$, values that are inserted in the columns (*1,375) and (*1/Pigreco) of the Table regarding the **universal music system based on Phi**.

According to (2.122), the integrand in (2.141) is the class $[d\mu] \in H^*(\mathcal{M}_0(C,\alpha))$ which describes the local contribution from $\mathcal{M}_0(C,\alpha)$ to the Seifert loop path integral $Z(\varepsilon; C, R)$,

$$[d\mu] = \frac{1}{|Z(G)|}\exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\times\frac{e^{\langle\alpha+\rho,\boldsymbol{u}\rangle}}{A_\rho(\boldsymbol{u})}\cdot\prod_{\beta>0}\langle\beta,\boldsymbol{u}\rangle\cdot q^*\left[\exp\left(\frac{1}{2\pi\varepsilon_r}(\Omega + i\frac{n}{2\pi}\Theta)\right)\cdot\hat{A}(\mathcal{M}_0)\right]. \quad (2.142)$$

Like the pair of expressions in (2.138) and (2.139), the integral over $\mathcal{M}_0(C,\alpha)$ in (2.141) should be compared to the corresponding localization result for the partition function $Z(\varepsilon)\big|_{\mathcal{M}_0}$,

$$Z(\varepsilon)\big|_{\mathcal{M}_0} = \frac{1}{|\mathrm{Z}(G)|}\exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\int_{\mathcal{M}_0}\hat{A}(\mathcal{M}_0)\cdot\exp\left[\frac{1}{2\pi\varepsilon_r}(\Omega + i\frac{n}{2\pi}\Theta)\right]. \quad (2.143)$$

The Seifert loop class $W_R(C)\big|_{\mathcal{M}_0}$ is then the element of $H^*(\mathcal{M}_0)$ such that the pushdown $q_*[d\mu]$ is given by the product of $W_R(C)\big|_{\mathcal{M}_0}$ with the integrand of the partition function $Z(\varepsilon)\big|_{\mathcal{M}_0}$ in (2.143), such that

$$Z(\varepsilon;C,R)\big|_{\mathcal{M}_0} = \frac{1}{|\mathrm{Z}(G)|}\exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\times\int_{\mathcal{M}_0}W_R(C)\big|_{\mathcal{M}_0}\cdot\exp\left[\frac{1}{2\pi\varepsilon_r}(\Omega + i\frac{n}{2\pi}\Theta)\right]\cdot\hat{A}(\mathcal{M}_0). \quad (2.144)$$

Comparing the Seifert integrand $[d\mu]\in H^*(\mathcal{M}_0(C,\alpha))$ in (2.142) to the preceding formula (2.144) for $Z(\varepsilon;C,R)\big|_{\mathcal{M}_0}$, we deduce

$$W_R(C)\big|_{\mathcal{M}_0} = q_*S_\alpha(\boldsymbol{u}), \qquad S_\alpha(\boldsymbol{u}) = \frac{e^{\langle\alpha+\rho,\boldsymbol{u}\rangle}}{A_\rho(\boldsymbol{u})}\cdot\prod_{\beta>0}\langle\beta,\boldsymbol{u}\rangle. \quad (2.145)$$

Thence, we have that

$$W_R(C)\big|_{\mathcal{M}_0} = q_*\frac{e^{\langle\alpha+\rho,\boldsymbol{u}\rangle}}{A_\rho(\boldsymbol{u})}\cdot\prod_{\beta>0}\langle\beta,\boldsymbol{u}\rangle. \quad (2.145b)$$

Recycling the result in the following expression

$$W_R(C)\big|_{\mathcal{N}(P)} = q_*S_\alpha(\boldsymbol{u}) = \frac{1}{|\mathrm{M}|}ch_R(\upsilon_p)\cdot\int_{\mathrm{O}_{-\lambda}}\eta = ch_R(\upsilon_p), \quad (2.145c)$$

we find the general description for the Seifert loop class,

$$W_R(C)\big|_{\mathcal{M}_0} = ch_R(\upsilon_p). \quad (2.146)$$

Equivalently,

$$Z(\varepsilon;C,R)\big|_{\mathcal{M}_0} = \frac{1}{|\mathrm{Z}(G)|}\exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\int_{\mathcal{M}_0}ch_R(\upsilon_p)\cdot\exp\left[\frac{1}{2\pi\varepsilon_r}\left(\Omega + i\frac{n}{2\pi}\Theta\right)\right]\cdot\hat{A}(\mathcal{M}_0). \quad (2.147)$$

Also here we have that $\frac{\pi}{2} = 1{,}570796 \cong 1{,}5756097$; $\frac{1}{2\pi} = 0{,}1591549 \cong 0{,}159649217$, values that are inserted in the columns (*1,375) and (*1/Pigreco) of the Table regarding the **universal music system based on Phi**.

The Todd class and the $\hat{A}$-genus of a complex manifold X generally satisfy the relation

$$Td(X) = \exp\left[\frac{1}{2}c_1(X)\right] \cdot \hat{A}(X), \quad (2.148)$$

This identity, applied to $\hat{A}(\mathcal{M}_0)$ in (2.134), implies that the contribution from $\mathcal{M}_0$ to $Z(\varepsilon; C, R)$ can be alternatively presented as

$$Z(\varepsilon; C, R)\Big|_{\mathcal{M}_0} = \frac{1}{|Z(G)|}\exp\left(-\frac{i\pi}{2}\eta_0(0)\right)\int_{\mathcal{M}_0} ch_R(\mathcal{V}_p) \cdot \exp\left[k\Omega_0 + i\frac{n}{4\pi^2\varepsilon_r}\Theta\right] \cdot Td(\mathcal{M}_0). \quad (2.149)$$

Also here we have that $\frac{\pi}{2} = 1{,}570796 \cong 1{,}5756097$; $\frac{1}{4\pi^2} = 0{,}025330295 \cong 0{,}02562136$, values that are inserted in the column (*1,375) of the Table regarding the **universal music system based on Phi**.

3. On some equations concerning the cusp anomaly and integrability from String theory. [3] [4]

The bosonic contribution to the coefficient of the Catalan constant K comes from the integral term in the following expression, concerning the bosonic sunset (“diminuzione”, “decline”)

$$W_{2Bsunset;m_x} = -\frac{1}{8\pi}I\left[\frac{2+\hat{v}^2}{4}\right]\left(2+\hat{v}^2 - 2\sqrt{1+\hat{v}^2} + 8\pi\hat{v}^2 I\left[\frac{1}{4}\left(1+\sqrt{1+\hat{v}^2}\right)^2\right]\right) + \\ + \int_0^1 du \frac{\left(1+\hat{v}^2\right)\arctan hu}{2\pi^2\left[\sqrt{1+\hat{v}^2+u^2} + \sqrt{1+\left(1+\hat{v}^2\right)u^2}\right]^2}, \quad (3.1)$$

whose small $\hat{v}$ expansion reads

$$\int_0^1 du \frac{8\left(1+\hat{v}^2\right)\arctan hu}{\left[\sqrt{1+\hat{v}^2+u^2} + \sqrt{1+\left(1+\hat{v}^2\right)u^2}\right]^2} = \left(1+\frac{1}{2}\hat{v}^2 - \frac{7}{32}\hat{v}^4 + \frac{7}{64}\hat{v}^6 - \frac{61}{1024}\hat{v}^8 + \ldots\right)K + \\ -\frac{1}{64}\hat{v}^4 + \frac{1}{128}\hat{v}^6 - \frac{11}{6144}\hat{v}^8 + \ldots \quad (3.2)$$

We note that this expression can be related with the Ramanujan modular equation concerning the superstrings and with the equation regarding the Palumbo-Nardelli model. Indeed, we have that:

$$\int_0^1 du \frac{8\left(1+\hat{\nu}^2\right)\arctan hu}{\left[\sqrt{1+\hat{\nu}^2+u^2}+\sqrt{1+\left(1+\hat{\nu}^2\right)u^2}\right]^2} = \left(1+\frac{1}{2}\hat{\nu}^2-\frac{7}{32}\hat{\nu}^4+\frac{7}{64}\hat{\nu}^6-\frac{61}{1024}\hat{\nu}^8+\ldots\right)K+$$

$$-\frac{1}{64}\hat{\nu}^4+\frac{1}{128}\hat{\nu}^6-\frac{11}{6144}\hat{\nu}^8+\ldots\Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] . \quad (3.2b)$$

Note that $1024=16\times 64$ and $6144=96\times 64$.

Similarly, the fermions contribute only through the integral $\mathcal{W}_1$ in the following expression concerning the fermionic sunset

$$W_{2Fsunset;m_x} = -\frac{\hat{\nu}^2}{8\pi}I\left[\frac{1}{4}\left(1+\hat{\nu}^2\right)\right]+\hat{\nu}^2 I\left[\frac{1}{4}\left(1+\hat{\nu}^2\right)\right]I\left[\frac{1}{4}\left(2+\hat{\nu}^2\right)\right]-\left(\hat{\nu}^2-\frac{1}{2}\right)I\left[\frac{1}{4}\left(1+\hat{\nu}^2\right)\right]^2+\mathcal{W}_1, \quad (3.3)$$

and we observe that their net effect is to simply change the sign of the coefficient of the bosonic contribution to K term in (3.2). Using the following expression

$$f_2 = \frac{\mathcal{F}_2(\ell)}{\sqrt{1+\ell^2}}+\frac{1}{2}\left(1+\ell^2\right)^{3/2}\left(\frac{df_1}{d\ell}\right)^2 \quad \text{where} \quad f_1 = \frac{\mathcal{F}_1(\ell)}{\sqrt{1+\ell^2}}, \quad (3.4)$$

to obtain $f_{2;K}$ we need to divide (3.2) by $\sqrt{1+\ell^2}$, while replacing $\hat{\nu}\to\ell$, and change the overall sign to account for the fermion contribution. Therefore, we find the following integral representation which can be expanded to any order in ℓ

$$f_{2;K}(\ell) = -\int_0^1 du \frac{8\sqrt{1+\ell^2}\arctan hu}{\left[\sqrt{1+\ell^2+u^2}+\sqrt{1+\left(1+\ell^2\right)u^2}\right]^2}\Big|_K = \left(-1+\frac{3}{32}\ell^4-\frac{3}{32}\ell^6+\frac{81}{1024}\ell^8+\ldots\right)K . \quad (3.5)$$

Also this expression can be related with the Ramanujan modular equation concerning the superstrings and with the equation regarding the Palumbo-Nardelli model. Indeed, we have that:

$$f_{2;K}(\ell) = -\int_0^1 du \frac{8\sqrt{1+\ell^2}\arctan hu}{\left[\sqrt{1+\ell^2+u^2}+\sqrt{1+(1+\ell^2)u^2}\right]^2}\Big|_K = \left(-1+\frac{3}{32}\ell^4-\frac{3}{32}\ell^6+\frac{81}{1024}\ell^8+\ldots\right)K \Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] \quad (3.5b)$$

The fact that the fermionic contribution simply changes the sign of the bosonic contribution to the coefficient of the Catalan constant was first observed for the ordinary cusp anomaly $(J=0)$.
A direct expansion of the integrand in equation (3.2) leads to divergent integrals at sufficiently high orders in $\hat{\nu}^{-1}$. A consistent expansion can be constructed by first using the identity

$$\frac{\arctan hu}{u} = \int_0^1 dy \frac{1}{1-u^2y^2} \quad (3.6)$$

to evaluate in closed form the u integral in equation (3.2). The integrand of the resulting y integral can be expanded at large $\hat{\nu}$, the integral of each term being finite. The absence of divergences indicates the consistency of this procedure. In this way we obtain

$$\int_0^1 du \frac{8(1+\hat{\nu}^2)\arctan hu}{\left[\sqrt{1+\hat{\nu}^2+u^2}+\sqrt{1+(1+\hat{\nu}^2)u^2}\right]^2} = 2+(6-\pi^2)\frac{1}{\hat{\nu}^2}+\frac{16}{3}\frac{1}{\hat{\nu}^3}+\left(4-\frac{\pi^2}{2}\right)\frac{1}{\hat{\nu}^4}-\frac{104}{45}\frac{1}{\hat{\nu}^5}+\ldots \quad (3.7)$$

Also this expression can be related with the Ramanujan modular equation concerning the superstrings and with the equation regarding the Palumbo-Nardelli model. Indeed, we have that:

$$\int_0^1 du \frac{8(1+\hat{\nu}^2)\arctan hu}{\left[\sqrt{1+\hat{\nu}^2+u^2}+\sqrt{1+(1+\hat{\nu}^2)u^2}\right]^2} = 2+(6-\pi^2)\frac{1}{\hat{\nu}^2}+\frac{16}{3}\frac{1}{\hat{\nu}^3}+\left(4-\frac{\pi^2}{2}\right)\frac{1}{\hat{\nu}^4}-\frac{104}{45}\frac{1}{\hat{\nu}^5}+\ldots$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$=\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (3.7b)$$

The asymptotic Bethe Ansatz expansion for f_2 can be written as ($g=\sqrt{\lambda}/4\pi$)

$$f_2^{ABA}=\frac{16\pi^2}{\sqrt{\ell^2+1}}\left[\frac{2g^2\partial_a\tilde{\mathcal{F}}^2(a_0)}{\sqrt{\ell^2+1}}-\frac{2g^2\tilde{\mathcal{F}}^2(a_0)}{\ell^2+1}+2g^2\delta\mathcal{F}-\left(\frac{5}{256\ell^6}+\frac{3}{64\ell^4}+\frac{1}{32\ell^2}\right)\right]. \quad (3.8)$$

Here $a_0=\sqrt{1+\ell^2}$. All the pieces in this formula can be analytically computed at large ℓ. The first terms in this expansion are

$$f_2^{ABA}=\frac{\pi^2}{3}\frac{1}{\ell^3}+\left(-\frac{32}{9}+\frac{\pi^2}{12}\right)\frac{1}{\ell^5}-\frac{232}{45}\frac{1}{\ell^6}+\frac{16}{5}\frac{1}{\ell^7}+\frac{20416}{1575}\frac{1}{\ell^8}-\left(\frac{3614}{1575}+\frac{\pi^2}{96}\right)\frac{1}{\ell^9}+\ldots \quad (3.9)$$

The only relevant contributions to the π^2 coefficient arise from the last term in parenthesis in (3.8), i.e. $-\left(\frac{5}{256\ell^6}+\frac{3}{64\ell^6}+\frac{1}{32\ell^2}\right)$ and a term in $\delta\mathcal{F}$

$$\delta\mathcal{F}=\ldots+\frac{1}{g^2}\left(\frac{5}{512\ell^6}+\frac{1}{32\ell^4}+\frac{5}{192\ell^2}\right)+\ldots \quad (3.10)$$

In the lecture of Riccardo Ricci "Cusp anomaly and integrability from string theory" (16.04.2011), from the partition function we can extract the generalized scaling:

$$f_2=-K+\ell^2\left(8\log^2\ell-6\log\ell-\frac{3}{2}\log 2+\frac{11}{4}\right)+\ell^4\left(-6\log^2\ell-\frac{7}{6}\log\ell+3\log 2\log\ell-\frac{9}{8}\log^2 2+\right.$$
$$\left.+\frac{11}{8}\log 2+\frac{3}{32}K-\frac{233}{576}\right)+O\left(\ell^6\right), \qquad (3.11)$$

that is in stupendous agreement with the Bethe-Ansatz prediction.
While, in the lecture of Benjamin Basso "Strong Coupling Expansion of Cusp Anomalous Dimension in Planar $\mathcal{N}=4$" (16.05.2008), with regard the weak coupling expansion of cusp from Beisert-Eden-Staudacher equation (BES equation), we have that the BES equation is

$$\sigma(t)=\frac{t}{e^t-1}\left(K(2gt,0)-4g^2\int_0^{+\infty}dt'K(2gt,2gt')\sigma(t')\right). \quad (3.12)$$

Thence, the solution at weak coupling is:

$$\sigma(t)=\frac{t}{e^t-1}\left[K(2gt,0)-4g^2\int_0^{\infty}dt'K(2gt,2gt')\frac{t'}{e^{t'}-1}K(2gt',0)+O\left(g^4\right)\right], \quad (3.13)$$

while the weak coupling expansion of the cusp anomaly is:

$$\Gamma_{cusp}(g) = 8g^2\sigma(0) = 4g^2 - \frac{4}{3}\pi^2 g^4 + \frac{44}{45}\pi^4 g^6 - 8\left(\frac{73}{630}\pi^6 + 4\zeta_3^2\right)g^8 +$$
$$+32\left(\frac{887}{14175}\pi^8 + \frac{4}{3}\pi^2\zeta_3^2 + 40\zeta_3\zeta_5\right)g^{10} + O\left(g^{12}\right). \quad (3.14)$$

With regard the strong coupling expansion from AdS/CFT correspondence, we have the following expression

$$\Gamma_{cusp}(g) = 2g - \frac{3\ln 2}{2\pi} + O(1/g). \quad (3.15)$$

The integral equation of the strong coupling expansion

$$s(t) = \sum_{n=1}^{M} s_n(g)\frac{J_n(2gt)}{2gt}, \quad (3.16)$$

becomes a finite-dimensional matrix equation for the coefficients $s_n(g)$. With regard the numerical result of the matrix equation and extract the cusp anomaly $\Gamma_{cusp}(g) = 4g^2 s_1(g)$, we have that

$$f(g) = 2\Gamma_{cusp}(g) = (4.000000 \pm 0.000001)g - (0.661907 \pm 0.000002) - \frac{0.0232 \pm 0.0001}{g} + \ldots \quad (3.17)$$

The first two terms are in remarkable agreement with the string theory result, and we have that

$$0.661907 = \frac{3\ln 2}{\pi}, \qquad 0.0232 = ? \qquad (3.18)$$

We can to observe that 0.0232 is very near to the following values: 0.02349 (system/2.71828) and 0.023292 (system $\times 1/\pi$) where "system" is the column where are defined the values of the musical system based on Phi (1.618033988); i.e. sum of exponents of Phi.
Thence, we can rewrite the results of (3.18) also as follows:

$$0.661907 = \frac{3\ln 2}{\pi}, \qquad 0.0232 = (\text{system} \times 1/\pi) \qquad (3.18b)$$

With regard the strong coupling expansion of cusp from BES equation, analytically the strong coupling solution was first analyzed at leading order and then in a more systematic approach, we have the following result

$$\Gamma_{cusp}(g + c_1) = 2g\left[1 - c_2 g^{-2} - c_3 g^{-3} - \left(c_4 + 2c_2^2\right)g^{-4} - \left(c_5 + 23c_2c_3\right)g^{-5} +\right.$$
$$\left. -\left(c_6 + \frac{166}{7}c_2c_4 + 54c_3^2 + 25c_2^3\right)g^{-6} + O\left(g^{-7}\right)\right], \quad (3.19)$$

where the expansion coefficients are given by

$$c_1 = \frac{3\ln 2}{4\pi}, \qquad c_2 = \frac{1}{16\pi^2} K, \qquad c_3 = \frac{27}{2^{11}\pi^3}\zeta(3),$$
$$c_4 = \frac{21}{2^{10}\pi^4}\beta(4), \qquad c_5 = \frac{43065}{2^{21}\pi^5}\zeta(5), \qquad c_6 = \frac{1605}{2^{15}\pi^6}\beta(6), \quad (3.20)$$

with the special functions

$$\zeta(x) = \sum_{n\geq 1} n^{-x} = \text{ Riemann zeta function;} \quad \beta(x) = \sum_{n\geq 0} (-1)^n (2n+1)^{-x} = \text{ Dirichlet zeta function}$$
$$K = \beta(2) = \text{ Catalan's constant.} \qquad (3.21)$$

In the Bethe ansatz approach, the cusp anomalous dimension is determined by the behaviour around the origin of the auxiliary function $\gamma(t)$ related to density of Bethe roots

$$\Gamma_{cusp}(g) = -8ig^2 \lim_{t\to 0} \gamma(t)/t. \quad (3.22)$$

The function $\gamma(t)$ depends on ‘t Hooft coupling and has the form

$$\gamma(t) = \gamma_+(t) + i\gamma_-(t), \quad (3.23)$$

where $\gamma_\pm(t)$ are real functions of t with a definite parity $\gamma_\pm(\pm t) = \pm\gamma_\pm(t)$. For arbitrary coupling the functions $\gamma_\pm(t)$ satisfy the (infinite-dimensional) system of integral equations

$$\int_0^\infty \frac{dt}{t} J_{2n-1}(t)\left[\frac{\gamma_-(t)}{1-e^{-t/(2g)}} + \frac{\gamma_+(t)}{e^{t/(2g)}-1}\right] = \frac{1}{2}\delta_{n,1}, \quad \int_0^\infty \frac{dt}{t} J_{2n}(t)\left[\frac{\gamma_+(t)}{1-e^{-t/(2g)}} - \frac{\gamma_-(t)}{e^{t/(2g)}-1}\right] = 0, \quad (3.24)$$

with $n \geq 1$ and $J_n(t)$ being the Bessel functions. These relations are equivalent to BES equation provided that $\gamma_\pm(t)$ verify certain analyticity conditions. The equations (3.24) can be significantly simplified with a help of the transformation $\gamma(t) \to \Gamma(t)$:

$$\Gamma(t) = \left(1 + i\coth\frac{t}{4g}\right)\gamma(t) \equiv \Gamma_+(t) + i\Gamma_-(t). \quad (3.25)$$

We find from (3.22) and (3.25) the following representation for the cusp anomalous dimension

$$\Gamma_{cusp}(g) = -2g\Gamma(0). \quad (3.26)$$

It follows from (3.23) and (3.24) that $\Gamma_\pm(t)$ are real functions with a definite parity, $\Gamma_\pm(-t) = \pm\Gamma_\pm(t)$, satisfying the system of integral equations

$$\int_0^\infty dt \cos(ut)[\Gamma_-(t) - \Gamma_+(t)] = 2; \qquad \int_0^\infty dt \sin(ut)[\Gamma_-(t) + \Gamma_+(t)] = 0, \quad (3.27)$$

with u being arbitrary real parameter such that $-1 \le u \le 1$. Since $\Gamma_{\pm}(t)$ take real values, we can rewrite these relations in a compact form

$$\int_0^{\infty} dt\left[e^{iut}\Gamma_{-}(t) - e^{-iut}\Gamma_{+}(t)\right] = 2 . \quad (3.28)$$

We have the following non perturbative scale $m_{O(6)}$ in the AdS/CFT. Its dependence on the coupling g follows univocally from FRS equation and it has the following form

$$m_{O(6)} = \frac{8\sqrt{2}}{\pi^2}e^{-\pi g} - \frac{8g}{\pi}e^{-\pi g}\,\mathrm{Re}\left[\int_0^{\infty}\frac{dte^{i(t-\pi/4)}}{t+i\pi g}\left(\Gamma_{+}(t)+i\Gamma_{-}(t)\right)\right], \quad (3.29)$$

where $\Gamma_{\pm}(t)$ are solutions to (3.28).

Also this expression can be related with the Ramanujan modular equation concerning the superstrings and with the equation concerning the Palumbo-Nardelli model. Indeed, we obtain:

$$m_{O(6)} = \frac{8\sqrt{2}}{\pi^2}e^{-\pi g} - \frac{8g}{\pi}e^{-\pi g}\,\mathrm{Re}\left[\int_0^{\infty}\frac{dte^{i(t-\pi/4)}}{t+i\pi g}\left(\Gamma_{+}(t)+i\Gamma_{-}(t)\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^{\infty}\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right] =$$

$$= \int_0^{\infty}\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_{\mu}\Phi\partial^{\mu}\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] . \quad (3.29b)$$

Furthermore, we have that $\frac{8}{\pi} = 2{,}546479 \cong 2{,}5493902$, value that is in the column (*1,375) of the Table regarding the **universal music system based on Phi**.

To fix the zero modes, we have to impose additional conditions on solutions to (3.28) and (3.24). These conditions follow unambiguously from BES equation and they can be formulated as a requirement that $\gamma_{\pm}(t)$ should be entire functions of t which admit a representation in the form of Neumann series over Bessel functions

$$\gamma_{-}(t) = 2\sum_{n\ge1}(2n-1)J_{2n-1}(t)\gamma_{2n-1}, \qquad \gamma_{+}(t) = 2\sum_{n\ge1}(2n)J_{2n}(t)\gamma_{2n}, \quad (3.30)$$

with the expansion coefficients γ_{2n-1} and γ_{2n} depending on the coupling constant. This implies in particular that the series on the right-hand side of (3.30) are convergent on the real axis. Using orthogonality conditions for the Bessel functions, we obtain from (3.30)

$$\gamma_{2n-1} = \int_0^\infty \frac{dt}{t} J_{2n-1}(t)\gamma_-(t), \qquad \gamma_{2n} = \int_0^\infty \frac{dt}{t} J_{2n}(t)\gamma_+(t). \quad (3.31)$$

Here we assumed that the sum over n in the right-hand side of (3.30) can be interchanged with the integral over t. We will show below that the relations (3.30) and (3.31) determine a unique solution to the system (3.24). The coefficient γ_1 plays a special role in our analysis since it determines the cusp anomalous dimension (3.22),

$$\Gamma_{cusp}(g) = 8g^2\gamma_1(g). \quad (3.32)$$

Here we applied (3.23) and (3.30) and took into account small-t behaviour of the Bessel functions, $J_n(t) \approx t^n$ as $t \to 0$. Let us now translate (3.30) and (3.31) into properties of the functions $\Gamma_\pm(t)$, or equivalently $\Gamma(t)$. It is convenient to rewrite the relation (3.25) as

$$\Gamma(it) = \gamma(it)\frac{\sin\left(\frac{t}{4g}+\frac{\pi}{4}\right)}{\sin\left(\frac{t}{4g}\right)\sin\left(\frac{\pi}{4}\right)} = \gamma(it)\sqrt{2}\prod_{k=-\infty}^{\infty}\frac{t-4\pi g\left(k-\frac{1}{4}\right)}{t-4\pi gk}. \quad (3.33)$$

We note that $\frac{\pi}{4} = 0{,}785398 \cong 0{,}786937$, value that is inserted in the column (*1/1,375) of the Table concerning the **universal music system based on Phi.**

Since $\gamma(it)$ is an entire function in the complex t-plane, we conclude from (3.33) that $\Gamma(it)$ has an infinite number of zeros, $\Gamma(it_{zeros}) = 0$, and poles, $\Gamma(it) \approx 1/(t - t_{poles})$, on real t-axis located at

$$t_{zeros} = 4\pi g\left(\ell - \frac{1}{4}\right), \qquad t_{poles} = 4\pi g\ell', \quad (3.34)$$

where $\ell, \ell' \in Z$ and $\ell' \neq 0$ so that $\Gamma(it)$ is regular at the origin (see. Eq.(3.22)).
To understand the relationship between analytical properties of $\Gamma(it)$ and properties of the cusp anomalous dimension, it is instructive to slightly simplify the problem and consider a "toy" model ("modello giocattolo") in which the function $\Gamma(it)$ is replaced with $\Gamma^{(toy)}(it)$. We require that $\Gamma^{(toy)}(it)$ satisfies the same integral equation (3.27) and define, following (3.26), the cusp anomalous dimension in the toy model as

$$\Gamma_{cusp}^{(toy)}(g) = -2g\Gamma^{(toy)}(0). \quad (3.35)$$

The only difference compared to $\Gamma(it)$ is that $\Gamma^{(toy)}(it)$ has different analytical properties dictated by the relation

$$\Gamma^{(toy)}(it) = \gamma^{(toy)}(it)\frac{t+\pi g}{t}, \quad (3.36)$$

while $\gamma^{(toy)}(it)$ has the same analytical properties as the function $\gamma(it)$. This relation can be considered as a simplified version of (3.33). Indeed, it can be obtained from (3.33) if we retained in the product only one term with $k=0$. As compared with (3.34), the function $\Gamma^{(toy)}(it)$ does not have poles and it vanishes for $t=-\pi g$.
Let us multiply both sides of the two relations in (3.24) by $2(2n-1)\gamma_{2n-1}$ and $2(2n)\gamma_{2n}$, respectively, and perform summation over $n\geq 1$. Then, we convert the sums into the functions $\gamma_\pm(t)$ using (3.30) and add the second relation to the first one to obtain

$$\gamma_1=\int_0^\infty \frac{dt}{t}\frac{(\gamma_+(t))^2+(\gamma_-(t))^2}{1-e^{-t/(2g)}}. \quad (3.37)$$

Since $\gamma_\pm(t)$ are real functions of t and the denominator is positively definite for $0\leq t<\infty$, this relation leads to the following inequality

$$\gamma_1\geq\int_0^\infty \frac{dt}{t}(\gamma_-(t))^2\geq 2\gamma_1^2\geq 0. \quad (3.38)$$

Here we replaced the function $\gamma_-(t)$ by its Bessel series (3.30) and made use of the orthogonality condition for the Bessel functions with odd indices. We deduce from (3.38) that

$$0\leq\gamma_1\leq\frac{1}{2} \quad (3.39)$$

and, then, apply (3.32) to translate this inequality into the following relation for the cusp anomalous dimension

$$0\leq\Gamma_{cusp}(g)\leq 4g^2. \quad (3.40)$$

Notice that the lower bound on the cusp anomalous dimension, $\Gamma_{cusp}(g)\geq 0$, holds in any gauge theory. It is upper bound $\Gamma_{cusp}(g)\leq 4g^2$ that is a distinguished feature of $\mathcal{N}=4$ theory. Let us verify the validity of (3.40). At weak coupling $\Gamma_{cusp}(g)$ admits perturbative expansion in powers of g^2

$$\Gamma_{cusp}(g)=4g^2\left[1-\frac{1}{3}\pi^2g^2+\frac{11}{45}\pi^4g^4-2\left(\frac{73}{630}\pi^6+4\zeta_3^2\right)g^6+...\right], \quad (3.41)$$

while at strong coupling it has the form

$$\Gamma_{cusp}(g)=2g\left[1-\frac{3\ln 2}{4\pi}g^{-1}-\frac{K}{16\pi^2}g^{-2}-\left(\frac{3K\ln 2}{64\pi^3}+\frac{27\zeta_3}{2048\pi^3}\right)g^{-3}+O(g^{-4})\right], \quad (3.42)$$

with K being the Catalan constant. It is easy to see that the relations (3.41) and (3.42) are in an agreement with (3.40).

We note, with regard the eqs. (3.41-3.42) that $\frac{11}{45}=0{,}24444\cong 0{,}243177$;

$\frac{73}{630} = 0{,}115873015 \cong 0{,}116101$; $\frac{27}{1048} = 0{,}013183593 \cong 0{,}01355179$, values that are inserted in the columns (*1/1,375 – *1/Pigreco) of the Table concerning the **universal music system based on Phi.**

Let us now construct the exact solution to the integral equations (3.28) and (3.24). To this end, it is convenient to Fourier transform the functions (3.23) and (3.25)

$$\tilde{\Gamma}(k) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{ikt} \Gamma(t), \qquad \tilde{\gamma}(k) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{ikt} \gamma(t). \quad (3.43)$$

According to (3.23) and (3.30), the function $\gamma(t)$ is given by the Neumann series over Bessel functions. Then, we perform the Fourier transform on both sides of (3.30) and use the well-known fact that the Fourier transform of the Bessel function $J_n(t)$ vanishes for $k^2 > 1$ to deduce that the same is true for $\gamma(t)$ leading to

$$\tilde{\gamma}(k) = 0, \quad \text{for } k^2 > 1. \quad (3.44)$$

This implies that the Fourier integral for $\gamma(t)$ only involves modes with $-1 \le k \le 1$ and, therefore, the function $\gamma(t)$ behaves at large (complex) t as

$$\gamma(t) \approx e^{|t|}, \quad \text{for } |t| \to \infty. \quad (3.45)$$

Let us now examine the function $\tilde{\Gamma}(k)$. We find from (3.43) and (3.33) that $\tilde{\Gamma}(k)$ admits the following representation

$$\tilde{\Gamma}(k) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{ikt} \frac{\sinh\left(\frac{t}{4g} + i\frac{\pi}{4}\right)}{\sinh\left(\frac{t}{4g}\right)\sin\left(\frac{\pi}{4}\right)} \gamma(t). \quad (3.46)$$

Here the integrand has poles along the imaginary axis at $t = 4\pi i g n$ (with $n = \pm 1, \pm 2, \ldots$). Taking into account the relation (3.45), we find that the contribution to (3.46) at infinity can be neglected for $k^2 > 1$ only. In this case, closing the integration contour into the upper (or lower) half-plane for $k > 1$ (or $k < -1$) we find

$$\tilde{\Gamma}(k) \overset{k^2>1}{=} \theta(k-1) \sum_{n \ge 1} c_+(n,g) e^{-4\pi n g(k-1)} + \theta(-k-1) \sum_{n \ge 1} c_-(n,g) e^{-4\pi n g(-k-1)}. \quad (3.47)$$

Here the notation was introduced for k-independent expansion coefficients

$$c_\pm(n,g) = \mp 4g \gamma(\pm 4\pi i g n) e^{-4\pi n g}, \quad (3.48)$$

where the factor $e^{-4\pi n g}$ is inserted to compensate exponential growth of $\gamma(\pm 4\pi i g n) \approx e^{4\pi n g}$ at large n (see eq. (3.45)). We recall that in the toy model (3.36), $\Gamma^{(toy)}(it)$ and $\gamma^{(toy)}(it)$ are entire functions of t . At large t they have the same asymptotic behaviour as the Bessel functions, $\Gamma^{(toy)}(it) \approx \gamma^{(toy)}(it) \approx e^{\pm it}$. Performing their Fourier transformation (3.43), we find

$$\tilde{\gamma}^{(toy)}(k) = \tilde{\Gamma}^{(toy)}(k) = 0, \quad \text{for} \quad k^2 > 1, \quad (3.49)$$

in a close analogy with (3.44). Comparison with (3.47) shows that the coefficients (3.48) vanish in the toy model for arbitrary n and g

$$c_+^{(toy)}(n,g) = c_-^{(toy)}(n,g) = 0. \quad (3.50)$$

The relation (3.47) defines the function $\tilde{\Gamma}(k)$ for $k^2 > 1$ but it involves the coefficients $c_\pm(n,g)$ that need to be determined. In addition, we have to construct the same function for $k^2 \le 1$. To achieve both goals, let us return to the integral equations (3.27) and replace $\Gamma_\pm(t)$ by Fourier integrals (see eqs. (3.43) and (3.25)

$$\Gamma_+(t) = \int_{-\infty}^{\infty} dk \cos(kt) \tilde{\Gamma}(k), \qquad \Gamma_-(t) = -\int_{-\infty}^{\infty} dk \sin(kt) \tilde{\Gamma}(k). \quad (3.51)$$

In this way, we obtain from (3.27) the following remarkably simple integral equation for $\tilde{\Gamma}(k)$

$$\int_{-\infty}^{\infty} \frac{dk \tilde{\Gamma}(k)}{k-u} + \pi \tilde{\Gamma}(u) = -2, \quad (-1 \le u \le 1), \quad (3.52)$$

where the integral is defined using the principal value prescription. Let us split the integral in (3.52) into $k^2 \le 1$ and $k^2 > 1$ and rewrite (3.52) in the form of singular integral equation for the function $\tilde{\Gamma}(k)$ on the interval $-1 \le k \le 1$

$$\tilde{\Gamma}(u) + \frac{1}{\pi} \int_{-1}^{1} \frac{dk \tilde{\Gamma}(k)}{k-u} = \phi(u), \qquad (-1 \le u \le 1), \quad (3.53)$$

where the inhomogeneous term is given by

$$\phi(u) = -\frac{1}{\pi}\left(2 + \int_{-\infty}^{-1} \frac{dk \tilde{\Gamma}(k)}{k-u} + \int_{1}^{\infty} \frac{dk \tilde{\Gamma}(k)}{k-u} \right). \quad (3.54)$$

Since integration in (3.54) goes over $k^2 > 1$, the function $\tilde{\Gamma}(k)$ can be replaced in the right-hand side of (3.54) by its expression (3.47) in terms of the coefficients $c_\pm(n,g)$. A general solution for the integral equation (3.53) for $\tilde{\Gamma}(k)$ reads (for $-1 \le k \le 1$)

$$\tilde{\Gamma}(k) = \frac{1}{2}\phi(k) - \frac{1}{2\pi}\left(\frac{1+k}{1-k}\right)^{1/4} \int_{-1}^{1} \frac{du \phi(u)}{u-k} \left(\frac{1-u}{1+u}\right)^{1/4} - \frac{\sqrt{2}}{\pi}\left(\frac{1+k}{1-k}\right)^{1/4} \frac{c}{1+k}, \quad (3.55)$$

where the last term describes the zero mode contribution with c being an arbitrary function of the coupling. We replace $\phi(u)$ by its expression (3.54), interchange the order of integration and find after some algebra

$$\tilde{\Gamma}(k) \overset{k^2 \le 1}{=} -\frac{\sqrt{2}}{\pi}\left(\frac{1+k}{1-k}\right)^{1/4}\left[1+\frac{c}{1+k}+\frac{1}{2}\int_{-\infty}^{\infty}\frac{dp\tilde{\Gamma}(p)}{p-k}\left(\frac{p-1}{p+1}\right)^{1/4}\theta(p^2-1)\right]. \quad (3.56)$$

With regard the eqs. (3.55-3.56) we have that $\frac{\sqrt{2}}{\pi} = 0,450158 \cong 0,45085$;

$\frac{1}{\pi} = 0,318309 \cong 0,318322$; $\frac{1}{2\pi} = 0,1591549 \cong 0,159649$, values that are inserted in the columns (*1/Pigreco), (*1/1,375) and (*1/Pigreco) of the Table concerning the **universal music system based on Phi.**

Furthermore, we note that the eq. (3.56) multiplied for $\frac{\pi^4}{4\sqrt{2}}$, can be related with the Jormakka's equation connected with the Ramanujan's equation concerning π (1.18), i.e.

$$\frac{\pi^4}{4\sqrt{2}}\tilde{\Gamma}(k) \overset{k^2 \le 1}{=} -\frac{\pi^3}{4}\left(\frac{1+k}{1-k}\right)^{1/4}\left[1+\frac{c}{1+k}+\frac{1}{2}\int_{-\infty}^{\infty}\frac{dp\tilde{\Gamma}(p)}{p-k}\left(\frac{p-1}{p+1}\right)^{1/4}\theta(p^2-1)\right] \Rightarrow$$

$$\Rightarrow \int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\int dy_1 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_1^2} = \int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dx_2 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_2^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dy_2 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_2^2} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$= \left[\int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2 = 8\frac{\pi^3}{4}\times\frac{1}{\left(\sqrt{2}\beta\right)^6} \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3\frac{\cos\pi x^2}{\sinh\pi x}dx + \int_0^\infty x^3\frac{\sin\pi x^2}{\sinh\pi x}dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right); \quad (3.56b)$$

We are now ready to write down a general expression for the function $\Gamma(t)$. According to (3.43), it is related to $\tilde{\Gamma}(k)$ through the inverse Fourier transformation

$$\Gamma(t) = \int_{-1}^{1} dk e^{-ikt}\tilde{\Gamma}(k) + \int_{-\infty}^{-1} dk e^{-ikt}\tilde{\Gamma}(k) + \int_{1}^{\infty} dk e^{-ikt}\tilde{\Gamma}(k), \quad (3.57)$$

where we split the integral into three terms since $\tilde{\Gamma}(k)$ has a different form for $k < -1, -1 \le k \le 1$ and $k > 1$. Then, we use the obtained expressions for $\tilde{\Gamma}(k)$, eqs. (3.47) and (3.56), to find after some algebra the following remarkable relation

$$\Gamma(it) = f_0(t)V_0(t) + f_1(t)V_1(t). \quad (3.58)$$

Here the notation was introduced for

$$f_0(t) = -1 + \sum_{n\geq 1} t\left[c_+(n,g)\frac{U_1^+(4\pi ng)}{4\pi ng - t} + c_-(n,g)\frac{U_1^-(4\pi ng)}{4\pi ng + t}\right],$$

$$f_1(t) = -c(g) + \sum_{n\geq 1} 4\pi ng\left[c_+(n,g)\frac{U_0^+(4\pi ng)}{4\pi ng - t} + c_-(n,g)\frac{U_0^-(4\pi ng)}{4\pi ng + t}\right]. \quad (3.59)$$

Also, V_n and $U_n^{\pm}$ (with $n = 0,1$) stand for integrals

$$V_n(x) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} du(1+u)^{1/4-n}(1-u)^{-1/4}e^{ux}\ ; \quad U_n^{\pm}(x) = \frac{1}{2}\int_1^{\infty} du(u\pm 1)^{-1/4}(u\mp 1)^{1/4-n}e^{-(u-1)x}\ , \quad (3.60)$$

which can be expressed in terms of Whittaker functions of 1st and 2nd kind.
Replacing $\Gamma(it)$ by its expression (3.58), we rewrite these relations in equivalent form

$$f_0(t_\ell)V_0(t_\ell) + f_1(t_\ell)V_1(t_\ell) = 0\,, \qquad t_\ell = 4\pi g\left(\ell - \frac{1}{4}\right). \quad (3.61)$$

Let us substitute (3.58) into the expression (3.26) for the cusp anomalous dimension. The result involves the functions $V_n(t)$ and $f_n(t)$ (with $n = 1,2$) evaluated at $t = 0$. It is easy to see from (3.60) that $V_0(0) = 1$ and $V_1(0) = 2$. In addition, we obtain from (3.59) that $f_0(0) = -1$ for arbitrary coupling leading to

$$\Gamma_{cusp}(g) = 2g[1 - 2f_1(0)]. \quad (3.62)$$

Replacing $f_1(0)$ by its expression (3.59) we find the following relation for the cusp anomalous dimension in terms of the coefficients c and $c_{\pm}$

$$\Gamma_{cusp}(g) = 2g\left\{1 + 2c(g) - 2\sum_{n\geq 1}\left[c_-(n,g)U_0^-(4\pi ng) + c_+(n,g)U_0^+(4\pi ng)\right]\right\}. \quad (3.63)$$

The following remarkable relation

$$\Gamma_{cusp}^{(toy)}(g) = 2g\left[1 - (2\pi g)^{-1/2}\frac{M_{1/4,1/2}(2\pi g)}{M_{-1/4,0}(2\pi g)}\right], \quad (3.63a)$$

defines the cusp anomalous dimension in the toy model for arbitrary coupling $g > 0$. At weak coupling, we find from (3.63a)

$$\Gamma_{cusp}^{(toy)}(g) = \frac{3}{2}\pi g^2 - \frac{1}{2}\pi^2 g^3 - \frac{1}{64}\pi^3 g^4 + \frac{5}{64}\pi^4 g^5 - \frac{11}{512}\pi^5 g^6 - \frac{3}{512}\pi^6 g^7 + O(g^8). \quad (3.63b)$$

The series (3.63b) has a finite radius of convergence $|g_0| = 0.796$

Furthermore, the eq. (3.56b) can be related with the Ramanujan's modular equation concerning the superstrings and the equation regarding the Palumbo-Nardelli model, i.e.

$$\Gamma_{cusp}^{(toy)}(g) = \frac{3}{2}\pi g^2 - \frac{1}{2}\pi^2 g^3 - \frac{1}{64}\pi^3 g^4 + \frac{5}{64}\pi^4 g^5 - \frac{11}{512}\pi^5 g^6 - \frac{3}{512}\pi^6 g^7 + O(g^8) \Rightarrow$$

$$\Rightarrow \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu \left(\left|F_2\right|^2 \right) \right] . \quad (3.63c)$$

Let us replace $f_0(t)$ and $f_1(t)$ in (3.61) by their explicit expressions (3.59) and rewrite the quantization conditions (3.61) as

$$V_0(4\pi g x_\ell) + c(g) V_1(4\pi g x_\ell) = \sum_{n \ge 1} \left[c_+(n,g) A_+(n, x_\ell) + c_-(n,g) A_-(n, x_\ell) \right], \quad (3.64)$$

where $x_\ell = \ell - \frac{1}{4}$ (with $\ell = 0, \pm 1, \pm 2, ...$) and the notation was introduced for

$$A_\pm(n, x_\ell) = \frac{n V_1(4\pi g x_\ell) U_0^\pm(4\pi n g) + x_\ell V_0(4\pi g x_\ell) U_1^\pm(4\pi n g)}{n \mp x_\ell} . \quad (3.65)$$

The relation (3.64) provides an infinite system of linear equations for $c_\pm(g,n)$ and $c(g)$. The coefficients in this system depend on $V_{0,1}(4\pi g x_\ell)$ and $U_{0,1}^\pm(4\pi n g)$ which are known functions. We examine (3.64) for $|x_\ell| >> 1$. In this limit, for $g = \text{fixed}$ we are allowed to replace the functions $V_0(4\pi g x_\ell)$ and $V_1(4\pi g x_\ell)$ in both sides of (3.64) by their asymptotic behaviour at infinity. We find for $|x_\ell| >> 1$

$$r(x_\ell) \equiv \frac{V_1(4\pi g x_\ell)}{V_0(4\pi g x_\ell)} = \begin{cases} -16\pi g x_\ell + ... (x_\ell < 0) \\ \frac{1}{2} + ..., (x_\ell > 0) \end{cases} \quad (3.66)$$

where ellipses denote terms suppressed by powers of $1/(g x_\ell)$ and $e^{-8\pi g |x_\ell|}$. We divide both sides of (3.64) by $V_1(4\pi g x_\ell)$ and observe that for $x_\ell \to -\infty$ the first term in the left-hand side of (3.64) is subleading and can be safely neglected. In the similar manner, one has $A_\pm(n, x_\ell)/V_1(4\pi g x_\ell) = O(1/x_\ell)$ for fixed n in the right-hand side of (3.64). Therefore, going to the limit $x_\ell \to -\infty$ in both sides of (3.64) we get

$$c(g) = 0 \quad (3.67)$$

for arbitrary g.

Arriving at (3.67), we tacitly assumed that the sum over n in (3.64) remains finite in the limit $x_\ell \to -\infty$. Taking into account large n behaviour of the functions $U_0^{\pm}(4\pi ng)$ and $U_1^{\pm}(4\pi ng)$, we obtain that this condition translates into the following condition for asymptotic behaviour of the coefficients at large n

$$c_+(n,g) = o\left(n^{1/4}\right), \qquad c_-(n,g) = o\left(n^{-1/4}\right). \quad (3.68)$$

These relations also ensure that the sum in the expression (3.63) for the cusp anomalous dimension is convergent.
In the following Table, we have showed the comparison of the numerical value of $\Gamma_{cusp}(g)/(2g)$ found from (3.64) and (3.63) for $n_{\max} = 40$ with the exact one for different values of the coupling constant g

g	0.1	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8
numer	0.1976	0.3616	0.5843	0.7096	0.7825	0.8276	0.8576	0.8787	0.8944	0.9065
exact	0.1939	0.3584	0.5821	0.7080	0.7813	0.8267	0.8568	0.8781	0.8938	0.9059

We note that there is the mathematical connections between these values and the following: 0,1967 (*1/1,375), 0,36067 (*Pigreco), 0,5835 (*Pigreco), 0,7081 (*Pigreco), 0,7869 (*1/Pigreco), 0,8333 (*1/Pigreco), 0,8498 (*Pigreco), 0,8753 (*Pigreco), 0,8989 (*1/1,375), 0,9017 (*1/Pigreco). All values inserted in the columns of the Table concerning the **universal music system based on Phi.**

At large g the integral in (3.29) receives a dominant contribution from $t \approx g$. In order to evaluate (3.29) it is convenient to change the integration variable as $t \to 4\pi git$

$$m_{O(6)} = \frac{8\sqrt{2}}{\pi^2} e^{-\pi g} - \frac{8g}{\pi} e^{-\pi g} \operatorname{Re}\left[\int_0^{-i\infty} dte^{-4\pi gt - i\pi/4} \frac{\Gamma(4\pi git)}{t + \frac{1}{4}}\right], \quad (3.69)$$

where integration goes along the imaginary axis. We find from (3.58) that $\Gamma(4\pi git)$ takes the form

$$\Gamma(4\pi git) = f_0(4\pi gt)V_0(4\pi gt) + f_1(4\pi gt)V_1(4\pi gt), \quad (3.70)$$

where $V_{0,1}(4\pi gt)$ are given by the Whittaker functions of first kind.

We note that $\frac{8\sqrt{2}}{\pi^2} = 1{,}146318 \cong 1{,}145833$ value that is inserted in the column (*1,375) of the Table concerning the **universal music system based on Phi.**

We have that $f_{0,1}(4\pi gt)$ admit the following representation (see eqs. (3.59) and (3.67))

$$f_0(4\pi gt)=\sum_{n\geq 1} t\left[c_+(n,g)\frac{U_1^+(4\pi ng)}{n-t}+c_-(n,g)\frac{U_1^-(4\pi ng)}{n+t}\right]-1,$$
$$f_1(4\pi gt)=\sum_{n\geq 1} n\left[c_+(n,g)\frac{U_0^+(4\pi ng)}{n-t}+c_-(n,g)\frac{U_0^-(4\pi ng)}{n+t}\right]. \quad (3.71)$$

Replacing $\Gamma(4\pi git)$ in (3.69) by its expression (3.70), we evaluate the t-integral and find after some algebra

$$m_{O(6)}=-\frac{16\sqrt{2}}{\pi}ge^{-\pi g}\left[f_0(-\pi g)U_0^-(\pi g)+f_1(-\pi g)U_1^-(\pi g)\right]. \quad (3.72)$$

This relation can be further simplified with a help of the quantization conditions (3.61). For $\ell=0$, we obtain from (3.61) that $f_0(-\pi g)V_0(-\pi g)+f_1(-\pi g)V_1(-\pi g)=0$. Together with the Wronskian relation for the Whittaker functions this leads to the following remarkable relation for the mass gap

$$m_{O(6)}=\frac{16\sqrt{2}}{\pi^2}\frac{f_1(-\pi g)}{V_0(-\pi g)}. \quad (3.73)$$

The functions $f_0(4\pi gt)$ and $f_1(4\pi gt)$ have the form

$$f_n(4\pi gt)=f_n^{(PT)}(4\pi gt)+\delta f_n(4\pi gt), \qquad (n=0,1). \quad (3.74)$$

Going through calculation of (3.71), we find after some algebra that perturbative corrections to $f_0(4\pi gt)$ and $f_1(4\pi gt)$ are given by linear combinations of the ratios of Euler gamma-functions

$$f_0^{(PT)}(4\pi gt)=-\frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{\Gamma\left(\frac{3}{4}-t\right)}+\frac{1}{4\pi g}\left[\left(\frac{3\ln 2}{4}+\frac{1}{8t}\right)\frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{\Gamma\left(\frac{3}{4}-t\right)}-\frac{\Gamma\left(\frac{1}{4}\right)\Gamma(1+t)}{8t\Gamma\left(\frac{1}{4}+t\right)}\right]+O\left(g^{-2}\right),$$

$$f_1^{(PT)}(4\pi gt)=\frac{1}{4\pi g}\left[\frac{\Gamma\left(\frac{1}{4}\right)\Gamma(1+t)}{4t\Gamma\left(\frac{1}{4}+t\right)}-\frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{4t\Gamma\left(\frac{3}{4}-t\right)}\right]-\frac{1}{(4\pi g)^2}\left[\frac{\Gamma\left(\frac{1}{4}\right)\Gamma(1+t)}{4t\Gamma\left(\frac{1}{4}+t\right)}\left(\frac{1}{4t}-\frac{3\ln 2}{4}\right)+\right.$$
$$\left.-\frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{4t\Gamma\left(\frac{3}{4}-t\right)}\left(\frac{1}{4t}+\frac{3\ln 2}{4}\right)\right]+O\left(g^{-3}\right). \quad (3.75)$$

In the similar manner, we compute non-perturbative corrections to (3.74)

$$\delta f_0(4\pi g t) = \Lambda^2 \left\{ \frac{1}{4\pi g} \left[\frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{2\Gamma\left(\frac{3}{4}-t\right)} - \frac{\Gamma\left(\frac{5}{4}\right)\Gamma(1+t)}{2\Gamma\left(\frac{5}{4}+t\right)} \right] + O\left(g^{-2}\right) \right\} + \dots,$$

$$\delta f_1(4\pi g t) = \Lambda^2 \left\{ \frac{1}{4\pi g} \frac{\Gamma\left(\frac{5}{4}\right)\Gamma(1+t)}{\Gamma\left(\frac{5}{4}+t\right)} + \frac{1}{(4\pi g)^2} \left[\frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{8t\Gamma\left(\frac{3}{4}-t\right)} - \frac{\Gamma\left(\frac{5}{4}\right)\Gamma(1+t)}{\Gamma\left(\frac{5}{4}+t\right)} \left(\frac{1}{8t} + \frac{3}{4}\ln 2 - \frac{1}{4} \right) \right] + O\left(g^{-3}\right) \right\} + \dots \quad (3.76)$$

where ellipses denote $O\left(\Lambda^4\right)$ terms.

Let us obtain the strong coupling expansion of the mass gap (3.73). We replace $V_0(-\pi g)$ by its asymptotic series

$$V_0(-\pi g) = \frac{(2\pi g)^{-5/4} e^{\pi g}}{\Gamma\left(\frac{3}{4}\right)} \left[F\left(\frac{1}{4}, \frac{5}{4} \middle| \frac{1}{2\pi g} + i\varepsilon \right) + \Lambda^2 F\left(-\frac{1}{4}, \frac{3}{4} \middle| -\frac{1}{2\pi g} \right) \right], \quad (3.77)$$

$$\begin{aligned}
U_0^+(x) &= (2x)^{-5/4}\Gamma\left(\frac{5}{4}\right) F\left(\frac{1}{4},\frac{5}{4}\middle| -\frac{1}{2x}\right) = (2x)^{-5/4}\Gamma\left(\frac{5}{4}\right)\left[1 - \frac{5}{32x} + \dots\right], \\
U_0^-(x) &= (2x)^{-3/4}\Gamma\left(\frac{3}{4}\right) F\left(-\frac{1}{4},\frac{3}{4}\middle| -\frac{1}{2x}\right) = (2x)^{-3/4}\Gamma\left(\frac{3}{4}\right)\left[1 + \frac{3}{32x} + \dots\right], \\
U_1^+(x) &= (2x)^{-1/4}\frac{1}{2}\Gamma\left(\frac{1}{4}\right) F\left(\frac{1}{4},\frac{1}{4}\middle| -\frac{1}{2x}\right) = (2x)^{-1/4}\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[1 - \frac{1}{32x} + \dots\right], \\
U_1^-(x) &= (2x)^{-3/4}\frac{1}{2}\Gamma\left(\frac{3}{4}\right) F\left(\frac{3}{4},\frac{3}{4}\middle| -\frac{1}{2x}\right) = (2x)^{-3/4}\frac{1}{2}\Gamma\left(\frac{3}{4}\right)\left[1 - \frac{9}{32x} + \dots\right], \quad (3.78)
\end{aligned}$$

and take into account (3.75) and (3.76) to get

$$m_{O(6)} = \frac{\sqrt{2}}{\Gamma\left(\frac{5}{4}\right)} (2\pi g)^{1/4} e^{-\pi g} \left\{ \left[1 + \frac{3 - 6\ln 2}{32\pi g} + \frac{-63 + 108\ln 2 - 108(\ln 2)^2 + 16K}{2048(\pi g)^2} + \dots \right] + \right.$$
$$\left. -\frac{\Lambda^2}{8\pi g}\left[1 - \frac{15 - 6\ln 2}{32\pi g} + \dots\right] + O\left(\Lambda^4\right) \right\}, \quad (3.79)$$

that can be connected with the eq. (3.69), obtaining the following mathematical connection:

$$m_{O(6)} = \frac{8\sqrt{2}}{\pi^2} e^{-\pi g} - \frac{8g}{\pi} e^{-\pi g} \operatorname{Re}\left[\int_0^{-i\infty} dt e^{-4\pi g t - i\pi/4} \frac{\Gamma(4\pi g i t)}{t + \frac{1}{4}} \right] =$$

$$= \frac{\sqrt{2}}{\Gamma\left(\frac{5}{4}\right)}(2\pi g)^{1/4} e^{-\pi g} \left\{ \left[1 + \frac{3 - 6\ln 2}{32\pi g} + \frac{-63 + 108\ln 2 - 108(\ln 2)^2 + 16K}{2048(\pi g)^2} + \dots \right] + \right.$$

$$\left. - \frac{\Lambda^2}{8\pi g}\left[1 - \frac{15 - 6\ln 2}{32\pi g} + \dots\right] + O\left(\Lambda^4\right) \right\}. \quad (3.80)$$

With regard this equation, we have that: $\sqrt{2}(2\pi)^{1/4} = 2{,}23903 \cong 2{,}24730$;
$1 - \frac{15 - 6\ln 2}{32\pi} = 0{,}892161415 \cong 0{,}898958$ that are inserted in the columns (*1/Pigreco) and (*1/1,375) of the Table concerning the **universal music system based on Phi.**
Furthermore, this equation can be related also with the Ramanujan modular equation regarding the superstrings and the equation concerning the Palumbo-Nardelli model. Thence, we obtain:

$$m_{O(6)} = \frac{8\sqrt{2}}{\pi^2} e^{-\pi g} - \frac{8g}{\pi} e^{-\pi g} \operatorname{Re}\left[\int_0^{-i\infty} dt e^{-4\pi g t - i\pi/4} \frac{\Gamma(4\pi g i t)}{t + \frac{1}{4}} \right] =$$

$$= \frac{\sqrt{2}}{\Gamma\left(\frac{5}{4}\right)}(2\pi g)^{1/4} e^{-\pi g} \left\{ \left[1 + \frac{3 - 6\ln 2}{32\pi g} + \frac{-63 + 108\ln 2 - 108(\ln 2)^2 + 16K}{2048(\pi g)^2} + \dots \right] + \right.$$

$$\left. - \frac{\Lambda^2}{8\pi g}\left[1 - \frac{15 - 6\ln 2}{32\pi g} + \dots\right] + O\left(\Lambda^4\right) \right\} \Rightarrow$$

$$\Rightarrow \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4}\right)} \right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x \sqrt{g} \left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr\left(G_{\mu\nu} G_{\rho\sigma}\right) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_v\left(\left|F_2\right|^2\right) \right]. \quad (3.81)$$

4. On some equations concerning the "fractal" behaviour of the partition function. [5] [6]

(see the following introduction: http://www.fractal.org/Bewustzijns-Besturings-Model/Partition-numbers-are-fractal.pdf)

The study of $p(n)$ has played a fundamental role in the number theory. The very famous mathematicians Hardy and Ramanujan invented the "circle method" in analytic number theory in their work on $p(n)$ asymptotics. They proved the asymptotic formula

$$p(n) \approx \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}}. \quad \text{(a)}$$

Rademacher subsequently perfected this method to derive his famous "exact" formula

$$p(n) = 2\pi(24n-1)^{-3/4} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{3/2}\left(\frac{\pi\sqrt{24n-1}}{6k}\right). \quad \text{(b)}$$

and this equation can be rewritten also as follows:

$$p(n) = \frac{2\pi}{\sqrt[4]{(24n-1)^3}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{3/2}\left(\frac{\pi\sqrt{24n-1}}{6k}\right). \quad \text{(c)}$$

In terms of congruences, $p(n)$ has served as a testing ground for fundamental constructions in the theory of modular forms. The theory of Ramanujan's celebrated congruences, assert that

$$\begin{aligned} p(5^m n + \delta_5(m)) &\equiv 0 \pmod{5^m}, \\ p(7^m n + \delta_7(m)) &\equiv 0 \pmod{7^{[m/2]+1}}, \\ p(11^m n + \delta_{11}(m)) &\equiv 0 \pmod{11^m}, \end{aligned} \quad (4.1)$$

where $0 < \delta_\ell(m) < \ell^m$ satisfies the congruence $24\delta_\ell(m) \equiv 1 \pmod{\ell^m}$. To prove these congruences, Atkin, Ramanujan and Watson made use of special modular equations to produce ℓ-adic expansions of generating functions

$$P_\ell(b;z) := \sum_{n=0}^{\infty} p\left(\frac{\ell^b n + 1}{24}\right) q^{\frac{n}{24}}. \quad (4.2)$$

(note that $q := e^{2\pi i z}$ throughout, $p(0)=1$, and $p(\alpha)=0$ if $\alpha < 0$ or $\alpha \notin Z$).
Little is known about the ℓ-adic properties of the $P_\ell(b;z)$, as $b \to +\infty$, for primes $\ell \geq 13$. Has been observed that these functions are nicely constrained ℓ-adically. Furthermore, they are "self-similar", i.e. have a "fractal" behaviour, with resolution that improves as one "zooms in" appropriately. Throughout, if $\ell \geq 5$ is prime and $m \geq 1$, then we let

$$b_\ell(m) := \begin{cases} m^2 & \text{if } \ell \geq 5 \text{ and } \ell \neq 7, \\ m^2 & \text{if } \ell = 7 \text{ and } m \text{ is even,} \\ m(m+1) & \text{if } \ell = 7 \text{ and } m \text{ is odd.} \end{cases} \qquad (4.3)$$

Theorem 1.1

Suppose that $5 \leq \ell \leq 31$ *is prime, and that* $m \geq 1$. *If* $b_1 \equiv b_2 \pmod 2$ *are integers for which* $b_2 > b_1 \geq b_\ell(m)$, *then there is an integer* $A_\ell(b_1, b_2, m)$ *such that for every non-negative integer* n *we have*

$$p\left(\frac{\ell^{b_2} n + 1}{24}\right) \equiv A_\ell(b_1, b_2, m) \cdot p\left(\frac{\ell^{b_1} n + 1}{24}\right) \quad (\text{mod } \ell^m). \quad (4.4)$$

If $\ell \in \{5, 7, 11\}$, *then* $A_\ell(b_1, b_2, m) = 0$.

Now we illustrate Theorem 1.1 with $\ell = 13$. For $m = 1$, Theorem 1.1 applies for every pair of positive integers $b_1 < b_2$ with the same parity. We let $b_1 := 1$ and $b_2 := 3$. It turns out that $A_{13}(1,3,1) = 6$, and so we have that

$$p(13^3 n + 1007) \equiv 6\, p(13n + 6) \qquad (\text{mod} 13).$$

By direct calculation, we find that

$$6\sum_{n=0}^{\infty} p(13n+6)q^n = 66 + 2940q + 50094q^2 + 534804q^3 + 4291320q^4 + 28183230q^5 + \ldots$$
$$\equiv 1 + 2q + 5q^2 + 10q^3 + 7q^4 + 10q^5 + \ldots \quad (\text{mod} 13),$$

and

$$\sum_{n=0}^{\infty} p(13^3 n + 1007)q^n = 31724668493728872881006491578226 +$$
$$+ 509916755043046677119363776450904149616258340615171112513 90q + \ldots$$
$$\equiv 1 + 2q + 5q^2 + 10q^3 + 7q^4 + 10q^5 + \ldots \quad (\text{mod} 13).$$

We note that (for n = 0, 1, 2, 3, 4, and 5, i.e. 13n + 6 = 6, 19, 32, 45, 58, and 71)

$$28183230 : \frac{p(71)}{13} = 78; \qquad 4291320 : \frac{p(58)}{13} = 78; \qquad 534804 : \frac{p(45)}{13} = 78;$$

$$50094 : \frac{p(32)}{13} = 78; \qquad 2940 : \frac{p(19)}{13} = 78; \qquad 66 : \frac{p(6)}{13} = 78,$$

and that

$$x = \frac{p(71) \times 78}{13} = 28183230; \qquad x = \frac{p(58) \times 78}{13} = 4291320; \qquad x = \frac{p(45) \times 78}{13} = 534804;$$

$$x = \frac{p(32)\times 78}{13} = 50094\,; \qquad x = \frac{p(19)\times 78}{13} = 2940\,; \qquad x = \frac{p(6)\times 78}{13} = 66\,.$$

From these expressions, it is evident the “fractal” behaviour of the partition numbers.
We zoom in and consider $m = 2$. It turns out that $b_1 := 2$ and $b_2 := 4$ satisfy the conclusion of Theorem 1.1 with $A_{13}(2,4,2) = 45$, which in turn implies that

$$p\left(13^4 n + 27371\right) \equiv 45\, p\left(13^2 n + 162\right) \quad \left(\bmod 13^2\right).$$

For $n = 0,1,$ and 2, we find that the smaller partition numbers give

$$\begin{aligned}
45\, p\left(13^2 \cdot 0 + 162\right) &= 5846125708665 \equiv 99 \quad \left(\bmod 13^2\right), \\
45\, p\left(13^2 \cdot 1 + 162\right) &= 3546056488619997675 \equiv 89 \quad \left(\bmod 13^2\right), \\
45\, p\left(13^2 \cdot 2 + 162\right) &= 1035074264658445797776215 \equiv 20 \quad \left(\bmod 13^2\right).
\end{aligned}$$

Although the other partition numbers are way too large to give here, we find

$$\begin{aligned}
p\left(13^4 \cdot 0 + 27371\right) &= 1058165383617801391727085615958122102244407 52... \equiv 99 \quad \left(\bmod 13^2\right), \\
p\left(13^4 \cdot 1 + 27371\right) &= 747061679432324321866969710089533207619136212... \equiv 89 \quad \left(\bmod 13^2\right), \\
p\left(13^4 \cdot 2 + 27371\right) &= 111777755456127388513960963128155705859381391... \equiv 20 \quad \left(\bmod 13^2\right).
\end{aligned}$$

We note, with regard the second expression, that 89 is a Fibonacci’s number.
We recall Dedekind’s eta-function

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 - q^n\right) = \sum_{k \in Z} (-1)^k q^{\frac{3k^2 + k}{2} + \frac{1}{24}}\,. \qquad (4.5)$$

If $\ell \geq 5$ is prime and $m \geq 1$, then we let $k_\ell(m) := \ell^{m-1}(\ell - 1)$. We define $\Omega_\ell(m)$ to be the $Z/\ell^m Z$-module of the reductions modulo ℓ^m of those forms which arise as images after applying at least the first $b_\ell(m)$ operators. We bound the dimension of $\Omega_\ell(m)$ independently of m, and we relate the partition generating functions to the forms in this space.

Theorem 1.2

If $\ell \geq 5$ is prime and $m \geq 1$, then $\Omega_\ell(m)$ is a $Z/\ell^m Z$-module with rank $\leq \left[\frac{\ell - 1}{12}\right]$. Moreover, if $b \geq b_\ell(m)$, then we have that

$$\begin{aligned}
P_\ell(b;z) &\equiv \frac{F_\ell(b;z)}{\eta(z)} \quad \left(\bmod \ell^m\right) \quad \textit{if } b \textit{ is even} \\
&\equiv \frac{F_\ell(b;z)}{\eta(\ell z)} \quad \left(\bmod \ell^m\right) \quad \textit{if } b \textit{ is odd,} \qquad (4.6)
\end{aligned}$$

where $F_\ell(b;z) \in \Omega_\ell(m)$.

Each form $F_\ell(b;z) \in \Omega_\ell(m)$ is congruent modulo ℓ to a cusp form in $S_{\ell-1} \cap Z[[q]]$. Since these spaces are trivial for $\ell \in \{5,7,11\}$, Theorem 1.2 for these ℓ follows immediately from the Ramanujan congruences. Conversely, if $\ell \in \{5,7,11\}$ and $m \geq 1$, then for $b \geq b_\ell(m)$ we have that

$$p\left(\ell^b n + \delta_\ell(b)\right) \equiv 0 \quad \left(\operatorname{mod} \ell^m\right). \quad (4.7)$$

Theorem 1.2 shows that the partition numbers are self-similar ℓ-adically with resolutions that improve as one zooms in properly using the stochastic process which defines the $P_\ell(b;z)$. Indeed, the $P_\ell(b;z)$ $\left(\operatorname{mod} \ell^m\right)$, for $b \geq b_\ell(m)$, form periodic orbits. Theorem 1.2 bounds the corresponding "Hausdorff dimensions", and these dimensions only depend on ℓ. For $\ell \in \{5,7,11\}$, the dimension is 0, a fact that is beautifully illustrated by Ramanujan's congruences*, and for $13 \leq \ell \leq 23$, the dimension is 1. Theorem 1.1 summarizes these observations for $5 \leq \ell \leq 23$ and include the primes $\ell = 29$ and 31.

*(*In mathematics, Ramanujan's congruences are some remarkable congruences for the partition function p(n). The Indian mathematician Srinivasa Ramanujan discovered the following*

$$\begin{aligned} p(5k+4) &\equiv 0 \pmod 5 \\ p(7k+5) &\equiv 0 \pmod 7 \\ p(11k+6) &\equiv 0 \pmod{11}. \end{aligned}$$

In his 1919 paper (Ramanujan, 1919), he gave proof for the first two congruences using the following identities (using q-Pochhammer symbol notation):

$$\sum_{k=0}^{\infty} p(5k+4)q^k = 5\frac{(q^5)_\infty^5}{(q)_\infty^6}$$

$$\sum_{k=0}^{\infty} p(7k+5)q^k = 7\frac{(q^7)_\infty^3}{(q)_\infty^4} + 49q\frac{(q^7)_\infty^7}{(q)_\infty^8}.$$

then stated that "It appears there are no equally simple properties for any moduli involving primes other than these") [from **Ramanujan's congruences - Wikipedia, the free encyclopedia**].

The following theorem gives the finite algebraic formula for $p(n)$.

Theorem 1.3

If n is a positive integer, then we have that

$$p(n) = \frac{1}{24n-1} \cdot Tr(n). \quad (4.8)$$

The numbers $P(\alpha_Q)$, as Q varies over $\mathcal{Q}_n$, form a multiset of algebraic numbers which is the union of Galois orbits for the discriminant $-24n+1$ ring class field. Moreover, for each $Q \in \mathcal{Q}_n$ we have that $6(24n-1)P(\alpha_Q)$ is an algebraic integer.

Theorem 1.3 gives an algorithm for computing $p(n)$, as well as the polynomial

$$H_n(x) = x^{h(-24n+1)} - (24n-1)p(n)x^{h(-24n+1)} + \ldots := \prod_{Q \in \mathcal{Q}_n} \left(x - P(\alpha_Q)\right) \in Q[x]. \quad (4.9)$$

Let L be the following lattice

$$L := \left\{ \begin{pmatrix} b & a/N \\ c & -b \end{pmatrix} : a,b,c \in Z \right\}. \quad (4.10)$$

For $k \in \frac{1}{2}Z$, we let $H_k(N)$ denote the space of harmonic Maass forms of weight k for $\Gamma := \Gamma_0(N)$. We let $H_k^{\infty}(N)$ denote the subspace of $H_k(N)$ consisting of those harmonic Maass forms whose principal parts at all cusps other than ∞ are constant. We write $M_k^{!\infty}(N) = M_k^{!}(N) \cap H_k^{\infty}(N)$. For a weak Maass form f of weight -2 for Γ we define

$$\Lambda(\tau, f) = L_{3/2,\tau} \int_M \left(R_{-2,z} f(z)\right) \Theta_L(\tau, z, \varphi_{KM}). \quad (4.11)$$

The Kudla-Millson theta kernel has exponential decay as $O\left(e^{-Cy^2}\right)$ for $y \to \infty$ at all cusps of Γ with some constant $C > 0$. Therefore the theta integral converges absolutely. It defines a $C[L'/L]$-valued function on H that transforms like a non-holomorphic modular form of weight – 1/2 for $\tilde{\Gamma}$. We denote by $\Lambda_h(\tau, f)$ the components of the lift $\Lambda(\tau, f)$ with respect to the standard basis $(\boldsymbol{e}_h)_h$ of $C[L'/L]$. The group $\mathrm{O}(L'/L)$ can be identified with the group generated by the Atkin-Lehner involutions. The following proposition, which is easily checked, shows that the theta lift is equivariant with respect to the action of $\mathrm{O}(L'/L)$.

Proposition 1.1

For $\gamma \in \mathrm{O}(L'/L)$ and $h \in L'/L$, we have

$$\Lambda_{\gamma h}(\tau, f) = \Lambda_h\left(\tau, f\big|_{-2}\gamma^{-1}\right). \quad (4.12)$$

Theorem 1.4

If m is a positive integer, then we have

$$\Lambda(\tau, F_m(z,s,-2)) = \frac{2^{2-s}\sqrt{\pi}Ns(1-s)}{\Gamma\left(\frac{s}{2} - \frac{1}{2}\right)} \sum_{n|m} n \cdot \mathcal{F}_{\frac{m^2}{4Nn^2}, \frac{m}{n}}\left(\tau, \frac{s}{2} + \frac{1}{4}, -\frac{1}{2}\right). \quad (4.13)$$

By definition we have

$$\Lambda(\tau, F_m(z,s,-2)) = L_{3/2,\tau} \int_M \left(R_{-2,z} F_m(z,s,-2)\right) \Theta_L(\tau, z, \varphi_{KM}). \quad (4.14)$$

Employing the following proposition

Proposition 1.2

$$\frac{1}{4\pi m} R_k F_m(z,s,k) = (s + k/2) F_m(z,s,k+2), \quad (4.15)$$

and

$$L_{3/2,\tau}\Theta_L(\tau,z,\varphi_{KM}) = -dd^c\Theta_L(\tau,z,\varphi_S) = \frac{1}{4\pi}\Delta_{0,z}\Theta_L(\tau,z,\varphi_S)\cdot\Omega, \quad (4.15b)$$

we see that this is equal to

$$m(s-1)\int_M F_m(z,s,0)\Delta_{0,z}\Theta_L(\tau,z,\varphi_S)\Omega. \quad (4.16)$$

Using the following definition

$$F_m(z,s,k) = \frac{1}{2\Gamma(2s)}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\left[\mathcal{M}_{s,k}(4\pi my)e(-mx)\right]_k\gamma, \quad (4.17)$$

we find, by the usual unfolding argument, that

$$\Lambda(\tau, F_m(z,s,-2)) = \frac{m(s-1)}{\Gamma(2s)}\int_{\Gamma_\infty\backslash H}\mathcal{M}_{s,0}(4\pi my)e(-mx)\Delta_{0,z}\Theta_L(\tau,z,\varphi_S)\Omega. \quad (4.18)$$

By the following proposition

Proposition 1.3

$$\Theta_L(\tau,z,\varphi_S) = \frac{1}{\sqrt{2\ell_z^2}}\cdot\Xi_K(\tau,0,0) + \frac{1}{2\sqrt{2\ell_z^2}}\sum_{n=1}^{\infty}\sum_{\gamma\in\tilde{\Gamma}_\infty\backslash\tilde{\Gamma}}\left[\exp\left(-\frac{\pi n^2}{2v\ell_z^2}\right)\Xi(\tau,n\mu(z),0)\right]\Bigg|_{-1/2,\rho_K}\gamma, \quad (4.19)$$

we may replace $\Delta_{0,z}\Theta_L(\tau,z,\varphi_S)$ by $\Delta_{0,z}\tilde{\Theta}_L(\tau,z,\varphi_S)$, where

$$\tilde{\Theta}_L(\tau,z,\varphi_S) = \frac{1}{2\sqrt{2\ell_z^2}}\sum_{n=1}^{\infty}\sum_{\gamma\in\tilde{\Gamma}_\infty\backslash\tilde{\Gamma}}\left[\exp\left(-\frac{\pi n^2}{2v\ell_z^2}\right)\Xi(\tau,n\mu(z),0)\right]\Bigg|_{-1/2,\rho_K}\gamma. \quad (4.20)$$

Recall that $\ell_z^2 = \frac{1}{2Ny^2}$. The function $\tilde{\Theta}_L(\tau,z,\varphi_S)$ and its partial derivatives have square exponential decay as $y\to\infty$. Therefore, for $\mathfrak{R}(s)$ large, we may move the Laplace operator to the Poincaré series and obtain

$$\Lambda(\tau, F_m(z,s,-2)) = \frac{m(s-1)}{\Gamma(2s)}\int_{\Gamma_\infty\backslash H}\left(\Delta_{0,z}\mathcal{M}_{s,0}(4\pi my)e(-mx)\right)\tilde{\Theta}_L(\tau,z,\varphi_S)\Omega =$$

$$= -\frac{ms(s-1)^2}{\Gamma(2s)}\int_{\Gamma_\infty\backslash H}\mathcal{M}_{s,0}(4\pi my)e(-mx)\tilde{\Theta}_L(\tau,z,\varphi_S)\Omega = -\frac{ms(s-1)^2}{\Gamma(2s)}\sum_{n=1}^{\infty}\sum_{\gamma\in\tilde{\Gamma}_\infty\backslash\tilde{\Gamma}} I(\tau,s,m,n)\Big|_{-1/2,\rho_K}\gamma,$$

(4.21)

where

$$I(\tau,s,m,n)=\int_{y=0}^{\infty}\int_{x=0}^{1}\mathcal{M}_{s,0}(4\pi my)e(-mx)\frac{1}{2\sqrt{2\ell_z^2}}\exp\left(-\frac{\pi n^2}{2v\ell_z^2}\right)\Xi(\tau,n\mu(z),0)\frac{dxdy}{y^2}. \quad (4.22)$$

If we use the fact that $K'=Z\begin{pmatrix}1/2N & 0\\ 0 & -1/2N\end{pmatrix}$, and identify $K'/K\cong Z/2NZ$, then we have

$$\Xi(\tau,n\mu(z),0)=\sqrt{v}\sum_{b\in Z}e\left(-\frac{b^2}{4N}\bar{\tau}-nbx\right)\boldsymbol{e}_b. \quad (4.23)$$

Inserting this in the formula for $I(\tau,s,m,n)$, and by integrating over x, we see that $I(\tau,s,m,n)$ vanishes when n†m. If n|m, then only the summand for $b=-m/n$ occurs and so

$$I(\tau,s,m,n)=\frac{\sqrt{Nv}}{2}\int_0^{\infty}\mathcal{M}_{s,0}(4\pi my)\exp\left(-\frac{\pi Nn^2y^2}{v}\right)\frac{dy}{y}e\left(-\frac{m^2}{4Nn^2}\bar{\tau}\right)\boldsymbol{e}_{-m/n}. \quad (4.24)$$

To compute this last integral, we note that

$$\mathcal{M}_{s,0}(4\pi my)=M_{0,s-1/2}(4\pi my)=2^{2s-1}\Gamma(s+1/2)\sqrt{4\pi my}\cdot I_{s-1/2}(2\pi my).$$

Substituting $t=y^2$ in the integral, we obtain

$$\int_0^{\infty}\mathcal{M}_{s,0}(4\pi my)\exp\left(-\frac{\pi Nn^2y^2}{v}\right)\frac{dy}{y}=2^{2s-1}\Gamma(s+1/2)\int_0^{\infty}\sqrt{4\pi my}I_{s-1/2}(2\pi my)\exp\left(-\frac{\pi Nn^2y^2}{v}\right)\frac{dy}{y}=$$

$$=2^{2s-1}\Gamma(s+1/2)\sqrt{\pi m}\int_0^{\infty}I_{s-1/2}\left(2\pi m\sqrt{t}\right)\exp\left(-\frac{\pi Nn^2t}{v}\right)t^{-3/4}dt. \quad (4.25)$$

The latter integral is a Laplace transform. Inserting the evaluation, we obtain

$$\int_0^{\infty}\mathcal{M}_{s,0}(4\pi my)\exp\left(-\frac{\pi Nn^2y^2}{v}\right)\frac{dy}{y}=2^{2s-1}\Gamma(s/2)\left(\frac{Nn^2}{\pi m^2v}\right)^{1/4}M_{1/4,s/2-1/4}\left(\frac{\pi m^2v}{Nn^2}\right)\exp\left(\frac{\pi m^2v}{2Nn^2}\right)=$$

$$=2^{2s-1}\Gamma(s/2)\left(\frac{Nn^2}{\pi m^2v}\right)^{1/2}\mathcal{M}_{s/2+1/4,-1/2}\left(\frac{\pi m^2v}{Nn^2}\right)\exp\left(\frac{\pi m^2v}{2Nn^2}\right). \quad (4.26)$$

Consequently, we have in the case $n|m$ that

$$I(\tau,s,m,n)=\frac{2^{2s-2}Nn}{\sqrt{\pi}m}\Gamma(s/2)\mathcal{M}_{s/2+1/4,-1/2}\left(\frac{\pi m^2v}{Nn^2}\right)e\left(-\frac{m^2}{4Nn^2}u\right)\boldsymbol{e}_{-m/n}. \quad (4.27)$$

Substituting this in (4.21), we find

$$\Lambda(\tau, F_m(z,s,-2)) = \frac{2^{2-s}\sqrt{\pi}Ns(1-s)}{\Gamma\left(\frac{s}{2}-\frac{1}{2}\right)} \sum_{n|m} n \cdot \mathcal{F}_{\frac{m^2}{4Nn^2},-\frac{m}{n}}\left(\tau, \frac{s}{2}+\frac{1}{4}, -\frac{1}{2}\right). \quad (4.28)$$

Since $\mathcal{F}_{m,h}(\tau,s,-1/2) = \mathcal{F}_{m,-h}(\tau,s,-1/2)$, this concludes the proof of the Theorem 1.4.

Corollary 1.5

If $f \in H_{-2}(N)$ is a harmonic Maass form of weight -2 for $\Gamma_0(N)$, then $\Lambda(\tau, f)$ belongs to $H_{-1/2,\rho_L}$. In particular, we have

$$\Lambda(\tau, F_m(z,2,-2)) = -2N \sum_{m|n} n \cdot \mathcal{F}_{\frac{m^2}{4Nn^2},\frac{m}{n}}\left(\tau, \frac{5}{4}, -\frac{1}{2}\right). \quad (4.29)$$

The formula for the image of the Poincaré series $F_m(z,2,-2)$ is a direct consequence of Theorem 1.4. These Poincaré series for $m \in Z_{>0}$ span the subspace $H_{-2}^{\infty}(N) \subset H_{-2}(N)$ of harmonic Maass forms whose principal parts at all cusps other than ∞ are constant. Consequently, we find that the image of $H_{-2}^{\infty}(N)$ is contained in $H_{-1/2,\rho_L}$. For simplicity, here we only prove that the image of the full space $H_{-2}(N)$ is contained in $H_{-1/2,\rho_L}$ in the special case when N is squarefree. When N is squarefree, then the group $\mathrm{O}(L'/L)$ of Atkin-Lehner involutions acts transitively on the cusps of $\Gamma_0(N)$. Consequently, we have

$$H_{-2}(N) = \sum_{\gamma \in \mathrm{O}(L'/L)} \gamma H_{-2}^{\infty}(N). \quad (4.30)$$

Using Proposition 1.1, we see that the whole space $H_{-2}(N)$ is mapped to $H_{-1/2,\rho_L}$.

Theorem 1.6

Let $f \in H_{-2}(N)$ and put $\partial f := \frac{1}{4\pi} R_{-2,z} f$. For $m \in Q_{>0}$ and $h \in L'/L$ the (m,h)-th Fourier coefficient of the holomorphic part of $\Lambda(\tau, f)$ is equal to

$$tr_f(m,h) = -\frac{1}{2m} \sum_{z \in Z(m,h)} \partial f(z). \quad (4.31)$$

Inserting the definition of the theta lifting and using (4.15b), we have

$$\Lambda(\tau, f) = 4\pi L_{3/2,\tau} \int_M \partial f(z) \Theta_L(\tau, z, \varphi_{KM}) = \int_M \partial f(z) \Delta_{0,z} \Theta_L(\tau, z, \varphi_S) \Omega. \quad (4.32)$$

For $X \in V(R)$ and $z \in D$ we define $\varphi_S^0(X,z) = e^{2\pi Q(X)} \varphi_S(X,z)$. Then the Fourier expansion of the Siegel theta function in the variable τ is given by

$$\Theta_L(\tau, z, \varphi_S) = \sum_{X \in L'} \varphi_S^0\left(\sqrt{v}X, z\right) q^{Q(X)} \mathbf{e}_X . \quad (4.33)$$

For $m \in Q_{>0}$ and $h \in L'/L$, we put $L_{m,h} = \{X \in L + h; Q(X) = m\}$. The group Γ acts on $L_{m,h}$ with finitely many orbits. We write $C(m,h)$ for the (m,h)-th Fourier coefficient of the holomorphic part of $\Lambda(\tau, f)$. Using (4.33), we see that

$$C(m,h) = \int_M \partial f(z) \Delta_{0,z} \sum_{X \in L_{m,h}} \varphi_S^0\left(\sqrt{v}X, z\right) \Omega . \quad (4.34)$$

For $Q(X) > 0$ the function $\varphi_S^0(X, z)$ has square exponential decay as $y \to \infty$. This implies that we may move the Laplacian in the integral to the function ∂f . Since $\Delta_0 \partial f = -2\partial f$, we see that

$$C(m,h) = -2\int_M \partial f(z) \sum_{X \in L_{m,h}} \varphi_S^0\left(\sqrt{v}X, z\right) \Omega . \quad (4.35)$$

Using the usual unfolding argument, we obtain

$$C(m,h) = -2 \sum_{X \in \Gamma \backslash L_{m,h}} \frac{1}{|\Gamma_X|} \int_D \partial f(z) \varphi_S^0\left(\sqrt{v}X, z\right) \Omega . \quad (4.36)$$

Furthermore, we have the following relationship:

$$C(m,h) = -2\int_M \partial f(z) \sum_{X \in L_{m,h}} \varphi_S^0\left(\sqrt{v}X, z\right) \Omega = -2 \sum_{X \in \Gamma \backslash L_{m,h}} \frac{1}{|\overline{\Gamma}_X|} \int_D \partial f(z) \varphi_S^0\left(\sqrt{v}X, z\right) \Omega . \quad (4.36b)$$

It is convenient to rewrite the integral over D as an integral over $G(R) = SL_2(R)$. If we normalize the Haar measure such that the maximal compact subgroup SO(2) has volume 1, we have

$$I(X) := \int_D \partial f(z) \varphi_S^0\left(\sqrt{v}X, z\right) \Omega = \int_{G(R)} \partial f(gi) \varphi_S^0\left(\sqrt{v}X, gi\right) dg . \quad (4.37)$$

Using the Cartan decomposition of $G(R)$ and the uniqueness of spherical functions, we find that

$$I(X) = \partial f(D_X) \cdot Y_\lambda\left(\sqrt{mv/N}\right), \quad (4.38)$$

where

$$Y_\lambda(t) = 4\pi \int_1^\infty \varphi_S^0\left(t\alpha(a)^{-1} X(i), i\right) \omega_\lambda(\alpha(a)) \frac{a^2 - a^{-2}}{2} \frac{da}{a} . \quad (4.39)$$

Here $\omega_\lambda(g)$ is the standard spherical function with eigenvalue $\lambda = -2$, and $\alpha(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. Note that $\omega_{-2}(\alpha(a)) = \frac{a^2 + a^{-2}}{2}$. It is easy computed that

$$\varphi_S^0\left(t\alpha(a)^{-1}X(i),i\right)=ve^{-\pi Nt^2\left(a^2-a^{-2}\right)^2}, \quad (4.40)$$

and therefore

$$Y_\lambda(t)=2\pi v\int_0^\infty e^{-4\pi Nt^2\sinh(r)^2}\cosh(r)\sinh(r)dr=\frac{v}{4Nt^2}. \quad (4.41)$$

Thence, we have the following relationship:

$$Y_\lambda(t)=4\pi\int_1^\infty \varphi_S^0\left(t\alpha(a)^{-1}X(i),i\right)\omega_\lambda(\alpha(a))\frac{a^2-a^{-2}}{2}\frac{da}{a}=2\pi v\int_0^\infty e^{-4\pi Nt^2\sinh(r)^2}\cosh(r)\sinh(r)dr=\frac{v}{4Nt^2}. \quad (4.41b)$$

Hence $Y_\lambda\left(\sqrt{mv/N}\right)=\frac{1}{4m}$. Inserting this into (4.36), we obtain the assertion.

Now we consider the theta integral (4.11) in the special case when $N=6$. We identify the discriminant form L'/L with $Z/12Z$ together with the Q/Z-valued quadratic form $r\mapsto -r^2/24$. The function $\eta(\tau)^{-1}$ can be viewed as a component of a vector valued modular form in $M^!_{-1/2,\rho_L}$ as follows. We define

$$G(\tau):=\sum_{r\in Z/12Z}\chi_{12}(r)\eta(\tau)^{-1}\mathbf{e}_r. \quad (4.42)$$

Using the transformation law of the eta-function under $\tau\mapsto\tau+1$ and $\tau\mapsto -1/\tau$, it is easily checked that $G\in M^!_{-1/2,\rho_L}$. The principal part of G is equal to $q^{-1/24}\left(\mathbf{e}_1-\mathbf{e}_{-5}-\mathbf{e}_7+\mathbf{e}_{11}\right)$. On the other hand, G can be obtained as a theta lift. Let $F\in M^!_{-2}(6)$ be the function defined in the following expression

$$F(z):=\frac{1}{2}\cdot\frac{E_2(z)-2E_2(2z)-3E_2(3z)+6E_2(6z)}{\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2}=q^{-1}-10-29q-\ldots \quad (4.42b)$$

It is invariant under the Fricke involution W_6, and under the Atkin-Lehner involution W_3 it is taken to its negative. Hence, in terms of Poincaré series we have

$$F=F_1(\cdot,2,-2)-F_1(\cdot,2,-2)|W_2-F_1(\cdot,2,-2)|W_3+F_1(\cdot,2,-2)|W_6. \quad (4.43)$$

The function P is given by $\frac{1}{4\pi}R_{-2}(F)$. Using Corollary 1.5 and Proposition 1.1, we see that $\Lambda(\tau,F)$ is an element of $M^!_{-1/2,\rho_L}$ with principal part $-4Nq^{-1/24}\left(\mathbf{e}_1-\mathbf{e}_{-5}-\mathbf{e}_7+\mathbf{e}_{11}\right)$. Consequently, we have

$$G=-\frac{1}{4N}\cdot\Lambda(\tau,F). \quad (4.44)$$

Now Theorem 1.6 tells us that for any positive integer n the coefficient of G with index $\left(\frac{24n-1}{24},1\right)$ is equal to

$$\frac{3}{N(24n-1)} \sum_{z \in Z\left(\frac{24n-1}{24},1\right)} P(z) = \frac{1}{24n-1} \sum_{Q \in \mathcal{Q}_n} P(\alpha_Q). \quad (4.45)$$

On the other hand, this coefficient is equal to $p(n)$ because

$$\frac{q^{\frac{1}{24}}}{\eta(z)} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n) q^n . \quad (4.46)$$

We have proved, with all these computations and proofs, that $p(n) = Tr(n)/(24n-1)$. To complete the proof of Theorem 1.3, we require results from the theory of complex multiplication, and some new general results which bound the denominators of singular moduli.
We first recall classical facts about Klein's j-function

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + ... \quad (4.47)$$

A point $\tau \in H$ is a CM point if it is a root of a quadratic equation over Z. The singular moduli for $j(z)$, its values at such CM points, play a central role in the theory of complex multiplication.

Theorem 1.7

Suppose that $Q = ax^2 + bxy + cy^2$ is a primitive positive definite binary quadratic form with discriminant $D = b^2 - 4ac < 0$ and let $\alpha_Q \in H$ be the point for which $Q(\alpha_Q, 1) = 0$. Then the following are true: (1) We have that $j(\alpha_Q)$ is an algebraic integer, and its minimal polynomial has degree h(D), the class number of discriminant D positive definite binary quadratic forms. (2) The Galois orbit of $j(\alpha_Q)$ consists of the j(z)-singular moduli associated to the h(D) classes of discriminant D forms. (3) If K = Q(√D), then the discriminant D singular moduli are conjugate to one another over K. Moreover, $K(j(\alpha_Q))$ is the discriminant – D Hilbert class field of K.

Theorem 1.7 and the properties of the weight 2 nonholomorphic Eisenstein series

$$E_2^*(z) := -\frac{3}{xy} + E_2(z) = 1 - \frac{3}{xy} - 24 \sum_{n=1}^{\infty} \sum_{d|n} dq^n \quad (4.48)$$

will play a cental role in the proof of Theorem 1.3.
For a positive integer N, we let ζ_N denote a primitive N-th root of unity. For a discriminant – D < 0 and r∈ Z with $r^2 \equiv -D \pmod{4N}$ we let $Q_{D,r,N}$ denote the set of positive definite integral binary quadratic forms [a,b,c] of discriminant – D with N | a and b ≡ r (mod 2N).
For Q = [a,b,c]∈ $Q_{D,r,N}$ we let $\alpha_Q = \frac{-b+\sqrt{-D}}{2a}$ be the corresponding Heegner point in H. We write O_D for the order of discriminant – D in Q(√-D).

Theorem 1.8

Let D > 0 be coprime to 6 and $r \in Z$ with $r^2 \equiv -D \pmod{24}$. If $Q \in Q_{D,r,6}$ is primitive, then $6D \cdot P(\alpha_Q)$ is an algebraic integer contained in the ring class field corresponding to the order $O_D \subset Q(\sqrt{-D})$.

By Theorem 1.8, the multiset of values $P(\alpha_Q)$ is a union of Galois orbits. Thereofore, Theorem 1.8 completes the proof of Theorem 1.3.

Conclusion

The main goal of this paper is that the "fractal" behaviour of the partition numbers can be the motivation whose very values of the equations concerning the string theory are connected with (*1,375) that is the mean value (σ factor) of the partition numbers.

L'angolo aureo è di ca. 137,5 Gradi. Il valore esatto è $360/\text{Phi}^2$ = 137,50776405. Il sistema musicale contiene esattamente questo valore con il numero puro di 1,90983006 moltiplicato per 72Hz, la frequenza base che è stata scelta.
C'è anche una connessione con PiGreco in quanto 432/PiGreco = 137,509870831 quasi identico al valore dell'angolo aureo. Notiamo che entrambi i valori 137,5077 e 137,5098 divisi per 100 forniscono 1,375077 e 1,375098 cioè ancora una volta il coefficiente medio (fattore σ) del numero delle partizioni.

Quindi, tutto è connesso: Pigreco, Phi e Sigma che si trovano tutti nel sistema musicale in base Phi e da essi scaturiscono le "note", quindi le vibrazioni "auree" dell'universo delle stringhe.

Appendix A. **(Francesco Di Noto) [7]**

Now we want to analyze the following 4 series of numbers that we have in various equations of the third Section of this paper. We show the mathematical connections with F, 2T, W, i.e. with the Fibonacci's numbers, the triangular numbers and the Witten's numbers.

First series: 3, 4, 8, 32, 40, 44, 45, 73, 630, 887, 14175

Second series: 1, 2, 3, 4, 7, 16, 21, 23, 25, 27, 54, 166, 1605, 43065

Third series: 1, 2, 3, 4, 5, 6, 7, 8, 11, 16, 32, 45, 61, 64, 81, 104, 128, 192, 512, 1024, 6144

Fourth series: 1, 2, 3, 5, 9, 12, 16, 32, 45, 64, 96, 232, 256, 1575, 3614, 20416

1°serie	2°serie	3°serie	4°serie	Fibonacci	Triangular	2T	W
	1	1	1	1	**1**	**2**	2
	2	2	2	2	**3**	**4**	**4**
3	3	3	3	3	**6**	**12**	**7**
4	4	4		5	10	20	**8**
	7	5	5	8	15	30	14
8		6	**9**	13	**21**	42	**16**
32	16	7	12	21	28	56	**21**
	21	8	16	34≈32	36	72	**32**
40	23	11	32	55≈54	**45**	90	105
44	25	16	45	89≈96	55	110	154

45	**27**	32	64	144≈166	66	112	175
73	54	45	96	233≈232	78	156	**256**
630	166	61	232	377	91	182	945
887	1605	64	256	610≈630	105	210	4096
14175	43065	**81**	1575	987≈887	120	240	8085
		104	3614	1597≈1605	136	272	10493
		128	20416	2584	153	306	74247
		192		4181	171	342	363825
		512		6765	190	380	
		1024		10946	210	420	
		6144		17711	231	462	
				28657	252	504	
				46368	275	550	
				75025	299	598	
				121393	324	648	
				196418	350	700	
				317811	377	754	
					405	810	
					434	868	
					464	928	
					495	990	
					527	1054	
					560	1120	
					594	1188	
					629	1258	
					665	1330	
					702	1404	
					740	1480	
					779	1558	
					819	1638	
					860	1720	
					902	1804	
					945	1890	
					989	1978	
					1034	2068	
					1080	2160	
					1127	2254	
					1175	2350	
					1224	2448	
					1274	2548	
					1325	2650	
					1377	2754	
					1430	2860	
					1484	2968	
					1539	3078	
					1595	3190	
					1652	3304	
					1710	3420	
					1769	3538	

					1829	3658	
					1890	3780	
					1952	3904	
					2015	4030	
					2079	4158	

In **blue** we have the numbers that are in two or most series: they are powers of 2: 2, 4, 8, 16, 32, 64.In **red** we have other powers of 2: 128, 256, 512, 1024. In **green** some powers of 3 (**9, 27, 81**) not repeated. In **clear brown** the Witten's numbers W that are in some of the four series : **2, 4, 7, 8, 16, 21, 32, 256** : that are all powers of 2 (except 7 and 21 =7*3) and with index 1, 2, 3, 4, 5, 8, that are all Fibonacci's numbers except 4.
All the numbers of the four series almost coincide with each other, and also with the Fibonacci' numbers, the triangular numbers T and 2T, and the Witten's numbers, after thin out more. It would be interesting to compare their graphics, similar at the begin and after more and more divergent.

Fibonacci's numbers:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 , 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811

The numbers in blue, coincide exactly with the Fibonacci's numbers, furthermore, also the other numbers of the four Tables are algebraic sums of Fibonacci's numbers. Indeed:

4 = 3 + 1; 6 = 5 + 1; 7 = 5 + 2; 9 = 8 + 1; 11 = 3 + 8; 12 = 5 + 5 +2; 16 = 3 + 13; 32 = 21 + 8 + 3; 45 = 34 + 8 + 3; 61 = 55 + 5 + 1; 64 = 55 + 8 + 1; 81 = 89 – 8; 96 = 89 + 5 + 2; 104 = 89 + 13 + 2; 128 = 89 + 34 + 5; 192 = 144 + 34 + 13 + 1; 232 = 233 – 1; 256 = 233 + 21 + 2; 512 = 2 x 256; 1024 = 2 x 512; 1575 = 1597 – 21 – 1; 3614 = 2584 + 987 + 34 + 8 + 1;
6144 = 6765 – 610 – 8 – 3; 20416 = 17711 + 1597 + 987 + 89 + 21 + 8 + 3.

23 = 21 + 2; 25 = 21 +3 + 1; 27 = 21 + 5 + 1; 40 = 34 + 5 + 1; 44 = 34 + 8 + 2; 54 = 55 – 1; 73 = 55 + 13 + 5; 166 = 144 + 21 + 1; 630 = 610 + 13 + 5 + 2; 887 = 987 – 89 – 8 – 3; 1605 = 1597 + 8; 14175 = 10946 + 2584 + 610 + 34 + 1; 43065 = 46368 – 2584 – 610 – 89 – 13 – 5 – 2.

Also with the Witten's numbers can be done the same reasoning.
Indeed, for example, we have that:

1605 = 945 + 256 + 256 + 105 + 32 + 7 + 4; 1575 = 945 + 256 + 256 + 32 + 32 + 21 + 21 + 8 + 4; 6144 = 4096 + 945 + 945 + 154 + 4; 3614 = 4096 – 256 – 175 – 32 – 8 – 7 – 4.

Also here, thence, numbers of the Tables that are algebraic sums of Witten's numbers.

In conclusion, also for the Triangular numbers can be done the same reasoning of the algebraic sums. Indeed:

166 = 110 + 56; 1605 = 1558 + 45 + 2; 3614 = 3538 + 72 + 4;
6144 = 4158 + 1890 + 90 + 6.

Also here, thence, numbers of the Tables that are algebraic sums of triangular numbers T and 2T

Partitions of numbers, p(n)

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, 6842, 8349, 10143, 12310, 14883, 17977, 21637, 26015, 31185, 37338, 44583, 53174, 63261, 75175, 89134

Numbers 4° series coinciding with p(n)- **1, 2, 3, 5, 1575**

Numbers 4° series very near to p(n)   **12**   =**11** + *1*
16 = **15** + *1*
32= **30** + *2*
45 = **42** + *3*
…
232 = **231** +*1*
3614 = **3718 - 104** (104 = **101 + *3***)
20416 = **21637**- 20416= 1221 = (1221= **1255 –** *34*

We note that, five numbers of the 4° series are also numbers p(n) (1,2,3,5,1575) while the others (12,16,32,45,232,3614 and 201416) are very near to others p(n) bigger and with differences corresponding to small Fibonacci's numbers)

Table

Numbers 4°serie	**p(n)**	**d =. N 4° serie -P(n)**	**Fibonacci**
1	**1**	**0**	**yes**
2	**2**	**0**	**yes**
3	**3**	**0**	**yes**
5	**5**	**0**	**yes**
12	**11**	***1***	**yes**
16	**15**	***1***	**yes**
32	**30**	***2***	**yes**
45	**42**	***3***	**yes**
232	**231**	***1***	**yes**
1575	**1575**	***0***	**yes**
3614	**3718**	-104	**104 -101 = *3***
20416	**21637**	-1221	1221**-1255=-*34***

101 and 1255 are p(n) numbers (see series at the begin of the present page)

All this look very interesting and not casual...

Connections between prime numbers, Lie's numbers, Triangular numbers T, 2T, Fibonacci's numbers, partitions of numbers, String Theory

There exist various mathematical connections that concerning the prime numbers and the String Theory. Indeed, we have as follows:

Prime Numbers:

In the equation of the projective geometry n^2+n+1, the number n must be prime or power of prime

This equation is also the equation of the Lie's numbers $L(n) = n^2+n+1$, to the base of the Lie's groups:

$L(2) = 2^2+2+1 =$ **7**, 2*7 = **14** number of dimension of the Lie group G2
$L(3) = 3^2+3+1 =$ **13**, 4*13 = **52** number of dimension of the Lie group F4
6*13 = **78** number of dimension of the Lie group E6
$L(5) = 5^2+5+1 =$ **31**= 8*31 =**248** number of dimens. of the Lie group E8
$L(11) = 11^2+11+1 = 133$ **133** number of dimens. of the Lie group E7

The equation $L(n) = n^2+n+1 = 2T$, where T are the **triangular numbers** (binomial coefficients (the second diagonal of the Tartaglia triangle)) the Lie's prime numbers 7, 13, 31 are of the form 6k+1, and also 133 = 7*19 with also 19 of the form 6k+1, and this suggests that Nature may prefer this form with respect to the form 6k -1 (both the forms are the forms of the prime numbers, except the 2 and 3, for many values of k.

We note that 2T is also the **sum of the first n even numbers**

L(n) = 2T+1 is to the base also of the Fibonacci's numbers F and also of the partitions of the numbers p(n), that are both in many natural phenomena (quantistics and cosmologic), giving them some stability and regularity (especially in the processes of growth).
The prime numbers are the base of the Riemann zeta function, that is connected to the string theory, as also the Fibonacci's numbers (connected to the zeta of Fibonacci, thence also connected to the string theory).

These **five exceptional Lie's groups** are to the base of the string theory, and are the **groups of simmetry**, especially E8, candidate in whole or in part to the Teory of Everything (see Garrett Lisi).

Appendix B **(Christian Lange)**

Here, we have showed the various columns concerning the **universal music system based on Phi**

Sistema	*PiGreco	*1/PiGreco	*1,375
0,0131556174964	0,0413295912802	0,0041875631080	0,0180889740576
0,0135510312596	0,0425718202538	0,0043134272179	0,0186326679820
0,0142348954757	0,0447202430511	0,0045311079587	0,0195729812791

0,0143953404412	0,0452242957761	0,0045821791774	0,0197935931067
0,0150749962219	0,0473594973837	0,0047985203316	0,0207281198051
0,0155281000757	0,0487829651221	0,0049427477677	0,0213511376041
0,0162612375116	0,0510861843046	0,0051761126615	0,0223592015784
0,0167499958021	0,0526216637597	0,0053316892574	0,0230312442279
0,0172744085295	0,0542691549314	0,0054986150129	0,0237523117281
0,0177385302102	0,0557272361938	0,0056463495323	0,0243904790390
0,0186337200909	0,0585395581465	0,0059312973213	0,0256213651249
0,0191937872550	0,0602990610349	0,0061095722366	0,0263914574756
0,0200999949626	0,0631459965116	0,0063980271088	0,0276374930735
0,0203265468895	0,0638577303808	0,0064701408269	0,0279490019730
0,0212862362522	0,0668726834325	0,0067756194387	0,0292685748468
0,0219260291607	0,0688826521337	0,0069792718466	0,0301482900960
0,0230325447060	0,0723588732418	0,0073314866839	0,0316697489707
0,0232921501136	0,0731744476831	0,0074141216516	0,0320267064062
0,0243918562674	0,0766292764569	0,0077641689923	0,0335388023676
0,0251249937032	0,0789324956395	0,0079975338860	0,0345468663419
0,0263112349929	0,0826591825604	0,0083751262159	0,0361779481152
0,0271020625193	0,0851436405077	0,0086268544359	0,0372653359640
0,0279505801363	0,0878093372197	0,0088969459819	0,0384320476874
0,0287015447905	0,0901685622606	0,0091359854556	0,0394646240870
0,0301499924438	0,0947189947674	0,0095970406632	0,0414562396103
0,0310562001514	0,0975659302441	0,0098854955355	0,0427022752082
0,0325224750231	0,1021723686092	0,0103522253230	0,0447184031568
0,0328890437411	0,1033239782005	0,0104689077699	0,0452224351440
0,0344418537486	0,1082022747127	0,0109631825467	0,0473575489044
0,0354770604203	0,1114544723875	0,0112926990645	0,0487809580780
0,0372674401817	0,1170791162929	0,0118625946426	0,0512427302498
0,0376874905548	0,1183987434592	0,0119963008291	0,0518202995129
0,0394668524893	0,1239887738406	0,0125626893239	0,0542669221728
0,0406530937789	0,1277154607615	0,0129402816538	0,0558980039460
0,0425724725044	0,1337453668650	0,0135512388774	0,0585371496936
0,0438520583214	0,1377653042674	0,0139585436932	0,0602965801919
0,0452249886658	0,1420784921511	0,0143955609949	0,0621843594154
0,0464400750007	0,1458957984544	0,0147823349878	0,0638551031260
0,0487837125347	0,1532585529139	0,0155283379845	0,0670776047352
0,0502499874064	0,1578649912790	0,0159950677721	0,0690937326838
0,0526224699857	0,1653183651208	0,0167502524319	0,0723558962303
0,0532155906305	0,1671817085813	0,0169390485968	0,0731714371170
0,0557280900008	0,1750749581452	0,0177388019854	0,0766261237512
0,0574030895811	0,1803371245212	0,0182719709111	0,0789292481740
0,0602999848877	0,1894379895348	0,0191940813265	0,0829124792206
0,0609796406684	0,1915731911423	0,0194104224807	0,0838470059190
0,0638587087566	0,2006180502975	0,0203268583162	0,0878057245404
0,0657780874821	0,2066479564010	0,0209378155398	0,0904448702879
0,0688837074973	0,2164045494254	0,0219263650934	0,0947150978087

0,0709541208407	0,2229089447751	0,0225853981291	0,0975619161560
0,0731755688021	0,2298878293708	0,0232925069768	0,1006164071028
0,0751416197912	0,2360643607150	0,0239183204434	0,1033197272129
0,0789337049786	0,2479775476812	0,0251253786478	0,1085338443455
0,0813061875578	0,2554309215231	0,0258805633076	0,1117960078920
0,0851449450088	0,2674907337301	0,0271024777549	0,1170742993871
0,0861046343716	0,2705056867818	0,0274079563667	0,1183938722609
0,0901699437495	0,2832772328580	0,0287019845321	0,1239836726555
0,0928801500014	0,2917915969087	0,0295646699757	0,1277102062519
0,0975674250694	0,3065171058277	0,0310566759691	0,1341552094704
0,0986671312232	0,3099719346015	0,0314067233097	0,1356673054319
0,1033255612459	0,3246068241382	0,0328895476401	0,1420726467131
0,1064311812610	0,3343634171626	0,0338780971936	0,1463428742339
0,1114561800017	0,3501499162905	0,0354776039708	0,1532522475023
0,1148061791621	0,3606742490424	0,0365439418223	0,1578584963479
0,1184005574678	0,3719663215219	0,0376880679717	0,1628007665183
0,1215816947919	0,3819601591693	0,0387006554313	0,1671748303389
0,1277174175133	0,4012361005951	0,0406537166323	0,1756114490807
0,1315561749643	0,4132959128021	0,0418756310796	0,1808897405758
0,1377674149945	0,4328090988509	0,0438527301867	0,1894301956175
0,1393202250021	0,4376873953631	0,0443470049635	0,1915653093779
0,1458980337503	0,4583521910032	0,0464407865175	0,2006097964067
0,1502832395825	0,4721287214299	0,0478366408868	0,2066394544259
0,1578674099571	0,4959550953625	0,0502507572956	0,2170676886910
0,1596467718916	0,5015451257439	0,0508171457904	0,2195143113509
0,1671842700025	0,5252248744357	0,0532164059562	0,2298783712535
0,1722092687432	0,5410113735636	0,0548159127334	0,2367877445219
0,1803398874990	0,5665544657159	0,0574039690642	0,2479673453111
0,1857603000028	0,5835831938175	0,0591293399514	0,2554204125039
0,1915761262699	0,6018541508926	0,0609805749485	0,2634171736211
0,1967233145832	0,6180245198843	0,0626189758747	0,2704945575518
0,2066511224918	0,6492136482763	0,0657790952801	0,2841452934262
0,2128623625221	0,6687268343252	0,0677561943872	0,2926857484679
0,2229123600034	0,7002998325809	0,0709552079416	0,3065044950046
0,2254248593737	0,7081930821449	0,0717549613302	0,3099591816388
0,2360679774998	0,7416294238611	0,0751427710496	0,3245934690622
0,2431633895839	0,7639203183387	0,0774013108625	0,3343496606778
0,2554348350265	0,8024722011902	0,0813074332647	0,3512228981614
0,2583139031148	0,8115170603454	0,0822238691001	0,3551816167828
0,2705098312484	0,8498316985739	0,0861059535963	0,3719510179666
0,2786404500042	0,8753747907262	0,0886940099270	0,3831306187558
0,2917960675006	0,9167043820064	0,0928815730350	0,4012195928134
0,3005664791649	0,9442574428599	0,0956732817736	0,4132789088518
0,3099766837377	0,9738204724145	0,0986686429202	0,4262179401393
0,3183050093751	0,9999846790536	0,1013196313059	0,4376693878907
0,3343685400051	1,0504497488714	0,1064328119124	0,4597567425069

0,3444185374863	1,0820227471272	0,1096318254669	0,4735754890437
0,3606797749979	1,1331089314318	0,1148079381284	0,4959346906221
0,3647450843758	1,1458804775080	0,1161019662938	0,5015244910167
0,3819660112501	1,1999816148643	0,1215835575671	0,5252032654689
0,3934466291663	1,2360490397686	0,1252379517493	0,5409891151037
0,4133022449836	1,2984272965527	0,1315581905602	0,5682905868525
0,4179606750063	1,3130621860893	0,1330410148906	0,5746959281337
0,4376941012510	1,3750565730096	0,1393223595525	0,6018293892201
0,4508497187474	1,4163861642898	0,1435099226605	0,6199183632776
0,4721359549996	1,4832588477223	0,1502855420992	0,6491869381244
0,4863267791677	1,5278406366773	0,1548026217250	0,6686993213556
0,5015528100076	1,5756746233071	0,1596492178687	0,6896351137604
0,5150283239582	1,6180091989379	0,1639386071806	0,7081639454426
0,5410196624969	1,6996633971477	0,1722119071926	0,7439020359332
0,5572809000084	1,7507495814524	0,1773880198541	0,7662612375116
0,5835921350013	1,8334087640128	0,1857631460700	0,8024391856267
0,5901699437495	1,8540735596529	0,1878569276240	0,8114836726555
0,6180339887499	1,9416110387255	0,1967263286167	0,8497967345311
0,6366100187502	1,9999693581072	0,2026392626118	0,8753387757815
0,6687370800101	2,1008994977428	0,2128656238249	0,9195134850139
0,6762745781211	2,1245792464347	0,2152648839907	0,9298775449165
0,7082039324994	2,2248882715834	0,2254283131488	0,9737804071866
0,7294901687516	2,2917609550160	0,2322039325875	1,0030489820334
0,7639320225002	2,3999632297287	0,2431671151342	1,0504065309378
0,7868932583326	2,4720980795372	0,2504759034987	1,0819782302074
0,8115294937453	2,5494950957216	0,2583178607888	1,1158530538998
0,8333333333333	2,6179938779915	0,2652582384865	1,1458333333333
0,8753882025019	2,7501131460191	0,2786447191050	1,2036587784401
0,9016994374948	2,8327723285796	0,2870198453209	1,2398367265553
0,9442719099992	2,9665176954446	0,3005710841984	1,2983738762489
0,9549150281253	2,9999540371608	0,3039588939177	1,3130081636722
1,0000000000000	3,1415926535898	0,3183098861838	1,3750000000000
1,0300566479165	3,2360183978758	0,3278772143612	1,4163278908852
1,0820393249937	3,3993267942955	0,3444238143851	1,4878040718663
1,0942352531274	3,4376414325239	0,3483058988813	1,5045734730501
1,1458980337503	3,5999448445930	0,3647506727013	1,5756097964067
1,1803398874990	3,7081471193057	0,3757138552480	1,6229673453111
1,2360679774998	3,8832220774509	0,3934526572334	1,6995934690622
1,2732200375004	3,9999387162144	0,4052785252237	1,7506775515630
1,3130823037529	4,1251697190287	0,4179670786575	1,8054881676602
1,3483616572916	4,2360030769294	0,4291968456671	1,8539972787759
1,4164078649987	4,4497765431669	0,4508566262976	1,9475608143733
1,4589803375032	4,5835219100319	0,4644078651750	2,0060979640669
1,5278640450004	4,7999264594573	0,4863342302684	2,1008130618756
1,5450849718747	4,8540275968137	0,4918158215417	2,1244918363278
1,6180339887499	5,0832036923152	0,5150362148005	2,2247967345311

1,6666666666667	5,2359877559830	0,5305164769730	2,2916666666667
1,7507764050038	5,5002262920383	0,5572894382100	2,4073175568802
1,7705098312484	5,5622206789585	0,5635707828720	2,4344510179666
1,8541019662497	5,8248331161764	0,5901789858501	2,5493902035933
1,9098300562505	5,9999080743217	0,6079177878355	2,6260163273445
2,0000000000000	6,2831853071796	0,6366197723676	2,7500000000000
2,0601132958330	6,4720367957516	0,6557544287223	2,8326557817704
2,1246117974981	6,6746648147503	0,6762849394464	2,9213412215599
2,1816949906249	6,8539969549208	0,6944550841536	2,9998306121092
2,2917960675006	7,1998896891860	0,7295013454026	3,1512195928134
2,3606797749979	7,4162942386115	0,7514277104960	3,2459346906221
2,4721359549996	7,7664441549019	0,7869053144668	3,3991869381244
2,5000000000000	7,8539816339745	0,7957747154595	3,4375000000000
2,6180339887499	8,2247963459050	0,8333461009843	3,5997967345311
2,6967233145832	8,4720061538587	0,8583936913341	3,7079945575518
2,8328157299975	8,8995530863337	0,9017132525952	3,8951216287466
2,8647450843758	8,9998621114824	0,9118766817532	3,9390244910167
3,0000000000000	9,4247779607693	0,9549296585514	4,1250000000000
3,0901699437495	9,7080551936273	0,9836316430835	4,2489836726555
3,2360679774998	10,1664073846305	1,0300724296010	4,4495934690622
3,3333333333333	10,4719755119659	1,0610329539460	4,5833333333333
3,4376941012510	10,7998345337790	1,0942520181039	4,7268293892201
3,5300566479165	11,0900000318502	1,1236519298206	4,8538278908851
3,7082039324994	11,6496662323528	1,1803579717002	5,0987804071866
3,8196601125011	11,9998161486433	1,2158355756710	5,2520326546890
4,0000000000000	12,5663706143591	1,2732395447352	5,5000000000000
4,0450849718747	12,7080092307881	1,2875905370012	5,5619918363277
4,2360679774998	13,3080000382202	1,3483823157848	5,8245934690622
4,3633899812498	13,7079939098417	1,3889101683071	5,9996612242185
4,5835921350013	14,3997793783719	1,4590026908052	6,3024391856267
4,6352549156242	14,5620827904409	1,4754474646252	6,3734755089832
4,8541019662497	15,2496110769457	1,5451086444014	6,6743902035933
5,0000000000000	15,7079632679489	1,5915494309190	6,8750000000000
5,2360679774998	16,4495926918100	1,6666922019685	7,1995934690622
5,3934466291663	16,9440123077174	1,7167873826683	7,4159891151036
5,5623058987491	17,4744993485292	1,7705369575502	7,6481706107799
5,7117516385413	17,9439969867709	1,8181070139742	7,8536585029943
6,0000000000000	18,8495559215387	1,9098593171027	8,2500000000000
6,1803398874990	19,4161103872547	1,9672632861669	8,4979673453111
6,4721359549995	20,3328147692609	2,0601448592019	8,8991869381244
6,5450849718747	20,5619908647625	2,0833652524607	8,9994918363277
6,8541019662496	21,5327963841251	2,1817284167690	9,4243902035932
7,0601132958329	22,1800000637003	2,2473038596412	9,7076557817703
7,4164078649987	23,2993324647056	2,3607159434003	10,1975608143733
7,4999999999999	23,5619449019232	2,3873241463784	10,3124999999999
7,8541019662496	24,6743890377149	2,5000383029528	10,7993902035932

8,0901699437494	25,4160184615762	2,5751810740024	11,1239836726555
8,4721359549995	26,6160000764404	2,6967646315695	11,6491869381243
8,7267799624996	27,4159878196832	2,7778203366142	11,9993224484369
9,0000000000000	28,2743338823080	2,8647889756541	12,3749999999999
9,2418082864578	29,0339970186210	2,9417589437948	12,7074863938794
9,7082039324993	30,4992221538913	3,0902172888029	13,3487804071865
10,0000000000000	31,4159265358978	3,1830988618379	13,7500000000000
10,4721359549995	32,8991853836199	3,3333844039371	14,3991869381243
10,5901699437494	33,2700000955505	3,3709557894619	14,5614836726554
11,0901699437493	34,8407964223452	3,5301107325537	15,2489836726553
11,4235032770827	35,8879939735419	3,6362140279484	15,7073170059887
11,9999999999999	37,6991118430773	3,8197186342055	16,4999999999999
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