# On some mathematical connections between the Cubic Equation and some sectors of String Theory and Relativistic Quantum Gravity 

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#### Abstract

In this paper we have described some interesting mathematical connections with various expressions of some sectors of String Theory and Relativistic Quantum Gravity, principally with the Palumbo-Nardelli model applied to the bosonic strings and the superstrings, and some parts of the theory of the Cubic Equation. In Appendix A, we have described the mathematical connections with some equations concerning the possible Relativistic Theory of Quantum Gravity. In conclusion In Appendix B, we have described a proof of Fermat's Last Theorem for the cubic equation case $\mathrm{n}=$ 3


## Cubic Equation

The most general cubic equation (equation of third degree):

$$
\begin{equation*}
x^{3}+r x^{2}+p x+q=0 \tag{1}
\end{equation*}
$$

can be reduced in the following form:

$$
\begin{equation*}
x^{3}+p x+q=0 \tag{2}
\end{equation*}
$$

Indeed:

$$
\begin{align*}
& (x-u)(x-v)(x-z)=\left(x^{2}-v x-u x+u v\right)(x-z)= \\
= & \left.x^{2}-(u+v) x+u v\right) \mid(x-z)=x^{3}-(u+v) x^{2}+u v x-z x^{2}+z(u+v) x-z u v= \\
= & x^{3}-(u+v+z) x^{2}+[u v+z(u+v)] x-z u v . \quad(2 \mathrm{~b}) \tag{2b}
\end{align*}
$$

Thence:

$$
\begin{equation*}
x^{3}-(u+v+z) x^{2}+[u v+z(u+v)] x-z u v=0 \tag{3}
\end{equation*}
$$

Putting:

$$
\left\{\begin{array}{l}
u+v+z=-r  \tag{4}\\
z(u+v)+u v=p \\
u v z=-q
\end{array}\right.
$$

because the term $r$ become null, is necessary and sufficient that $z=-(u+v)$. Thence:

$$
\begin{align*}
& x^{3}-(u+v+z) x^{2}+[u v+z(u+v)] x-z u v=0 \\
& x^{3}-[u+v-(u+v)] x^{2}+[u v-(u+v)(u+v)] x-[-(u+v) u v]=0 \\
& x^{3}-\left[u v-(u+v)^{2}\right] x-[-(u+v) u v]=0 \\
& x^{3}-\left[u v-\left(u^{2}+2 u v+v^{2}\right)\right] x+u^{2} v+u v^{2}=0 \\
& x^{3}-\left[u v-u^{2}-2 u v-v^{2}\right] x+u^{2} v+u v^{2}=0 \\
& x^{3}-\left(-u v-u^{2}-v^{2}\right) x+u v(u+v)=0 \\
& \quad x^{3}-\left(u^{2}+u v+v^{2}\right) x+u v(u+v)=0 \tag{5}
\end{align*}
$$

Putting:

$$
\left\{\begin{array}{l}
u^{2}+u v+v^{2}=-p  \tag{6}\\
u v(u+v)=q
\end{array}\right.
$$

the $x^{3}-\left(u^{2}+u v+v^{2}\right) x+u v(u+v)=0$, become:

$$
\begin{equation*}
x^{3}+p x+q=0 \tag{7}
\end{equation*}
$$

From the $u^{2}+u v+v^{2}=0$, we obtain two quadratic equations (of second degree):

$$
\begin{aligned}
& u=\frac{-v \pm \sqrt{v^{2}-4 v^{2}}}{2}=\frac{-v \pm \sqrt{-3 v^{2}}}{2}=-\frac{1}{2} v \pm v \frac{\sqrt{3 i^{2}}}{2}=-v\left(\frac{1}{2} \pm \frac{\sqrt{3 i^{2}}}{2}\right)=-v\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right) ; \\
& \text { 2) } v^{2}+u v+u^{2}=0 ; \\
& v=\frac{-u \pm \sqrt{u^{2}-4 u^{2}}}{2}=\frac{-u \pm \sqrt{-3 u^{2}}}{2}=-\frac{1}{2} u \pm u \frac{\sqrt{3 i^{2}}}{2}=-u\left(\frac{1}{2} \pm \frac{\sqrt{3 i^{2}}}{2}\right)=-u\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right) .
\end{aligned}
$$

Now, we take the eq. (6). We have that:

$$
\left\{\begin{array} { l } 
{ u ^ { 2 } + u v + v ^ { 2 } = - p } \\
{ u = - v ( \frac { 1 } { 2 } \pm \frac { i \sqrt { 3 } } { 2 } ) \quad \text { (7a } ) } \\
{ v = - u ( \frac { 1 } { 2 } \pm \frac { i \sqrt { 3 } } { 2 } ) }
\end{array} \quad \left\{\begin{array}{l}
u v=-u v\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right)^{2} \\
u^{2}=u \cdot u=-u v\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right)^{2} \\
v^{2}=v \cdot v=-u v\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right)^{2}
\end{array}\right.\right.
$$

thence:

$$
\begin{gather*}
-\left[u v\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right)^{2}+u v\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right)^{2}+u v\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right)^{2}\right]=-(u v+u v+u v)=p ; \\
3 u v=-p ; \quad u v=-\frac{p}{3} \tag{8}
\end{gather*}
$$

Indeed, we have, for example:

$$
\begin{aligned}
& \quad-\left[u v\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}+u v\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}+u v\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}\right]=p \\
& -\left[u v\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)+u v\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)+u v\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)\right]= \\
& =-\left[u v\left(\frac{1}{4}+\frac{i \sqrt{3}}{2}-\frac{3}{4}\right)\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)+u v\left(\frac{1}{4}+\frac{i \sqrt{3}}{2}-\frac{3}{4}\right)\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)+u v\left(\frac{1}{4}+\frac{i \sqrt{3}}{2}-\frac{3}{4}\right)\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)\right]= \\
& =-\left[u v\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)+u v\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)+u v\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)\right]= \\
& =-\left[u v\left(\frac{1}{4}+\frac{3}{4}\right)+u v\left(\frac{1}{4}+\frac{3}{4}\right)+u v\left(\frac{1}{4}+\frac{3}{4}\right)\right]= \\
& =-(u v+u v+u v)=p ; \quad 3 u v=-p ; \quad u v=-\frac{p}{3} .
\end{aligned}
$$

Now:

$$
\left\{\begin{array}{l}
u^{2} v+u v^{2}=q  \tag{8b}\\
u=-v\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right) \\
v=-u\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right)
\end{array}\right.
$$

$$
u^{2}(-u)\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right)+v^{2}(-v)\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right)=q
$$

that for the (8b) and for sign + , for example, become:

$$
\begin{align*}
& v^{2}\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2} \cdot v\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)+u^{2}\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2} \cdot u\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)=q \\
& v^{2}\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)(-v)+u^{2}\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)(-u)=q \\
&-v^{3}\left(\frac{1}{4}+\frac{3}{4}\right)-u^{3}\left(\frac{1}{4}+\frac{3}{4}\right)=q ; \quad-v^{3}-u^{3}=q \\
& u^{3}+v^{3}=-q \tag{9}
\end{align*}
$$

Finally, we have:

$$
\left\{\begin{array} { l } 
{ u ^ { 3 } + v ^ { 3 } = - q }  \tag{10}\\
{ u v = - \frac { p } { 3 } }
\end{array} \quad \text { thence: } \quad \left\{\begin{array}{l}
u^{3}+v^{3}=-q \\
u^{3} v^{3}=-\frac{p^{3}}{27}
\end{array}\right.\right.
$$

We obtain the same result putting in the (2) $x=u+v$. Indeed, we have:

$$
\begin{gathered}
x^{3}+p x+q=0 ; \quad x=u+v \\
(u+v)^{3}+p(u+v)+q=0 ; \quad u^{3}+3 u^{2} v+3 u v^{2}+v^{3}+p(u+v)+q=0 \\
u^{3}+v^{3}+3 u v(u+v)+p(u+v)+q=0 ;
\end{gathered}
$$

from this equation, we obtain:

$$
\begin{gather*}
u^{3}+v^{3}=-q ; \quad 3 u v(u+v)=-p(u+v) ; \quad \text { thence, in conclusion: } \\
\left\{\begin{array} { l } 
{ u ^ { 3 } + v ^ { 3 } = - q } \\
{ 3 u v = - p }
\end{array} \quad \left\{\begin{array} { l } 
{ u ^ { 3 } + v ^ { 3 } = - q } \\
{ u v = - \frac { p } { 3 } }
\end{array} \quad \left\{\begin{array}{l}
u^{3}+v^{3}=-q \\
u^{3} v^{3}=-\frac{p^{3}}{27}
\end{array}\right.\right.\right. \tag{11}
\end{gather*}
$$

From the relations (10) and (11), we obtain the sum and the product of $u^{3}$ and $v^{3}$. We have that $u^{3}$ and $v^{3}$ must be roots of the following quadratic equation:

$$
\begin{equation*}
t^{2}+q t-\frac{p^{3}}{27} \tag{12}
\end{equation*}
$$

This equation is defined the "resolving" of the cubic equation (2). Thence, we have:

$$
\begin{aligned}
\frac{-q \pm \sqrt{q^{2}-4(1)\left(-\frac{p^{3}}{27}\right)}}{2} & =\frac{t^{2}+q t-\frac{p^{3}}{27}}{-q \pm \sqrt{q^{2}+\frac{4 p^{3}}{27}}} \begin{aligned}
2 & \frac{q}{2} \pm \sqrt{\frac{1}{4} q^{2}+\frac{1}{4} \cdot \frac{4 p^{3}}{27}}= \\
& =-\frac{q}{2} \pm \sqrt{\frac{q^{2}}{4}+\frac{p^{3}}{27}}
\end{aligned} .
\end{aligned}
$$

Thence, we have that:

$$
\begin{align*}
& t_{1}=u^{3}=-\frac{q}{2}+\sqrt{\frac{q^{2}}{4}+\frac{p^{3}}{27}} \\
& t_{2}=v^{3}=-\frac{q}{2}-\sqrt{\frac{q^{2}}{4}+\frac{p^{3}}{27}} \tag{13}
\end{align*} ;
$$

from this, we obtain:

$$
u=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^{2}}{4}+\frac{p^{3}}{27}}} ;
$$

if $\quad u v=-\frac{p}{3} ; \quad v=-\frac{p}{3 u}$, we obtain:

$$
v=-\frac{p}{3 \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^{2}}{4}+\frac{p^{3}}{27}}}}
$$

From the $x=u+v$, we obtain:

$$
\begin{equation*}
x=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^{2}}{4}+\frac{p^{3}}{27}}}-\frac{p}{3 \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^{2}}{4}+\frac{p^{3}}{27}}}} \tag{14}
\end{equation*}
$$

The expression (14) is the "resolving" formula of the cubic equation (2), from this we have the three roots in correspondence of the three values of the cubic radical. If $p=0$ the eq. (2) become the binomial equation $x^{3}+q=0$, solved from the formula $x=\sqrt[3]{-q}$.
The expression (14) is written generally in the "Cardano" or "Tartaglia" form:

$$
\begin{equation*}
x=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^{2}}{4}+\frac{p^{3}}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^{2}}{4}+\frac{p^{3}}{27}}} . \tag{15}
\end{equation*}
$$

The solution that Cardano gives of the cubic equation of type $x^{3}+p x=q$, leads to the following formula:

$$
\begin{equation*}
x=\sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^{3}+\left(\frac{q}{2}\right)^{2}}+\frac{q}{2}}-\sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^{3}+\left(\frac{q}{2}\right)^{2}}-\frac{q}{2}} ; \tag{15b}
\end{equation*}
$$

while for the cubic equation of type $x^{3}+p x^{2}=q$, we have the following formula:

$$
\begin{equation*}
x=\sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^{2}-\left(\frac{p}{3}\right)^{3}}+\frac{q}{2}}-\sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^{2}-\left(\frac{p}{3}\right)^{3}}-\frac{q}{2}} . \tag{15c}
\end{equation*}
$$

## Mathematical connections with some sectors of string theory

Now we show two examples of solutions of cubic equations and the possible mathematical connections with the "modes" that correspond to the physical vibrations of the bosonic strings by the Ramanujan function.

1) Let

$$
\begin{equation*}
x^{3}-8 x=3 \tag{16}
\end{equation*}
$$

Putting

$$
\begin{gathered}
q=u-v=3 ; \quad\left(\frac{p}{3}\right)^{3}=u v=\left(-\frac{8}{3}\right)^{3}=-\frac{512}{27} ; \text { we apply the formula: } \\
x=\sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^{3}+\left(\frac{q}{2}\right)^{2}}+\frac{q}{2}}-\sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^{3}+\left(\frac{q}{2}\right)^{2}}-\frac{q}{2}}= \\
=\sqrt[3]{\sqrt{-\frac{512}{27}+\frac{9}{4}}+\frac{3}{2}}-\sqrt[3]{\sqrt{-\frac{512}{27}+\frac{9}{4}}-\frac{3}{2}}=\sqrt[3]{\sqrt{\frac{-2048+243}{108}}+\frac{3}{2}}-\sqrt[3]{\sqrt{\frac{-2048+243}{108}}-\frac{3}{2}}= \\
=\sqrt[3]{\sqrt{-\frac{1805}{108}}+\frac{3}{2}}-\sqrt[3]{\sqrt{-\frac{1805}{108}}-\frac{3}{2}}=\left(\sqrt[3]{\sqrt{\frac{1805}{108} i^{2}}+\frac{3}{2}}\right)^{3}-\left(\sqrt[3]{\sqrt{\frac{1805}{108} i^{2}}-\frac{3}{2}}\right)^{3}= \\
=\sqrt{\frac{1805}{108} i^{2}}+\frac{3}{2}-\sqrt{\frac{1805}{108} i^{2}}+\frac{3}{2}=\frac{3}{2}+\frac{3}{2}=\frac{6}{2}=3 .
\end{gathered}
$$

Indeed, for $x=3$, the eq. (16) become:

$$
x^{3}-8 x=3 ; \quad 3^{3}-8 \cdot 3=3 ; \quad 27-24=3 ; \quad 3=3
$$

But, we have also that:

$$
8 x=x^{3}-3 ; \quad 24=x^{3}-3 ; \quad 24=27-3 ; \quad 24=24
$$

We know that the number 24 is related to the "modes" that correspond to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$
\begin{equation*}
24=\frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]}, \tag{a}
\end{equation*}
$$

Thence, we have the following mathematical connection:

$$
\begin{aligned}
& 4\left[\text { anti } \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(t t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}} \\
& \log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]
\end{aligned} x^{3}-3 \quad \text { (17) } \quad \text { for } x=3
$$

A mathematical connection that link the cubic equation $x^{3}-8 x=3$ with the number 24, i.e. with the "modes" that correspond to the physical vibrations of the bosonic strings.
We note also that 3 and 8 , i.e. $p$ and $q$ are also Fibonacci's numbers.
2) Let

$$
\begin{equation*}
x^{3}+8 x=24 . \tag{18}
\end{equation*}
$$

Putting

$$
\begin{gathered}
q=u-v=24 ; \quad\left(\frac{p}{3}\right)^{3}=u v=\left(\frac{8}{3}\right)^{3}=\frac{512}{27} ; \text { for the formula: } \\
x=\sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^{3}+\left(\frac{q}{2}\right)^{2}}+\frac{q}{2}}-\sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^{3}+\left(\frac{q}{2}\right)^{2}}-\frac{q}{2}} ; \quad \text { we obtain: } \\
\sqrt[3]{\sqrt{\frac{512}{27}+\left(\frac{24}{2}\right)^{2}}+\frac{24}{2}}-\sqrt[3]{\sqrt{\frac{512}{27}+\left(\frac{24}{2}\right)^{2}}-\frac{24}{2}}= \\
=\sqrt[3]{\sqrt{\frac{512}{27}+144}+12}-\sqrt[3]{\sqrt{\frac{512}{27}+144}-12}=\sqrt[3]{\sqrt{162,962}+12}-\sqrt[3]{\sqrt{162,962}+12}= \\
=\sqrt[3]{12,765657+12}-\sqrt[3]{12,765657-12}=\sqrt[3]{24,765657}-\sqrt[3]{0,765657} \cong 2,914852-0,914852=2 .
\end{gathered}
$$

Thence, we have:

$$
\begin{aligned}
x^{3}+8 x=24 ; & 2^{3}+8 \cdot 2=24 ; \quad 8+16=24 ; \quad 24=24 ; \\
& 24=x^{3}+8 x ;
\end{aligned}
$$

$$
\frac{4\left[\text { anti } \log \frac{\int_{0}^{\infty} \frac{\cos \pi x w^{\prime}}{\cosh \pi x} e^{-\pi \pi^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]}=x^{3}+8 x ; \quad \text { (19) } \quad \text { for } x=2
$$

The further connection that link this cubic equation with the "modes" that correspond to the physical vibrations of the bosonic strings.

Creation and solution of cubic equation incomplete, without the first degree, defined "Premonica".

Let $u, v$ and $z$ three roots of a same equation. Obviously the equation is cubic and the factorization of it is:

$$
\begin{gather*}
x^{3}+p x^{2}+r x+q=0 \\
(x-u)(x-v)=x^{2}-u x-v x+u v=x^{2}-(u+v) x+u v ; \\
{\left[x^{2}-(u+v) x+u v\right](x-z)=x^{3}-(u+v) x^{2}+u v x-z x^{2}+z(u+v) x-z u v=} \\
=x^{3}-(u+v+z) x^{2}+[u v+z(u+v)] x-u v z ; \quad \text { thence: } \\
x^{3}-(u+v+z) x^{2}+[u v+z(u+v)] x-u v z=0 ; \tag{21}
\end{gather*}
$$

from this, putting

$$
\left\{\begin{array}{l}
u+v+z=-p \\
z(u+v)+u v=r \\
u v z=-q
\end{array}\right.
$$

we obtain the eq. (20).
to cancel the term $r$, necessary and sufficient condition is that:

$$
\begin{align*}
z(u+v)+u v & =0 ; \quad z(u+v)=-u v \quad \text { whence } \\
z & =-\frac{u v}{u+v} . \tag{22}
\end{align*}
$$

The (22) is defined the "fundamental relation". Substituting in the eq. (21) the value found of $z$, we obtain:

$$
\begin{gathered}
x^{3}-(u+v+z) x^{2}+[u v+z(u+v)] x-u v z=0 \\
x^{3}-\left(u+v-\frac{u v}{u+v}\right) x^{2}+\left[u v-\frac{u v}{u+v}(u+v)\right] x-u v\left(-\frac{u v}{u+v}\right)=0 \\
x^{3}-\frac{(u+v)^{2}-u v}{u+v} x^{2}+\frac{u^{2} v^{2}}{u+v}=0 ; \text { that we can rewrite also as follows: }
\end{gathered}
$$

$$
\begin{equation*}
(u+v) x^{3}-\left[(u+v)^{2}-u v\right] x^{2}+u^{2} v^{2}=0 . \tag{23}
\end{equation*}
$$

With this relation we can solve a cubic equation in "Premonica" form, i.e. without the first degree. $x^{3}+p x^{2}+q=0$. For example, $3 x^{3}-7 x^{2}+4=0$, we will use the following system:

$$
\left\{\begin{array} { l } 
{ u + v = 3 } \\
{ ( u + v ) ^ { 2 } - u v = 7 } \\
{ u ^ { 2 } v ^ { 2 } = 4 }
\end{array} \quad \text { (23b) } \quad \left\{\begin{array} { l } 
{ 9 - u v = 7 } \\
{ u ^ { 2 } v ^ { 2 } = 4 }
\end{array} \quad \left\{\begin{array} { l } 
{ u v = 2 } \\
{ u ^ { 2 } v ^ { 2 } = 4 }
\end{array} \quad \left\{\begin{array}{l}
u v=2 \\
u v= \pm 2
\end{array}\right.\right.\right.\right.
$$

The $u v$ will be equal only if $u v=2$, thence

$$
\left\{\begin{array}{l}
u+v=3 \\
u v=2
\end{array} \quad \text { whence } \quad t^{2}-3 t+2=0 ; \quad t=\frac{3 \pm \sqrt{9-8}}{2}=\frac{3 \pm 1}{2}=\left\{\begin{array}{l}
t_{1}=2=u=x_{1} \\
t_{2}=1=v=x_{2}
\end{array}\right.\right.
$$

Because the equation not admit the term of first degree, the third root must be necessarily

$$
z=-\frac{u v}{u+v}=-\frac{1 \cdot 2}{2+1}=-\frac{2}{3}=x_{3}
$$

Now, we have the following cubic equation:

$$
\begin{equation*}
x^{3}-2 x+1=0 \tag{24}
\end{equation*}
$$

We want solve this equation by a system of relations between roots and coefficients. This procedure will be valid for each equation in form "Monica" form, i.e. without the second degree, whatever the coefficients.
Let $u$ and $v$ two of the roots of the equation, the third root will be necessarily $u+v$ and thence we have

$$
\begin{equation*}
x_{1}=u ; \quad x_{2}=v ; \quad x_{3}=u+v ; \tag{25}
\end{equation*}
$$

$$
\left\{\begin{array}{l}
x_{1}=-(u+v)+u=-v  \tag{25b}\\
x_{2}=-(u+v)+v=-u
\end{array}\right.
$$

Now:

$$
\begin{gather*}
\begin{aligned}
&(x-u)(x-v)=x^{2}-(u+v) x+u v ; \\
& {\left[x^{2}-(u+v) x+u v\right][x+(u+v)] }=x^{3}-(u+v) x^{2}+u v x+(u+v) x^{2}-(u+v)^{2} x+u v(u+v)= \\
&=x^{3}-\left\lfloor(u+v)^{2}-u v \mid x+u v(u+v) ; \quad\right. \text { from this, we have: } \\
& x^{3}-\left\lfloor(u+v)^{2}-u v \mid x+u v(u+v)=0 .\right.
\end{aligned}
\end{gather*}
$$

This equation is a cubic equation in the form "Monica", i.e. of type $x^{3}+p x+q=0$, where we put:

$$
\left\{\begin{array}{l}
(u+v)^{2}-u v=-p \\
u v(u+v)=q
\end{array}\right.
$$

Now, we put $u+v=t$ and $u v=z$, we have:

$$
\left\{\begin{array} { l } 
{ t ^ { 2 } - z = - p } \\
{ t z = q }
\end{array} \quad \left\{\begin{array} { l } 
{ t ^ { 2 } = z - p } \\
{ t z = q }
\end{array} \quad \left\{\begin{array}{l}
t= \pm \sqrt{z-p} \\
t z=q
\end{array} \quad t z=q ; \quad \pm z(\sqrt{z-p})=q\right.\right.\right.
$$

we raise all to the square: $z^{2}(z-p)=q^{2}$; from which developing:

$$
\begin{equation*}
z^{3}-p z^{2}=q^{2} ; \quad z^{3}-p z^{2}-q^{2}=0 \tag{27}
\end{equation*}
$$

This last equation is a "Premonica" and we can find $u$ and $v$ applying the Caccioppoli-Grablovitz method (comparison system between roots and coefficients). We resume the eq. (24)

$$
x^{3}-2 x+1=0
$$

Decompose the equation. We have that:

$$
\begin{gathered}
\left(x^{2}+x-1\right)(x-1)=x^{3}-x^{2}+x^{2}-x-x+1=x^{3}-2 x+1 ; \quad \text { thence: } \\
\left(x^{2}+x-1\right)(x-1)=0 ; \quad x-1=0 ; \quad x=1 \\
x^{2}+x-1=0 ; \quad \frac{-1 \pm \sqrt{1-4(1)(-1)}}{2}=\frac{-1 \pm \sqrt{1+4}}{2}=\frac{-1 \pm \sqrt{5}}{2}\left\{\begin{array}{l}
x_{1}=\frac{-1+\sqrt{5}}{2} \\
x_{2}=\frac{-1-\sqrt{5}}{2}
\end{array}\right.
\end{gathered}
$$

The three roots are: $1 ; \frac{-1+\sqrt{5}}{2} ; \frac{-1-\sqrt{5}}{2}$.
Now, $x^{3}-2 x+1=0$ applying the relation (27), we have:

$$
\begin{aligned}
& z^{3}-p z^{2}-q^{2}=0 ; \\
& \left\{\begin{array}{l}
z^{3}-2 z^{2}-1=0 \\
(u+v=1 \\
(u+v)^{2}-u v=2 \\
u^{2} v^{2}=1
\end{array}\right. \\
& \left\{\begin{array}{l}
u+v=1 \\
1-u v=2 \\
u v= \pm 1
\end{array}\right. \\
& \text { whence, from (23b) } \\
& u v= \pm 1
\end{aligned} \quad\left\{\begin{array}{l}
u+v=1 \\
u v=-1 \\
u v=-1
\end{array} \quad\left\{\begin{array}{l}
u+v=1 \\
u v=-1
\end{array}\right]\right.
$$

Thence:

$$
t^{2}-t-1=0 ; \quad \frac{1 \pm \sqrt{1+4}}{2}=\left\{\begin{array}{l}
u=\frac{1+\sqrt{5}}{2} \\
v=\frac{1-\sqrt{5}}{2}
\end{array}\right.
$$

For the relations (25b), the third root will be $x_{3}=u+v=\frac{1}{2}+\frac{\sqrt{5}}{2}+\frac{1}{2}-\frac{\sqrt{5}}{2}=1$. The other two roots, will be: $\quad x_{1}=-u=\frac{-1-\sqrt{5}}{2}$; and $x_{2}=-v=\frac{-1+\sqrt{5}}{2}$.
We note that the $x_{2}=\frac{\sqrt{5}-1}{2}=0,61803398 \ldots$ is equal to the aurea section.

## Mathematical connection with some parts of relativistic quantum gravity.

Now we want to analyze some mathematical connection with some parts of relativistic quantum gravity.
Starting from the Schrodinger equation we may obtain the expression for the energy of a particle in periodic motion inside a cubical box of edge length $L$. The result is:

$$
\begin{equation*}
E_{n}=\frac{n^{2} h^{2}}{8 m_{g} L^{2}}, \quad n=1,2,3, \ldots \tag{28}
\end{equation*}
$$

We note that the term $h^{2} / 8 m_{g} L^{2}$ (energy) will be minimum for $L=L_{\max }$ where $L_{\max }$ is the maximum edge length of a cubical box whose maximum diameter $d_{\max }=L_{\max } \sqrt{3}$ is equal to the maximum length scale of the Universe. Now, we have that $L=n L_{\min }$ or $L=\frac{L_{\max }}{n}$. Multiplying these expressions by $\sqrt{3}$ and reminding that $d=L \sqrt{3}$, we obtain $d=n d_{\min }$ or $d=\frac{d_{\max }}{n}$. These equations show that the length (and therefore the space) is quantized.
The minimum energy of a particle is obviously its inertial energy at rest $m_{g} c^{2}=m_{i} c^{2}$. Therefore we can write

$$
\begin{equation*}
\frac{n^{2} h^{2}}{8 m_{g} L^{2}{ }_{\max }}=m_{g} c^{2} . \tag{29}
\end{equation*}
$$

Then, from the equation above it follows that

$$
\begin{equation*}
m_{g}= \pm \frac{n h}{c L_{\max } \sqrt{8}}, \tag{30}
\end{equation*}
$$

Now, if we take the eq. (29), after some steps, we obtain that:

$$
\begin{equation*}
8=\frac{n^{2} h^{2}}{c^{2} L_{\max }^{2}} \cdot \frac{1}{m_{g}^{2}} . \tag{30b}
\end{equation*}
$$

This formula contain the number 8 that is a Fibonacci's number and is connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$
\begin{equation*}
8=\frac{1}{3} \frac{\left.4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cos 2 x}}{\frac{-\pi x^{2} w^{\prime}}{} d x}\right] \cdot \frac{\sqrt{142}}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right]}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]} . \tag{b}
\end{equation*}
$$

Thence, we can rewrite the eq.(30b) also as follows:

$$
\begin{equation*}
\frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]}=\frac{n^{2} h^{2}}{c^{2} L_{\max }^{2}} \cdot \frac{1}{m_{g}^{2}} . \tag{30c}
\end{equation*}
$$

This is a new possible relationship of a relativistic theory of quantum gravity. (See also Appendix A)

Now, we take the $d_{\max }=L_{\max } \sqrt{3}$. For the (7a) or (8b):

$$
\begin{gather*}
u=-v\left(\frac{1}{2}-\frac{i \sqrt{3}}{2}\right) ; \quad u=-\frac{v}{2}+\frac{i \sqrt{3} v}{2} ; \quad 2 u=-v+i \sqrt{3} v ; \\
2 u+v=i v \sqrt{3}, \tag{31}
\end{gather*}
$$

we have that:

$$
d_{\max }=L_{\max } \sqrt{3} \quad\left\{\begin{array}{l}
d_{\max }=(2 u+v) \\
L_{\max }=i v
\end{array}\right.
$$

Now, we have the following equation:

$$
\begin{equation*}
8=(2 u+v)-\left(\frac{1-\sqrt{5}}{2}\right) \tag{32}
\end{equation*}
$$

We note that $\left(\frac{1-\sqrt{5}}{2}\right)$ is equal to the aurea section changed of sign, i.e. $-0,61803398 \ldots$ or $0,61803398 \ldots \times i^{2}$, where $i$ is the imaginary unit. Furthermore, from the eq. (24), where $x_{2}=-v=\frac{-1+\sqrt{5}}{2}$, we note that $\left(\frac{1-\sqrt{5}}{2}\right)$ is equal to $i^{2} \cdot(-v)$

We can rewrite the above equation also as follows:

$$
d_{\max }-\left(\frac{1-\sqrt{5}}{2}\right)=8 ; \text { whence } d_{\max }=8+\left(\frac{1-\sqrt{5}}{2}\right) \text { or also } L_{\max } \sqrt{3}=8+\left(\frac{1-\sqrt{5}}{2}\right)
$$

In conclusion, we have the following three expressions:

$$
\begin{align*}
& L_{\max } \sqrt{3}=\frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right.}+\left(\frac{1-\sqrt{5}}{2}\right) ;  \tag{33}\\
& \frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi \pi^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]}=L_{\max } \sqrt{3}-\left(\frac{1-\sqrt{5}}{2}\right) ;  \tag{33b}\\
& d_{\max }=\frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi \pi^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]}+\left(\frac{1-\sqrt{5}}{2}\right) ; \tag{34}
\end{align*}
$$

But $\quad d_{\max }=2 u+v ; \quad d_{\max }=-2 v\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right)-u\left(\frac{1}{2} \pm \frac{i \sqrt{3}}{2}\right)$. Thence, we have also the following interesting relationship:

$$
\begin{equation*}
-2 v\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)-u\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)=\frac{1}{3} \frac{4\left[\text { anti } \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right.}+\left(\frac{1-\sqrt{5}}{2}\right) . \tag{35}
\end{equation*}
$$

This last relationship, link the maximum length scale of the Universe, the roots of a cubic equation, the "modes" that correspond to the physical vibrations of a superstring and the aurea section changed of sign, i.e. $-0,61803398 \ldots$ or $0,61803398 \ldots \times i^{2}$, where $i$ is the imaginary unit.

From (35), we have also that:

$$
\begin{equation*}
\frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi \pi^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]}=-2 v\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)-u\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)-\left(\frac{1-\sqrt{5}}{2}\right) . \tag{35b}
\end{equation*}
$$

Multiplying the expression (35) for $\left(\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)$, and since

$$
\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)\left(\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)=\frac{1}{4}+\frac{3}{4}=1 ; \quad\left(i^{4}=1\right)
$$

we obtain the equivalent relationship:

$$
\begin{equation*}
-2 v-u=\left(\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)\left\{\frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]}+\left(\frac{1-\sqrt{5}}{2}\right)\right\} . \tag{36}
\end{equation*}
$$

*We note that, with regard the roots of the cubic equation:

$$
\begin{equation*}
\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}+\left(\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)^{2}=-\frac{1}{2}+\frac{i \sqrt{3}}{2}-\frac{1}{2}-\frac{i \sqrt{3}}{2}=-\frac{2}{2}=-1=i^{2} . \tag{a}
\end{equation*}
$$

Thence:

$$
\begin{equation*}
\sqrt{\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}+\left(\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)^{2}}=\sqrt{i^{2}}=i=\sqrt{-1} \tag{b}
\end{equation*}
$$

## On some equation concerning the thermodynamics of black holes in anti-de sitter space

The metric of the covering space of anti-de Sitter space can be written in the static form
$d s^{2}=-V d t^{2}+V^{-1} d r^{2}+r^{2}\left(d \theta^{2}+\sin ^{2} \theta d \varphi^{2}\right)$
$V=1+\frac{r^{2}}{b^{2}}$
$b \equiv \sqrt{\left(-\frac{3}{\Lambda}\right)}$
The substitution $\tau=i t$, where $i=\sqrt{-1}$ is the imaginary unit, makes the metric $d s^{2}$ in (37) Euclidean, i.e. positive definite.
The path integral over the matter fields and metric fluctuations on the anti-de Sitter background can be regarded as giving the contribution of thermal radiation in anti-de Sitter space to the partition function $Z$. For a conformally invariant field this will be:

$$
\begin{equation*}
\log Z=3 \pi^{2} b^{3} \int_{0}^{T} T^{-2} f(T) d T=\frac{\pi^{4}}{90} g\left(\frac{b}{\beta}\right)^{3}+O\left(\frac{b}{\beta}\right) \tag{38}
\end{equation*}
$$

The energy of the thermal radiation will be:

$$
\begin{equation*}
\langle E\rangle=-\frac{\partial}{\partial \beta} \log Z=3 \pi^{2} b^{3} f(T) \approx \frac{\pi^{4}}{30} g T^{4} b^{3} . \tag{39}
\end{equation*}
$$

The partition function $Z(\beta)$ is the Laplace transform of the density of states $N(E)$,

$$
\begin{equation*}
Z(\beta)=\int_{0}^{\infty} N(E) e^{-\beta E} d E \tag{40}
\end{equation*}
$$

Thus $N(E)$ is the inverse Laplace transform

$$
\begin{equation*}
N(E)=\frac{1}{2 \pi i} \int_{-i \infty}^{+i \infty} Z(\beta) e^{\beta E} d \beta \tag{41}
\end{equation*}
$$

The second derivative of the logarithm of the integrand in (41) is:

$$
\begin{equation*}
\frac{2 \pi^{4}}{15} g b^{3} \beta^{-5} \tag{42}
\end{equation*}
$$

Thus the path of steepest descent is parallel to the imaginary axis. This means that $N(E)$ is real and is given approximately by:

$$
\begin{equation*}
N(E) \cong \exp \left[\frac{4 \pi}{3}\left(\sqrt[4]{\frac{g b^{3}}{30}}\right)\left(\sqrt[4]{E^{3}}\right)\right] \tag{43}
\end{equation*}
$$

This equation will hold for $E<E_{2} \approx m_{p}^{2} b$.
The Euclidean action of a black hole of period $\beta$ is:

$$
\begin{equation*}
I_{ \pm}=\frac{\pi}{9} m_{p}^{2} b^{2}\left[1-\frac{2\left(\beta_{0}^{2}-\beta^{2}\right)\left(\beta_{0} \pm \sqrt{\beta_{0}^{2}-\beta^{2}}\right)}{\beta_{0} \beta^{2}}\right] \tag{44}
\end{equation*}
$$

The one-loop term about the black hole metrics will contribute a factor of order one or $i \exp \left(\frac{\pi^{4}}{90} g b^{3} \beta^{-3}\right)$ respectively. The factor of $i$ arises in the lower mass case from the negative nonconformal mode.
For (b), concerning the roots of a cubic equation, we have that the expression $i \exp \left(\frac{\pi^{4}}{90} g b^{3} \beta^{-3}\right)$ can be rewritten also as follows

$$
\begin{equation*}
\sqrt{\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}+\left(\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)^{2}} \exp \left(\frac{\pi^{4}}{90} g b^{3} \beta^{-3}\right) \tag{44b}
\end{equation*}
$$

In the higher mass case, if $E>M_{0}$, the stationary phase point in eq. (41) will be at

$$
\begin{equation*}
\beta \cong \frac{4 \pi b^{2} r_{+}}{b^{2}+3 r_{+}^{2}} \tag{45}
\end{equation*}
$$

where $r_{+}$is the solution of

$$
\begin{equation*}
E=\frac{1}{2} m_{p}^{2} r_{+}\left(1+\frac{r_{+}^{2}}{b^{2}}\right) . \tag{46}
\end{equation*}
$$

The second derivative of the logarithm of the integrand is $T^{2} \partial M / \partial T>0$. Thus the path of steepest descent will be parallel to the imaginary axis and $N(E)$ will be real and given by

$$
\begin{equation*}
N(E) \cong \exp \left(\pi m_{p}^{2} r_{+}^{2}\right) \cong \exp \left[\pi\left(\sqrt[3]{2 m_{p} b^{2} E}\right)^{2}\right], \text { for } E \gg M_{0}=\frac{1}{\sqrt{3^{3}}} 2 m_{p}^{2} b \tag{47}
\end{equation*}
$$

If $E_{0}<E \ll M_{0}$, the stationary phase point will be at the larger root of

$$
\begin{equation*}
E=M+E_{\text {rad }} \cong \frac{m_{p}^{2}}{8 \pi}+\frac{\pi^{4}}{30} g b^{3} \beta^{-4} \tag{48}
\end{equation*}
$$

where a black hole of energy $M$ is in equilibrium with thermal radiation of energy $E_{\text {rad }}$. Furthermore, we note that we can rewrite the eq. (48) also as follows:
$8\left(\frac{\pi^{5}}{30} g b^{3} \beta^{-4}-\pi\left(M+E_{r a d}\right)\right)=-m_{p}^{2}, \quad$ and remembering the expression (b)

$$
\begin{align*}
& 8=\frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi \pi^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]} \text {, we have that: } \\
& \frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]}\left(\frac{\pi^{5}}{30} g b^{3} \beta^{-4}-\pi\left(M+E_{r a d}\right)\right)=-m_{p}^{2} . \tag{49}
\end{align*}
$$

This expression, for (35b), can be rewritten also as follows:

$$
\begin{equation*}
-2 v\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)-u\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)-\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{\pi^{5}}{30} g b^{3} \beta^{-4}-\pi\left(M+E_{r a d}\right)\right)=i^{2} m_{p}^{2} \tag{49b}
\end{equation*}
$$

where a black hole of energy $M$ is in equilibrium with thermal radiation of energy $E_{r a d}$ and is connected with the "modes" corresponding to the physical vibrations of the superstring for the eq. (49) and with the roots of the cubic equation, thence with the imaginary unit for the eq. (49b). In both the expressions is present $\pi$, the transcendental number always present in the string theory.

The second derivative of the logarithm of the integrand of (41) will be negative at each of these saddle points. Thus the path of steepest descent will be parallel to the real axis. This will introduce a factor $i$ which will cancel the factor of $i$ arising from the negative nonconformal mode. Thus $N(E)$ will be real and will be given by:

$$
\begin{equation*}
N(E) \cong \exp \left[4 \pi m_{p}^{-2} M^{2}+\frac{4 \pi}{3} \sqrt[4]{\frac{g b^{3}}{30}} \sqrt[4]{E_{\text {rad }}^{3}}\right] \tag{50}
\end{equation*}
$$

where $M$ and $E_{\text {rad }}$ are the two terms of (48) that add up to $E$. If $E<E_{0}$, eq. (48) has no solution for a black hole in equilibrium with radiation, so one obtains only the contribution (43) of pure thermal radiation.
For eq. (48), can be rewritten the eq. (50) also as follows:

$$
\begin{align*}
& N(E) \cong \exp \left[4 \pi m_{p}^{-2}\left(\frac{m_{p}^{2} \beta}{8 \pi}\right)^{2}+\frac{4 \pi}{3} \sqrt[4]{\frac{g b^{3}}{30}} \sqrt[4]{E_{\text {rad }}^{3}}\right]= \\
& \quad N(E) \cong \exp \left[\frac{m_{p}^{2} \beta^{2}}{16 \pi}+\frac{4 \pi}{3} \sqrt[4]{\frac{g b^{3}}{30}} \sqrt[4]{E_{\text {rad }}^{3}}\right], \tag{51}
\end{align*}
$$

that for (35b), we can rewrite also as follows:

$$
\begin{equation*}
N(E) \cong \exp \left\{\frac{m_{p}^{2} \beta^{2}}{2 \pi\left[-2 v\left(\frac{1}{2}+\frac{1 \sqrt{3}}{2}\right)-u\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)-\left(\frac{1-\sqrt{5}}{2}\right)\right]}+\frac{4 \pi}{3} \sqrt[4]{\frac{g b^{3}}{30}} \sqrt[4]{E_{\text {rad }}^{3}}\right\} . \tag{52}
\end{equation*}
$$

We have thence, the following two expression:

$$
\begin{align*}
& N(E)=\frac{1}{2 \pi i} \int_{-i \infty}^{+i \infty} Z(\beta) e^{\beta E} d \beta \cong \exp \left[4 \pi m_{p}^{-2}\left(\frac{m_{p}^{2} \beta}{8 \pi}\right)^{2}+\frac{4 \pi}{3} \sqrt[4]{\frac{g b^{3}}{30}} \sqrt[4]{E_{\text {rad }}^{3}}\right] ; \quad(52 \mathrm{a}) \\
& N(E)=\frac{1}{2 \pi i} \int_{-i \infty}^{+i \infty} Z(\beta) e^{\beta E} d \beta \cong \exp \left\{\frac{m_{p}^{2} \beta^{2}}{2 \pi\left[-2 v\left(\frac{1}{2}+\frac{1 \sqrt{3}}{2}\right)-u\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)-\left(\frac{1-\sqrt{5}}{2}\right)\right]}+\frac{4 \pi}{3} \sqrt[4]{\frac{g b^{3}}{30}} \sqrt[4]{E_{\text {rad }}^{3}}\right\} . \tag{52b}
\end{align*}
$$

In the first expression is present 8 , i.e. the number of the "modes" corresponding to the physical vibrations of the superstrings, while in the second expression with the roots of the cubic equation, thence with the imaginary unit. In both the expressions is present $\pi$, the transcendental number always present in the string theory.

We note that there are in the equations of this section, some mathematical connections with some frequencies connected to the aurea ratio i.e. $\frac{\sqrt{5}+1}{2}=1,61803398 \ldots$
We have that:

$$
\begin{array}{ll}
\frac{\pi^{4}}{30}=3,2469 \ldots \cong 3,2461 ; & \frac{2 \pi^{4}}{15}=12,9878=\frac{\pi^{4}}{30} \times 4 ; \quad \frac{\pi^{4}}{90}=1,0823 \cong 1,0820 \\
\frac{4 \pi}{3}=4,18879 \cong 4,18885 ; \quad \frac{\pi}{9}=0,349065 \cong 0,349072,
\end{array}
$$

where 3,2461 12,9878 1,0820 4,18885 and 0,349072 are all frequencies connected with the aurea ratio, i.e. with $\frac{\sqrt{5}+1}{2}=1,61803398 \ldots$

## On some equations concerning the Stephen Hawking's theory of imaginary time and the HartleHawking wave-function

We start from the Pythagoras's Theorem, $\mathrm{c}^{2}=\mathrm{a}^{2}+\mathrm{b}^{2} \rightarrow c=\sqrt{a^{2}+b^{2}}$. Well, it turns out that's is just a specific application of the distance formula in 3D space. Generally, if a point has coordinates $(X, Y, Z)$, then its distance from the origin, the point $(0,0,0)$, is $D^{2}=X^{2}+Y^{2}+Z^{2}$.

Now let's go to relativity. Instead of points in space, we have events in spacetime. If an event has coordinates ( $T, X, Y, Z$ ), then its spacetime interval distance is $S^{2}=-T^{2}+X^{2}+Y^{2}+Z^{2}$. Notice the negative sign for the time component. The sign difference between space and time is crucial for all the weird things that happen in relativity.

But suppose we invent a new variable $t$ that is related to the old T by $\mathrm{T}=\mathrm{it}$, where i is the imaginary unit $(i=\sqrt{-1})$. Then $T^{2}=(i)^{2} t^{2}=-t^{2}$. So the spacetime interval becomes $S^{2}=t^{2}+X^{2}+Y^{2}+Z^{2}$. All the signs are the same.

By using imaginary time, we convert the time dimension into a space dimension indistinguishable from the other spatial dimensions. So spacetime becomes a 4D space. This allow us to study spacetime as a pure geometric object.

Why is this useful? It eliminates the question of "what happened before the Big Bang?" Let's suppose that after transforming spacetime into a geometric spatial object, we get a sphere. So the time dimension becomes just a space dimension on the sphere, say, the direction of north-south. Asking what happened before the beginning of time is thus like asking what's north of the north pole: simply it does not make sense. So, we tend to say that the universe simply "is", i.e. "exists".

The universe is assumed to have the geometry

$$
\begin{equation*}
d s^{2}=N(t)^{2} d t^{2}-a(t)^{2}\left(d r^{2}+\sin ^{2}(r)\left(d \theta^{2}+\sin ^{2}(\theta) d \varphi^{2}\right)\right) \tag{53}
\end{equation*}
$$

with a homogeneous scalar field $\phi(t)$.
The action will be assumed to be the standard Einstein Hamiltonian action,

$$
\begin{equation*}
S=\int\left(\pi^{i j} \dot{\gamma}_{i j}+\pi_{\phi} \dot{\phi}-N H_{0}-N_{i} H^{i}\right) d^{3} x d t \tag{54}
\end{equation*}
$$

where

$$
\begin{aligned}
& H_{0}=\left(\frac{1}{\sqrt{\gamma}} \pi^{i j} \pi^{k l}\left(\gamma_{i k} \gamma_{j l}-\frac{1}{2} \gamma_{i j} \gamma_{k l}\right)-\sqrt{\gamma} R+\left(\frac{1}{2} \pi_{\phi}^{2}+\sqrt{\gamma}\left(\frac{1}{2 \sqrt{\gamma}} \phi_{i} \phi_{j} \gamma^{i j}+V(\phi)\right)\right)\right), \\
& H_{i}=\pi_{j}^{i j}+\Gamma_{j k}^{i} \pi^{j k}+\pi_{\phi} \phi_{j} \gamma^{i j} .
\end{aligned}
$$

We note that, if we put $\gamma=3$, in the eq. concerning $H_{0}$, we have:

$$
\frac{1}{\sqrt{\gamma}}=\frac{1}{\sqrt{3}}=0,57735 ; \quad \frac{1}{2 \sqrt{3}}=\frac{\sqrt{3}}{6}=0,28867 ; \quad \frac{\sqrt{3}}{2}=0,86602 ; \quad \frac{\sqrt{3}}{2 \sqrt{3}}=\frac{1}{2} .
$$

But we have also that: $0,57735+0,28867=0,86602$ and that:

$$
\begin{equation*}
\frac{1}{\sqrt{3}}+\frac{1}{2 \sqrt{3}}=\frac{2+1}{2 \sqrt{3}}=\frac{3}{2 \sqrt{3}}=\frac{3 \sqrt{3}}{2 \sqrt{3} \sqrt{3}}=\frac{3 \sqrt{3}}{6}=\frac{\sqrt{3}}{2}=0,86602 . \tag{55}
\end{equation*}
$$

We have, thence, the value corresponding to the fix number of a regular hexagon that is just 0,86602 . Furthermore, we note that:
$\frac{1}{2}(0,866 \times 6)=2,598$ that is the constant of the regular hexagon and is very near to the number 2,618 that is $\left(\frac{\sqrt{5}+1}{2}\right)^{2}=\Phi^{2}=2,61803398 \ldots$ (see eqs. (A52-55) Appendix A)

Because of the symmetry we have assumed for our fields, $H_{i}$ is automatically equal to zero, and we can write the action in terms of $a, \phi$ and their conjugate momenta $\pi_{a}$ and $\pi_{\phi}$ so that

$$
\begin{array}{r}
S=\int \pi_{a} \dot{a}+\pi_{\phi} \dot{\phi}-N H d t, \\
H=-\frac{1}{24 a} \pi_{a}^{2}-6 a+\left(\frac{1}{2 a^{3}} \pi_{\phi}^{2}+a^{3} V(\phi)\right) \tag{57}
\end{array}
$$

Thence, we can rewrite eq. (56) also as follows:

$$
\begin{equation*}
S=\int \pi_{a} \dot{a}+\pi_{\phi} \dot{\phi}-N\left[-\frac{1}{24 a} \pi_{a}^{2}-6 a+\left(\frac{1}{2 a^{3}} \pi_{\phi}^{2}+a^{3} V(\phi)\right)\right] d t \tag{57b}
\end{equation*}
$$

We consider important to highlight that the Hartle-Hawking procedure is often characterized as allowing time to become imaginary.
We note that $N$ occurs in these equations only in the combination $N d t$. We can thus define a new variable $\tau$ by

$$
\begin{equation*}
\tau=\int N(t) d t \tag{58}
\end{equation*}
$$

where we will take the zero of $\tau$ to occur at the "time" when $a$ is zero. Since we are allowing $N$ to be complex, $\tau$ will also be a complex function of $t$.
We remember that the complex numbers allow for solutions to certain equations that have no solutions in real numbers. For example, the equation

$$
(x+1)^{2}=-9
$$

has no real solution, since the square of a real number cannot be negative. Complex numbers provide a solution to this problem. The idea is to extend the real numbers with the imaginary unit i where $i=\sqrt{ }-1 ; \quad i^{2}=-1$, so that solutions to equations like the preceding one can be found. In this case the solutions are $-1+3 i$ and $-1-3 i$, as can be verified using the fact that $i^{2}=-1$ :

$$
\begin{gathered}
(x+1)^{2}=-9 ; \quad x^{2}+2 x+1+9=0 ; \quad x^{2}+2 x+10=0 ; \\
\frac{-2 \pm \sqrt{4-40}}{2}=\frac{-2 \pm \sqrt{-36}}{2}=\frac{-2 \pm \sqrt{36 i^{2}}}{2}=\frac{-2 \pm 6 i}{2}=\left\{\begin{array}{l}
-1+3 i \\
-1-3 i
\end{array}\right.
\end{gathered}
$$

and

$$
\begin{aligned}
& ((-1+3 i)+1)^{2}=(3 i)^{2}=(3)^{2}(i)^{2}=9(-1)=-9, \\
& ((-1-3 i)+1)^{2}=(-3 i)^{2}=(-3)^{2}(i)^{2}=9(-1)=-9 .
\end{aligned}
$$

According to the fundamental theorem of algebra, all polynomial equations with real or complex coefficients in a single variable have a solution in complex numbers.

A complex function is one in which the independent variable and the dependent variable are both complex numbers. More precisely, a complex function is a function whose domain and range are subsets of the complex plane.
For any complex function, both the independent variable and the dependent variable may be separated into real and imaginary parts:

$$
\begin{aligned}
& z=x+i y \text { and } \\
& w=f(z)=u(x, y)+i v(x, y)
\end{aligned}
$$

where $x, y \in \mathbb{R}$ and $u(x, y), v(x, y)$ are real-valued functions.
In other words, the components of the function $f(z)$,

$$
\begin{aligned}
& u=u(x, y) \text { and } \\
& v=v(x, y)
\end{aligned}
$$

can be interpreted as real-valued functions of the two real variables, $x$ and $y$.
The zeroth order action is just:

$$
\begin{align*}
S\left(a_{f}, \phi_{f}\right) & =\int_{0}^{\tau_{0}} \pi_{a}^{(0)} a^{(0)} d \tau=-\int_{0}^{\tau_{0}} 12 a^{(0)}\left(a^{(0)^{\prime}}\right)^{2} d \tau  \tag{59}\\
& =\frac{24 \beta}{V_{0}} \gamma\left(\sqrt{\frac{V_{0} a_{f}^{2}}{6}-1}\right)^{3}-i \frac{24 \beta}{V_{0}}, \tag{60}
\end{align*}
$$

Thence:

$$
\begin{equation*}
S\left(a_{f}, \phi_{f}\right)=\int_{0}^{\tau_{0}} \pi_{a}^{(0)} a^{(0)} d \tau=-\int_{0}^{\tau_{0}} 12 a^{(0)}\left(a^{(0)}\right)^{2} d \tau=\frac{24 \beta}{V_{0}} \gamma\left(\sqrt{\frac{V_{0} a_{f}^{2}}{6}-1}\right)^{3}-i \frac{24 \beta}{V_{0}}, \tag{60b}
\end{equation*}
$$

That for (b), $\sqrt{\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}+\left(\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)^{2}}=\sqrt{i^{2}}=i=\sqrt{-1}$, we can rewrite also as follows:

$$
\left.\begin{array}{rl}
S\left(a_{f}, \phi_{f}\right) & =\int_{0}^{\tau_{0}} \pi_{a}^{(0)} a^{(0)} d \tau=-\int_{0}^{\tau_{0}} 12 a^{(0)}\left(a^{(0)}\right)^{2} d \tau=\frac{24 \beta}{V_{0}} \gamma\left(\sqrt{\frac{V_{0} a_{f}^{2}}{6}}-1\right.
\end{array}\right)^{3}-i \frac{24 \beta}{V_{0}}=.
$$

But, we know that 24 is related to the "modes" that correspond to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$
24=\frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]} .
$$

Thence, in conclusion, we obtain:

$$
\begin{gather*}
S\left(a_{f}, \phi_{f}\right)=\int_{0}^{\tau_{0}} \pi_{a}^{(0)} a^{(0)} d \tau=-\int_{0}^{\tau_{0}} 12 a^{(0)}\left(a^{(0)^{\prime}}\right)^{2} d \tau=\frac{24 \beta}{V_{0}} \gamma\left(\sqrt{\frac{V_{0} a_{f}^{2}}{6}-1}\right)^{3}-i \frac{24 \beta}{V_{0}}= \\
=\frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right.} \beta\left[\frac{\gamma}{V_{0}}\left(\sqrt{\frac{V_{0} a_{f}^{2}}{6}-1}\right)^{3}-\sqrt{\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}+\left(\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)^{2}} \frac{1}{V_{0}}\right] . \tag{60c}
\end{gather*}
$$

This expression link complex path to zeroth order, physical vibrations of the bosonic strings and the roots of the cubic equation, thence with the imaginary unit.
The function, $S_{0}\left(a_{f}, \phi_{f}\right)$ is a Hamilton-Jacobi function for the $\varepsilon=0$ system, in that it obeys the equation

$$
\begin{equation*}
-\frac{1}{24 a}\left(\frac{\partial S_{0}}{\partial a}\right)^{2}-6 a+\left(\frac{1}{2 a^{2}} \frac{\partial S_{0}}{\partial \phi}\right)^{2}+a^{3} V_{0}=0 \tag{61}
\end{equation*}
$$

(see eq.(57)) which is the Hamilton-Jacobi form of the constant equation for $\varepsilon=0$.
Now, the first order equations for $a$ and $\phi$ are required to find the second order action. The equations are:

$$
\begin{array}{r}
-12 a^{(0)} \delta a^{\prime}+2 V_{0} a^{(0)} \delta a+a^{(0) 2} \phi_{f}=0 \\
\left(a^{(0) 3} \delta \phi^{\prime}\right)^{\prime}=-a^{(0) 3} . \tag{63}
\end{array}
$$

The solution for $a$ is

$$
\begin{equation*}
\delta a=\beta \frac{\phi_{f}}{2 V_{0}}\left(i \tau \cos (i v \tau)-\frac{1}{v} \sin (i v \tau)\right) \tag{64}
\end{equation*}
$$

while for $\delta \phi$ we get first that

$$
\begin{equation*}
\delta \phi^{\prime}=-\frac{1}{a^{(0) 3}} \int_{0}^{\tau} a^{(0) 3} d \tau=\frac{\frac{i}{v}\left(\frac{1}{3} \cos ^{3}(i v \tau)-\cos (i v \tau)+\frac{2}{3}\right)}{\sin ^{3}(i v \tau)} \tag{65}
\end{equation*}
$$

Because $\delta \phi\left(\tau_{0}\right)=0$, we have

$$
\begin{array}{r}
\delta \phi(\tau)=\int_{\tau}^{\tau_{0}} \delta \phi^{\prime} d \widetilde{\tau} \\
\delta \phi^{\prime}=-\frac{1}{a^{(0) 3}} \int_{0}^{\tau} a^{(0) 3} d \tau . \tag{67}
\end{array}
$$

Because of the $1 / a^{3}$ term in the integrand for $\delta \phi$, its value will depend, among other things, on the way in which the path in $\tau$ space wraps around the poles where $a=0$. We get

$$
\begin{equation*}
\delta \phi(\tau)=\frac{2}{V_{0}}\left[\frac{1}{1+\cos (i v \tau)}+\ln \left(\frac{1+\cos (i v \tau)}{1+\cos \left(i v \tau_{0}\right)}\right)-\frac{1}{1+\cos \left(i v \tau_{0}\right)}\right] . \tag{68}
\end{equation*}
$$

Thence, we have, in conclusion:

$$
\begin{equation*}
\delta \phi(\tau)=\int_{\tau}^{\tau_{0}} \delta \phi^{\prime} d \tilde{\tau}=\frac{2}{V_{0}}\left[\frac{1}{1+\cos (i v \tau)}+\ln \left(\frac{1+\cos (i v \tau)}{1+\cos \left(i v \tau_{0}\right)}\right)-\frac{1}{1+\cos \left(i v \tau_{0}\right)}\right] \tag{68b}
\end{equation*}
$$

Given the first order solutions, we can calculate the action to second order in $\varepsilon$.

$$
\begin{equation*}
\delta^{2} S=\phi_{f}^{2} \frac{d^{2}}{d V_{0}^{2}} S_{0}\left(a_{f}, \phi_{f}, V_{0}\right)-\int_{0}^{\tau_{0}} a^{(0) 3}\left(\delta \phi^{\prime}\right)^{2} d \tau \tag{69}
\end{equation*}
$$

We have

$$
\begin{align*}
-\int_{0}^{\tau_{0}} a^{(0) 3}\left(\delta \phi^{\prime}\right)^{2} d \tau & =\beta \int_{0}^{\tau_{0}} \frac{1}{v^{5} \sin ^{3}(i v \tau)}\left(\frac{1}{3} \cos ^{3}(i v \tau)-\cos (i v \tau)+\frac{2}{3}\right)^{2} d \tau= \\
& =\frac{i \beta}{9 v^{6}} \int_{1}^{\cos \left(i v \tau_{0}\right)}\left(\frac{(1-z)(z+2)}{1+z}\right)^{2} d z \quad(70) \tag{70}
\end{align*}
$$

with $z=\cos (i v \tau)$ which has poles of order 2 in the integrand at $z=-1$ or $\tau=i(2 m+1) \pi / v$. Each time the integrand circles one of these poles, we accumulate a residue, and it is clear that the residue is the same at each of the poles in $\tau$ space, since it is the same pole $z=-1$ in $z$ space. Thus, we have

$$
\begin{equation*}
-\int_{0}^{\tau_{0}} a^{(0) 3}\left(\delta \phi^{\prime}\right)^{2} d \tau=\frac{i \beta}{9 v^{6}} \int_{\Gamma_{0}}\left(\frac{(1-z)(z+2)}{1+z}\right)^{2} d z+2 \pi \beta k \frac{96}{V_{0}^{3}} \tag{71}
\end{equation*}
$$

where $\Gamma_{0}$ is some fiducial path from the point $\tau=0$ to $\tau=\tau_{0}$, and $k$ is the total net number of times that the actual path wraps around the various pole.
Assuming that $v a_{f}>1$, we can take the fiducial path as a straight line connecting $\tau=0$ to $\tau=\tau_{f}$. We finally get

$$
\begin{equation*}
\delta^{2} S=\beta\left[\phi_{f}^{2} \frac{\partial^{2} S_{0}\left(a_{f}, \phi_{f}, V_{0}\right)}{\partial V_{0}^{2}}-\pi k \frac{192}{V_{0}^{3}}+\frac{8 i}{V_{0}^{3}}\left(-\frac{12}{Z+1}+12 \ln \left(\frac{Z+1}{2}\right)+17-12 Z+Z^{3}\right)\right], \tag{72}
\end{equation*}
$$

where $Z=\cos \left(i v \tau_{0}\right)=\psi\left|\sqrt{\frac{V_{0} a_{f}^{2}}{6}-1}\right|$.
We have the equation (72):

$$
\delta^{2} S=\beta\left[\phi_{f}^{2} \frac{\partial^{2} S_{0}\left(a_{f}, \phi_{f}, V_{0}\right)}{\partial V_{0}^{2}}-\pi k \frac{192}{V_{0}^{3}}+\frac{8 i}{V_{0}^{3}}\left(-\frac{12}{Z+1}+12 \ln \left(\frac{Z+1}{2}\right)+17-12 Z+Z^{3}\right)\right] .
$$

For $\quad \sqrt{\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}+\left(\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)^{2}}=\sqrt{i^{2}}=i=\sqrt{-1} ; \quad$ and the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$
8=\frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi \pi^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]},
$$

We have that

$$
\begin{align*}
\delta^{2} S= & \beta\left[\phi_{f}^{2} \frac{\partial^{2} S_{0}\left(a_{f}, \phi_{f}, V_{0}\right)}{\partial V_{0}^{2}}-\pi k \frac{192}{V_{0}^{3}}+\frac{8 i}{V_{0}^{3}}\left(-\frac{12}{Z+1}+12 \ln \left(\frac{Z+1}{2}\right)+17-12 Z+Z^{3}\right)\right] \Rightarrow \\
\delta^{2} S & =\beta\left[\phi_{f}^{2} \frac{\partial^{2} S_{0}\left(a_{f}, \phi_{f}, V_{0}\right)}{\partial V_{0}^{2}}-\pi k \frac{192}{V_{0}^{3}}+\frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right] V_{0}^{3}}\right. \\
& \left.\cdot \sqrt{\left(\frac{1}{2}+\frac{i \sqrt{3}}{2}\right)^{2}+\left(\frac{1}{2}-\frac{i \sqrt{3}}{2}\right)^{2}}\left(-\frac{12}{Z+1}+12 \ln \left(\frac{Z+1}{2}\right)+17-12 Z+Z^{3}\right)\right] . \tag{73}
\end{align*}
$$

This expression link complex path to second order action, the "modes" corresponding to the physical vibrations of the superstrings and the roots of the cubic equation, thence with the imaginary unit.
Further, we note that

$$
192 \times 12=2304 ; \quad 2304-(963 \cdot 2)=2304-1926=378=369+9 .
$$

And that:

$$
963+852+471=2286=2304-18 ; \quad 936+825+714-147=2328=2304+24,
$$

where 24, is related to the "modes" that correspond to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$
24=\frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi \pi^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]} .
$$

From (57)

$$
H=-\frac{1}{24 a} \pi_{a}^{2}-6 a+\left(\frac{1}{2 a^{3}} \pi_{\phi}^{2}+a^{3} V(\phi)\right)
$$

we have that:

$$
\begin{equation*}
H_{\mathrm{mod}}=-\frac{1}{24 a} \pi_{a}^{2}-6 a+\left(\frac{1}{2 a^{3}} \pi_{\phi}^{2}+a^{3} V(\phi)\right)-288 \frac{\varepsilon^{2}}{V_{0}^{4} a^{3}} . \tag{74}
\end{equation*}
$$

Let us examine the classical solutions obtained by using $S_{R}$ as Hamilton Jacobi function for the solution. We are interested in the motion for $v\left|a_{f}\right|>1$. Furthermore, we expect $V_{0}$ to enter into the complete action only in the combination $V=V\left(\phi_{f}\right)=V_{0}+\varepsilon \phi_{f}$. Thus, we define

$$
\begin{equation*}
y=\left|\sqrt{\frac{V a_{f}^{2}}{6}-1}\right| \tag{75}
\end{equation*}
$$

and have

$$
\begin{equation*}
S_{R} \cong \beta \gamma \frac{24}{V} y^{3}+\varepsilon^{2} \beta \frac{4}{V^{3}}\left(-12 \frac{\gamma y}{1+y^{2}}+6 i \ln \frac{1+i \gamma y}{1-i \gamma y}+12 \gamma y+\not y^{3}-24 \pi k\right) . \tag{76}
\end{equation*}
$$

We note that

$$
\begin{aligned}
& 288=285+3 ; \quad 288+288-9=576-9=567 \\
& 24 \times 12=288=285+3 ; \quad 24 \times 24=576=582-6
\end{aligned}
$$

The equations relating the derivatives with respect to time of the dynamic variables and the Hamilton Jacobi function are

$$
\begin{equation*}
\pi_{a}=-12 a \dot{a}=\frac{\partial S_{R}}{\partial a} ; \quad \pi_{\phi}=a^{3} \dot{\phi}=\frac{\partial S_{R}}{\partial \phi} . \tag{77}
\end{equation*}
$$

For large $a$, we have

$$
\begin{equation*}
S_{R} \cong \beta \gamma|a|^{3} 4 \sqrt{\frac{V(\phi)}{6}}\left[1+\frac{1}{6 V^{2}} \varepsilon^{2}\right] . \tag{78}
\end{equation*}
$$

We can rewrite the expression for $S_{R}$ as:

$$
\begin{equation*}
S_{R} \cong \beta \gamma \frac{24}{V}\left(\left(1+\varepsilon^{2} \frac{17}{6 V^{2}}\right)\left(\sqrt{y^{2}-\varepsilon^{2} \frac{4}{3 V^{2}}}\right)^{3}\right)+\varepsilon^{2} O\left(y^{5}\right)+O\left(\varepsilon^{3}\right) . \tag{79}
\end{equation*}
$$

Since

$$
\begin{equation*}
y^{2}-\varepsilon^{2} \frac{4}{3 V^{2}} \cong \frac{V_{0}}{6}\left(a-a_{T}\right)\left(2 a_{T}\right)+\varepsilon \frac{1}{6} \phi a_{T}^{2}+O\left(\left(a-a_{T}\right)^{2}\right), \tag{80}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{T}=\sqrt{\frac{6}{V_{0}}}\left(1+\varepsilon^{2} \frac{2}{3 V_{0}^{2}}\right), \tag{81}
\end{equation*}
$$

we can write the action as

$$
\begin{equation*}
S_{R} \cong \beta \gamma \frac{24}{V}\left(\left(1+\frac{17}{6 V^{2}} \varepsilon^{2}\right)\left(\sqrt{\frac{V_{0} a_{T}}{3}\left(|a|-a_{T}\right)+\varepsilon \frac{a_{T}^{2}}{6}} \phi\right)^{3}\right) . \tag{82}
\end{equation*}
$$

We note, with regard the eq. (81), that:
$\sqrt{6} \frac{2}{3}=\sqrt{\frac{24}{9}}=\sqrt{\frac{8}{3}}=2 \sqrt{\frac{2}{3}}=1,6329 \ldots$ that is a value very near to the aurea ratio, i.e. $\left(\frac{\sqrt{5}+1}{2}\right)=1,61803398$.
Defining $\xi=|a|-a_{T}$, we have

$$
\begin{gather*}
\pi_{a}=-12 a \dot{a} \cong-12 a_{T} \xi \cong \beta \gamma \frac{24}{V_{0}}\left(1+\varepsilon^{2} \frac{17}{6 V_{0}^{2}}\right) \frac{3}{2} \frac{V_{0} a_{T}}{3} \sqrt{\frac{V_{0} a_{T}}{3} \xi+\varepsilon \frac{a_{T}^{2}}{6} \phi} \cong \\
\cong \beta \gamma 12 a_{T}\left(1+\varepsilon^{2} \frac{17}{6 V_{0}^{2}}\right) \sqrt{\frac{V_{0} a_{T}}{3} \xi+\varepsilon \frac{a_{T}^{2}}{6}} \phi .  \tag{83}\\
\pi_{\phi}=a_{T}^{3} \dot{\phi} \cong \varepsilon \beta \gamma \frac{24}{V_{0}} \frac{3}{2} \frac{a_{T}^{2}}{6} \sqrt{\frac{V_{0} a_{T}}{3} \xi+\varepsilon \frac{a_{T}^{2}}{6}} \phi \cong \beta \gamma \varepsilon \frac{6 a_{T}^{2}}{V_{0}} \sqrt{\frac{V_{0} a_{T}}{3} \xi+\varepsilon \frac{a_{T}^{2}}{6} \phi,} \tag{84}
\end{gather*}
$$

where we have only kept terms to lowest order in $\xi$ and $\phi$. Solving these equations we find that

$$
\begin{equation*}
\sqrt{\frac{V_{0} a_{T}}{3} \xi+\varepsilon \frac{a_{T}^{2}}{6}} \phi=-\beta \gamma \frac{V_{0} a_{T}}{6}\left(1+\varepsilon^{2} \frac{35}{6 V_{0}^{2}}\right) t . \tag{85}
\end{equation*}
$$

Now, from (82), we have that:

$$
24 \times 17=408 ; \quad 408=396+12=417-9 .
$$

From (79), we have that:

$$
\begin{gathered}
24 \times 17 \times 4=1632 ; \quad 1632-936=696=693+3 ; \\
1632-(243 \times 6)=1632-1458=174 ; \quad 693-54=639 ; \quad 408+174=582 ; \quad 582-54=528 .
\end{gathered}
$$

Thence, in conclusion:

$$
\begin{gathered}
(24 \times 17 \times 4)-(243 \times 6)+(24 \times 17)-54=528 . \text { But } 528=24 \times 22, \text { thence: } \\
(24 \times 17 \times 4)-(243 \times 6)+(24 \times 17)-54=24 \times 22 ;
\end{gathered}
$$

We know that 24 is related to the "modes" that correspond to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$
24=\frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi x x w^{\prime}}{\cosh \pi x} e^{-\pi \pi^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]} .
$$

In conclusion, we have:

$$
\begin{equation*}
\frac{4\left[\text { anti } \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(t t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}} \times 22}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]}=(24 \times 17 \times 4)-(243 \times 6)+(24 \times 17)-54 \tag{86}
\end{equation*}
$$

Now, we have the following two series:
a) $\mathbf{2 , 3}, 4,6,8,9,12,17,24,35,96,192,288$
b) $\mathbf{3}, \mathbf{1 5}, \mathbf{3 0}, 90$

For the first series, we note a certain proximity with the Fibonacci's numbers, that we have compared in Table 1

TABLE 1 :

| Numbers first series S | Fibonacci's numbers near to F | Differences <br> S - F <br> near or equal to the numbers $F$ | Some numbers S are the double or about of $S$ or multiples of precedent |  |
| :---: | :---: | :---: | :---: | :---: |
| 2 | 2 | 0 |  |  |
| 3 | 3 | 0 |  |  |
| 4 | 5 | -1 | $4=2 * 2$ |  |
| 6 | 5 | 1 | 6=2*3 |  |
| 8 | 8 | 0 | $8=2 * 4$ |  |
| 9 | 8 | 1 | $9=2 * 4+1$ |  |
| 12 | 13 | -1 | $12=2 * 6$ |  |
| 17 | 17 mean <br> of 13 and 21 | 0 | $17=2 * 8+1$ |  |
| 24 | 21 | 3 | $24=2 * 12$ |  |
| 35 | 34 | 1 | $35=2 * 17+1$ |  |
| 96 | 89 | 7 $\approx 8$ | $96=4 * 24$ |  |
| 192 | 188,5 <br> mean <br> 144 and 233 | $3,5 \approx 3$ | $\begin{aligned} & 192=2 * 96 \\ & 192=8 * 24 \end{aligned}$ |  |
| 288 | 337 | 89 <br> Fibonacci's number | $\begin{aligned} & 288=2 * 144 \\ & 288=12 * 24 \end{aligned}$ |  |
| 288 | 233 | 55 <br> Fibonacci's number |  |  |

Besides the obvious coincidence (for 2,3 and 8 ) or proximity (for $4,6,9,12,24,35$ and 96 ) with the Fibonacci's numbers or their means (17, 192), which is also repeated for the differences $\mathrm{S}-\mathrm{F}$, we also note that in the fourth column, as multiples of $S$, we notice often 8 and 24 , already, in itself, same numbers of the $S$ series, andimportant in the modes of vibrations of the strings, as we already know. Hence two important connections of these numbers both with the Fibonacci numbers, and with the modes of vibration of the strings.

Now we see some possible connections with partitions numbers
$1,1,2,3,5,7,11,15,22,30,42,56,77,101,135,176,231$,
$297,385,490,627,792,1002,1255,1575,1958,2436,3010$,
$3718,4565,5604, \ldots$
TABLE 2

| Numbers first <br> series S | Numbers of <br> Partitions P near <br> to S | Differences |
| :--- | :--- | :--- |
| 2 | 2 | S - P |
| 3 | 3 | 0 |
| 4 | 5 | 0 |
| 6 | 7 | -1 |
| 8 | 7 | -1 |
| 9 | 15 | 1 |
| 12 | 22 | 2 |
| 17 | 30 | 2 |
| 24 | 101 | -5 |
| 35 | 176 | $16 \approx 15$ |
| 96 | 297 | $-9 \approx 7$, |
| 192 | $79 \approx 11$ |  |
| 288 | 7 | 5 |
|  | 7 | -5 |

Also in this comparison, up to 96 the differences S - P are very small, and also all of these equal or very near to other partition numbers smaller. Therefore also this comparison shows a connection with the partition numbers, although weaker than that with the Fibonacci's numbers. Finally, we try also with the numbers Lie, of form $\mathrm{n}^{2}+\mathrm{n}+1$, halfway between Fibonacci's numbers and partition numbers (and important in the Standard Model because at the basis of the five sporadic groups of symmetry)

Numbers L $=1,3,7,13,21,31,43,57,73,91,121$

TABLE 3

| Numbers first series S | Numbers of Lie Lequal or near to $S$ | Differences S - L |
| :---: | :---: | :---: |
| 2 | 1 | 1 |
| 3 | 3 | 0 |
| 4 | 3 | 1 |
| 6 | 7 | -1 |
| 8 | 7 | 1 |
| 9 |  |  |
| 12 | 13 | -1 |
| 17 |  |  |
| 24 | 21 | 3 |
| 35 | 31 | 4 $\approx 3$ |
| 96 | 91 | $5 \approx 7$ |
| 192 | 183 | $9 \approx 7$ |
| 288 | 273 | $15 \approx 13$ |

Also here, the numbers S are very near to the Lie's numbers, and also the their differences $\mathrm{S}-\mathrm{L}$. We can therefore say that the numbers $S$ are connected especially with the Fibonacci's numbers and
with the modes of vibration of the strings, and in a somewhat weaker also with the partition of numbers and the numbers of Lie, all numbers related to many natural phenomena, starting with by strings.

## Second series 3, 15, 30,90

First, we note that $15=3 * 5,30=15 * 2$ e $90=30 * 3$, and that the factors are respectively 2,3 and 5 , i.e. only the first three prime numbers, which are also Fibonacci's numbers:
$3=3$
$15=3 * 5$,
$30=2 * 3 * 5$,
$90=2 * 3 * 3 * 5$
Further, also these few numbers of this series are connected to Fibonacci, to partitions and to the Lie's numbers

TABLE 4

| Numbers S' | Fibonacci | Partitions | Lie |
| :--- | :--- | :--- | :--- |
| 3 | 3 | 3 | 3 |
| 15 | 13 | 15 | 13 |
| diff. 2 | 34 | 30 | diff. 2 |
| 30 | diff. -4 | diff. 1 |  |
| 90 | diff. 1 | diff.-11 | 91 |

Also here small differences, and for the partitions also three perfect coincidences: 3, 15 and 30 . Obviously, between the Fibonacci numbers missing 5 and 8 between 3 and 15, the 21 between 13 and 34 , the 55 between 34 and 89 .

Thence, also the Table 4 we could give some numerical information and therefore also useful mathematics on the second series.

Now, we analyze further two series.
Numbers of the first series:
741, 417, 528, 852, 396, 639, 243, 324, 111, 1269
Some numbers are permutations of other, for example, 741 and 417, 528 and 852, 396 and 639, 243 and 324 (see the next numerical series)

TABLE 1
Subsequent ratios in the order indicated

| Numbers | Subsequent ratios | Connection with Fibonacci <br> (in green the values more near) | Factors | Prime numbers $>3$ of form $6 \mathrm{k}-1$ <br> (-) <br> Form 6k+1 (+) |
| :---: | :---: | :---: | :---: | :---: |
| 741 | 741/417=1,77 | >1,618 | 3, 13, 19 | ++ |
| 417 | 0,78 | >0,618 | 3,139 | + |
| 528 | 0,619 | $\approx 0,618$ | 2^4, 3, 11 | - |
| 852 | 2,15 | $\approx 2,61=1,618{ }^{\wedge} 2$ | 2^2, 3, 71 | - |
| 396 | 0,619 | $\approx 0,618$ | $2^{\wedge} 2,3 \wedge 2,11$ | - |
| 639 | 2,62 | $\approx 2,61=1,618^{\wedge} 2$ | 3^2,71 | - |
| 243 | 0,75 | >0,618 | 3^5 |  |
| 324 | 2,91 | $\approx 2,61=1,618 \wedge 2$ | $2^{\wedge} 2,3^{\wedge} 4$ |  |
| 111 | 0,08 | $\begin{aligned} & \approx \sqrt{ } \sqrt{ } 1,618 \\ & =\mathbf{1 , 0 6 - 1} \end{aligned}$ | 3, 37 | + |
| 1269 |  |  | 3^3, 47 | - |

## TABLE 2

Subsequent ratios in the natural order of the numbers involved

| Numbers | Subsequent ratios | Connection with Fibonacci or pigreco | Factors | Prime numbers $>3$ of form $6 \mathrm{k}-1$ -) $\text { form } \mathbf{6 k + 1}(+)$ |
| :---: | :---: | :---: | :---: | :---: |
| 111 |  |  | 3,37 | + |
| 243 | 243/111 $=2,18$ | $\approx 2,61=1,618{ }^{\wedge} 2$ | 3^5 |  |
| 324 | 1,33... | $\begin{aligned} & \approx \mathbf{1 , 2 7}=\sqrt{ } 1,618 \\ & \approx 1,33=\sqrt{ } \sqrt{ } 3,14 \end{aligned}$ | $2^{\wedge} 2,3^{\wedge} 4$ |  |
| 396 | 1,053 | $\begin{aligned} & \approx \sqrt{ } \sqrt{ } 1,618 \\ & =1,06 \end{aligned}$ | $2^{\wedge} 2,3^{\wedge} \mathbf{2 , 1 1}$ | - |
| 417 | 1,26... | $\approx 1,27=\sqrt{ } 1,618$ | 3, 139 | + |
| 528 | 1,21... | $\begin{array}{ll} \approx 1,19 \text { mean } & \\ \sqrt{ } 1,618 & \text { and } \\ \sqrt{ } \sqrt{1,618} & \end{array}$ | $2^{\wedge} 4,3,11$ | - |
| 639 | 1,15... | $\begin{array}{ll} \hline \approx \mathbf{1 , 1 9} \text { mean } & \\ & \\ \sqrt{ } \mathbf{1 , 6 1 8} & \text { and } \\ \sqrt{ } \sqrt{1,618} & \end{array}$ | $3^{\wedge} 2,71$ | - |
| 741 | 1,14... | $\begin{array}{ll} \approx 1,19 \text { mean } & \\ \sqrt{ } 1,618 & \text { and } \\ \sqrt{ } 1,618 & \end{array}$ | 3, 13, 19 | + |
| 852 | 1,48... | $\approx 1,33=\sqrt{ } \sqrt{ }$, 14 | 2^2, 3, 71 | - |
| 1269 |  |  | $3^{\wedge} 3,47$ | - |

The connections with Fibonacci seem obvious, some connections there are also with pigreco About the factors> 3 and their numerical form $6 \mathrm{k}-1$ and $6 \mathrm{k}+1$ does not appear to be any particular connections, only a slight alternation in TABLE 2

Second series (permutations of 147, 258 and 396, with difference 3 between their consecutive digits with difference 3 or 6 , instead, for their permutations)

396 417, 528, 639, 741, 852
963, 174, 285, 396, 417, 528, 639, 741, 852, 963, 174, 285, 693, 714, 825, 936, 147, 258, 369, 471, 582, 693, 714, 825, 936, 147, 258, 369, 471, 582

The numbers in red are the permutations of the base number $\underline{369}$
The numbers in blue are the permutations of the base number $\underline{147}$
The numbers in black are the permutations of the base number $\underline{258}$
In these numbers of the second series is missing the base number $\underline{234}$ present only in the first series with his permutation 324

## Interpretation through the permutations

The six permutations of 147 are:
147
174
417 present also in the first series
471
714
741 present also in the first series
here in $3^{\text {rd }}$ and $6^{\text {th }}$ position

The six permutations of 234 are: 234

243 present also in the first series
324 present also in the first series
342
423
432
here in $2^{\text {nd }}$ and $3^{\text {rd }}$ position

The six permutations of 258 are:
258
285

528 present also in the first series
582
825
852 present also in the first series
here in $3^{\text {rd }}$ and $6^{\text {th }}$ position

The six permutations of 369 are:
369
396 present also in the first series
639 present also in the first series
693
936
963
here in $2^{\text {nd }}$ and $3^{\text {rd }}$ position

So we see this regularly: two numbers in 3rd and 6th position and in 2nd and 3rd position a way alternated from the base number smaller to the base number largest. We also note that the base numbers, and obviously also their permutations, are all divisible by 3 , have sum of the digit that is divisible by 3 , always increasing, though not in the same order:

147 and its permutations: digit sum $1+4+7=12$
234 and its permutations : digit sum $2+3+4=9$
258 and its permutations : digit sum $2+5+8=15$
369 and its permutations : digit sum $3+6+9=18$
This is also a small regularity that might be useful in some way in this work on gravity and quantum physics

## Interpretation by the Fibonacci's numbers

Now, we see only the four base numbers, the their connections with Fibonacci, the factors and the subsequent ratios:

TABLE 3

| Base numbers | Fibonacci's numbers near <br> And small differences = Fibonacci's number smaller | Factors and connections with Fibonacci (in green) | Subsequent ratios between the base number and the precedent |
| :---: | :---: | :---: | :---: |
| 147 | $144$ <br> Diff . 3 | $\begin{aligned} & 3,7 \wedge 2 \\ & 7 \approx 8 \end{aligned}$ | - |
| 234 | $233$ <br> Diff. 1 | $\begin{aligned} & 2,3^{\wedge} 2,13 \\ & 3^{\wedge} 2=9 \approx 8 \end{aligned}$ | $\begin{aligned} & 1,5918 \ldots \\ & \approx 1,618 \end{aligned}$ |
| 258 | $233$ <br> Diff. $25 \approx 23+2$ | $\begin{aligned} & 2,3,43 \\ & 43 \approx 2 * 21 \end{aligned}$ | $\begin{aligned} & 1,1025 \ldots \\ & \approx 1,12=\sqrt{ } 1,618 \end{aligned}$ |
| 369 | $\begin{aligned} & 377 \\ & \text { Diff. - } 8 \end{aligned}$ | $\begin{aligned} & 3 \wedge 2,41 \\ & 41 \approx 2 * 21 \\ & 41 \approx 44,5=(34+55) / 2 \end{aligned}$ | $\begin{aligned} & 1,4302 \ldots \\ & \approx 1.4440 \ldots= \\ & (1,618+\sqrt{ } 1,618) / 2 \end{aligned}$ |

Also here we note the mathematical connections with Fibonacci, specially in the first and in the fourth column.
We note also that:
$852 / 528=1,613636 \ldots ; \quad 639 / 396=1,613636 \ldots$ value very near to the aurea ratio that is $\frac{\sqrt{5}+1}{2}=1,61803398 \ldots$
We have also that:
$\sqrt{852} \cong 29,18$
$\sqrt{528} \cong 22,97 ; ~ 29,18+22,97=52,15 ;$
$\sqrt{639} \cong 25,27$
$\sqrt{396} \cong 19,89 ; \quad 25,27+19,89=45,16 ; \quad 52,15-45,16=6,99 ; \quad \sqrt[4]{6,99} \cong 1,6259 \ldots$
$\sqrt[3]{852} \cong 9,480$
$\sqrt[3]{528} \cong 8,082 ; \quad 9,480+8,082=17,562$;
$\sqrt[3]{639} \cong 8,613$
$\sqrt[3]{396} \cong 7,343 ; \quad 8,613+7,343=15,956 ; \quad 17,562-15,956=1,606 \ldots$
The mean between 1,6259 and 1,606 is $1,61595 \ldots$ value that is very near to the value of the aurea ratio $\frac{\sqrt{5}+1}{2}=1,61803398 \ldots$
With regard the approximation to $\pi$, we note that:
$9,480 / 3=3,16 ; \quad 29,18 / 9=3,24 ; \quad 29,18-22,97=6,21 ; 6,21 / 2=3,105$.
The mean between 3,16 and 3,105 is 3,1325 value very near to $\pi$.

## Appendix A

## On some equations concerning the possible Relativistic Theory of Quantum Gravity

In the thesis ${ }^{1}$, the generalized expression for the action of a free-particle will have the following form:

$$
\begin{equation*}
S=-m_{g} c \int_{a}^{b} d s \tag{A1}
\end{equation*}
$$

or

[^0]\[

$$
\begin{equation*}
S=-\int_{t_{1}}^{t_{2}} m_{g} c^{2} \sqrt{1-\frac{V^{2}}{c^{2}}} d t \tag{A2}
\end{equation*}
$$

\]

where the Lagrange's function is

$$
\begin{equation*}
L=-m_{g} c^{2} \sqrt{1-\frac{V^{2}}{c^{2}}} . \tag{A3}
\end{equation*}
$$

The integral $S=\int_{t_{1}}^{t_{2}} m_{g} c^{2} \sqrt{1-\frac{V^{2}}{c^{2}}} d t$, preceded by the plus sign, cannot have a minimum. Thus, the integrand of eq. (A2) must be always positive.
The momentum is the vector $\vec{p}=\partial L / \partial \vec{V}$. Thus, from eq. (A3) we obtain

$$
\vec{p}=\frac{m_{g} \vec{V}}{ \pm \sqrt{1-\frac{V^{2}}{c^{2}}}}=M_{g} \vec{V}
$$

The ( + ) sign in the equation above will be used when $m_{g}>0$ and the (-) sign if $m_{g}<0$. Consequently, we will express the momentum $\vec{p}$ in the following form

$$
\begin{equation*}
\vec{p}=\frac{m_{g} \vec{V}}{\sqrt{1-\frac{V^{2}}{c^{2}}}}=M_{g} \vec{V} . \tag{A4}
\end{equation*}
$$

We know that $\vec{p} \cdot \vec{V}-L$ denotes the energy of the particle. Thus, we can write

$$
\begin{equation*}
E_{g}=\vec{p} \cdot \vec{V}-L=\frac{m_{g} c^{2}}{\sqrt{1-\frac{V^{2}}{c^{2}}}}=M_{g} c^{2} \tag{A5}
\end{equation*}
$$

Equation (A5) can be rewritten also in the following form:

$$
\begin{equation*}
E_{g}=\frac{m_{g}}{m_{i}}\left[m_{i} c^{2}+\left(\frac{m_{i} c^{2}}{\sqrt{1-V^{2} / c^{2}}}-m_{i} c^{2}\right)\right]=\frac{m_{g}}{m_{i}}\left(E_{i 0}+E_{K i}\right)=\frac{m_{g}}{m_{i}} E_{i}, \tag{A6}
\end{equation*}
$$

where

$$
\begin{gathered}
E_{i 0}=m_{i} c^{2} \\
E_{K i}=\frac{m_{i} c^{2}}{\sqrt{1-\frac{V^{2}}{c^{2}}}}-m_{i} c^{2} .
\end{gathered}
$$

We have that $E_{i}=E_{i 0}+E_{K i}$ is the total inertial energy, where $E_{K i}$ is the kinetic initial energy. From eqs. (A5) and (A6) we thus obtain

$$
\begin{equation*}
E_{i}=\frac{m_{i 0} c^{2}}{\sqrt{1-\frac{V^{2}}{c^{2}}}}=M_{i} c^{2} \tag{A7}
\end{equation*}
$$

The gravity, $\vec{g}$, in a gravitational field produced by a particle of gravitational mass $M_{g}$, depends on the particle's gravitational energy, $E_{g}$ (given by eq. (A5)), because we can write

$$
\begin{equation*}
g=-G \frac{E_{g}}{r^{2} c^{2}}=-G \frac{M_{g} c^{2}}{r^{2} c^{2}} . \tag{A8}
\end{equation*}
$$

Due to $g=\partial \Phi / \partial r$, the expression of the relativistic gravitational potential, $\Phi$, is given by

$$
\begin{equation*}
\Phi=-\frac{G M_{g}}{r}=-\frac{G m_{g}}{r \sqrt{1-\frac{V^{2}}{c^{2}}}} . \tag{A9}
\end{equation*}
$$

Then, we get

$$
\begin{equation*}
\frac{\partial \Phi}{\partial r}=\frac{G m_{g}}{r^{2} \sqrt{1-\frac{V^{2}}{c^{2}}}} \tag{A10}
\end{equation*}
$$

The gravitational potential energy per unit of gravitational mass of a particle inside a gravitational field is equal to the gravitational potential $\Phi$ of the field. Thus, we can write that

$$
\begin{equation*}
\Phi=\frac{U(r)}{m_{g}^{\prime}} \tag{A11}
\end{equation*}
$$

The, it follows that

$$
\begin{equation*}
F_{g}=-\frac{\partial U(r)}{\partial r}=-m_{g}^{\prime} \frac{\partial \Phi}{\partial r}=-G \frac{m_{g} m_{g}^{\prime}}{r^{2} \sqrt{1-\frac{V^{2}}{c^{2}}}} . \tag{A12}
\end{equation*}
$$

Comparison between (A5) and (A7) shows that $E_{g 0}=E_{i 0}$ i.e., $m_{g 0}=m_{i 0}$. Consequently, we have

$$
\begin{equation*}
E_{g}+E_{i}=E_{g 0}+E_{i 0}=2 E_{i 0} \tag{A13}
\end{equation*}
$$

However $E_{i}=E_{i 0}+E_{K i}$. Thus, (A13) becomes

$$
\begin{equation*}
E_{g}=E_{i 0}-E_{K i} . \tag{A14}
\end{equation*}
$$

Note the symmetry in the equations of $E_{i}$ and $E_{g}$. Substitution of $E_{i 0}=E_{i}-E_{K i}$ into (A14) yields

$$
\begin{equation*}
E_{i}-E_{g}=2 E_{K i} \tag{A15}
\end{equation*}
$$

Squaring the eqs. (A4) and (A5) and comparing the result, we find the following correlation between gravitational energy and momentum:

$$
\begin{equation*}
\frac{E_{g}^{2}}{c^{2}}=p^{2}+m_{g}^{2} c^{2} . \tag{A15b}
\end{equation*}
$$

The energy expressed as a function of the momentum is called Hamilton's function:

$$
\begin{equation*}
H_{g}=c \sqrt{p^{2}+m_{g}^{2} c^{2}} . \tag{A16}
\end{equation*}
$$

Starting from the Schrodinger equation we may obtain the expression for the energy of a particle in periodic motion inside a cubical box of edge length $L$. The result now is:

$$
\begin{equation*}
E_{n}=\frac{n^{2} h^{2}}{8 m_{g} L^{2}} \quad n=1,2,3, \ldots \tag{A17}
\end{equation*}
$$

We note that the term $h^{2} / 8 m_{g} L^{2}$ (energy) will be minimum for $L=L_{\max }$ where $L_{\max }$ is the maximum edge length of a cubical box whose maximum diameter

$$
d_{\max }=L_{\max } \sqrt{3} \quad(\mathrm{~A} 17 \mathrm{~b})
$$

## is equal to the maximum length scale of the Universe.

The minimum energy of a particle is obviously its inertial energy at rest $m_{g} c^{2}=m_{i} c^{2}$. Therefore we can write

$$
\begin{equation*}
\frac{n^{2} h^{2}}{8 m_{g} L^{2} \max }=m_{g} c^{2} . \tag{A18}
\end{equation*}
$$

Then, from the equation above it follows that

$$
\begin{equation*}
m_{g}= \pm \frac{n h}{c L_{\max } \sqrt{8}} \tag{A19}
\end{equation*}
$$

whence we see that there is a minimum value for $m_{g}$ given by

$$
\begin{equation*}
m_{g(\min )}= \pm \frac{h}{c L_{\max } \sqrt{8}} . \tag{A20}
\end{equation*}
$$

Now, if we take the eq. (A19), after some steps, we obtain that:

$$
\begin{equation*}
8=\frac{n^{2} h^{2}}{c^{2} L_{\max }^{2}} \cdot \frac{1}{m_{g}^{2}} \tag{A20b}
\end{equation*}
$$

This formula contain the number 8 that is a Fibonacci's number and is connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$
8=\frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]} . \text { (a) }
$$

Thence, we can rewrite the eq.(A20b) also as follows:

$$
\begin{equation*}
\frac{1}{3} \frac{4\left[\text { anti } \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(t t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]}=\frac{n^{2} h^{2}}{c^{2} L_{\max }^{2}} \cdot \frac{1}{m_{g}^{2}} . \tag{A20c}
\end{equation*}
$$

This is a new possible relationship of a relativistic theory of quantum gravity.
The relativistic gravitational mass $M_{g}=\frac{m_{g}}{\sqrt{1-\frac{V^{2}}{c^{2}}}}$, defined in eq. (A4), shows that

$$
\begin{equation*}
M_{g(\min )}=m_{g(\min )} . \tag{A21}
\end{equation*}
$$

The propagation number $k=|\vec{k}|=2 \pi / \lambda$ is restricted to the values $k=2 \pi n / L$. This is deducted assuming an arbitrarily large but finite cubical box of volume $L^{3}$. Thus, we have

$$
L=n \lambda . \quad(\mathrm{A} 22)
$$

From this equation, we conclude that

$$
\begin{equation*}
n_{\max }=\frac{L_{\max }}{\lambda_{\min }} \tag{A23}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{\min }=n_{\min } \lambda_{\min }=\lambda_{\min }, \tag{A24}
\end{equation*}
$$

since $n_{\text {min }}=1$.
Therefore, we can write that

$$
\begin{equation*}
L_{\max }=n_{\max } L_{\min } . \tag{A25}
\end{equation*}
$$

From this equation, we thus conclude that

$$
\begin{equation*}
L=n L_{\min } \tag{A26}
\end{equation*}
$$

or

$$
\begin{equation*}
L=\frac{L_{\max }}{n} . \tag{A27}
\end{equation*}
$$

Multiplying (A26) and (A27) by $\sqrt{3}$ and reminding that $d=L \sqrt{3}$, (see eq. A17b) we obtain

$$
\begin{equation*}
d=n d_{\min } \quad \text { or } \quad d=\frac{d_{\max }}{n} \tag{A28}
\end{equation*}
$$

that we can rewrite also as follows:

$$
\begin{equation*}
L \sqrt{3}=n L_{\min } \sqrt{3} \quad \text { or } \quad L \sqrt{3}=\frac{L_{\max } \sqrt{3}}{n} \tag{A28b}
\end{equation*}
$$

Equation above show that the length (and therefore the space) is quantized. By analogy to (A19) we can also conclude that

$$
\begin{equation*}
M_{g(\max )}=\frac{n_{\max } h}{c L_{\min } \sqrt{8}}, \tag{A29}
\end{equation*}
$$

and as for the (A20c), after some calculation, we obtain:

$$
\begin{equation*}
\frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi \pi^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right.}=\frac{n_{\max }^{2} h^{2}}{c^{2} L_{\min }^{2}} \cdot \frac{1}{M_{g(\max )}^{2}} . \tag{A29b}
\end{equation*}
$$

Equation (A25) tells us that $L_{\text {min }}=L_{\max } / n_{\max }$. Thus eq. (A29) can be rewritten as follows:

$$
\begin{equation*}
M_{g(\max )}=\frac{n_{\max }^{2} h}{c L_{\max } \sqrt{8}} \tag{A30}
\end{equation*}
$$

and thence, for the (A29b):

$$
\begin{equation*}
\frac{1}{3} \frac{4\left[\text { anti } \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cos \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(t t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]}=\frac{n_{\max }^{4} h^{2}}{c^{2} L_{\max }^{2}} \cdot \frac{1}{M_{g(\max )}^{2}} . \tag{A30b}
\end{equation*}
$$

Comparison of (A30) with (A20) shows that

$$
\begin{equation*}
M_{g(\max )}=n_{\max }^{2} m_{g(\min )} \tag{A31}
\end{equation*}
$$

which leads to following conclusion that

$$
\begin{equation*}
M_{g}=n^{2} m_{g(\min )} \tag{A32}
\end{equation*}
$$

This equation shows that the gravitational mass is quantized. The substitution of eq. (A32) into eq. (A8) leads to the quantization of gravity, i.e.,

$$
\begin{equation*}
g=\frac{G M_{g}}{r^{2}}=n^{2}\left(\frac{G m_{g(\min )}}{\left(r_{\max } / n\right)^{2}}\right)=n^{4} g_{\min } . \tag{A33}
\end{equation*}
$$

But for the eq. (A31b), we note that:

$$
\begin{equation*}
n^{4}=\frac{1}{3} \frac{4\left[\operatorname{anti} \log \frac{\int_{0}^{\infty} \frac{\cos \pi t x w^{\prime}}{\cosh \pi x} e^{-\pi x^{2} w^{\prime}} d x}{e^{-\frac{\pi^{2}}{4} w^{\prime}} \phi_{w^{\prime}}\left(i t w^{\prime}\right)}\right] \cdot \frac{\sqrt{142}}{t^{2} w^{\prime}} \cdot c^{2} L^{2} M_{g}^{2}}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]} \cdot \frac{1}{h^{2}}, \tag{A33b}
\end{equation*}
$$

thence:

$$
\begin{equation*}
g=\frac{G M_{g}}{r^{2}}=n^{2}\left(\frac{G m_{g(\min )}}{\left(r_{\max } / n\right)^{2}}\right)=\frac{4\left[\operatorname{anti\operatorname {log}\frac {\int _{0}^{\infty }\frac {\operatorname {cos}\pi txw^{\prime }}{\operatorname {cosh}\pi x}e^{-\pi x^{2}w^{\prime }}dx}{3}\frac {e^{-\frac {\pi ^{2}}{4}w^{\prime }}\phi _{w^{\prime }}(itw^{\prime })}{}]\cdot \frac {\sqrt {142}}{t^{2}w^{\prime }}\cdot c^{2}L^{2}M_{g}^{2}}\right.}{\log \left[\sqrt{\left(\frac{10+11 \sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7 \sqrt{2}}{4}\right)}\right]} \cdot \frac{1}{h^{2}} g_{\min } \tag{A33c}
\end{equation*}
$$

A Master Equation of a possible relativistic theory of quantum gravity, that show the quantization of a gravitational mass and that include also the "modes" that correspond to the physical vibrations of a superstring.

The redshift $z$ is often described as a redshift velocity, which is the recessional velocity that would produce the same redshift if it were caused by a linear Doppler effect. To determine the redshift velocity $v_{r s}$ the relation:

$$
\begin{equation*}
v_{r s} \equiv c z, \tag{A34}
\end{equation*}
$$

is used.
The redshift velocity agrees with the velocity from a low-velocity simplification of the so-called Fizeau-Doppler formula

$$
\begin{equation*}
z=\frac{\lambda_{o}}{\lambda_{e}}-1=\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}-1 \approx \frac{v}{c} \tag{A35}
\end{equation*}
$$

Here $\lambda_{o}, \lambda_{e}$ are the observed and emitted wavelengths respectively.
Suppose $R(t)$ is called the scale factor of the Universe, and increases as the Universe expands in a manner that depends upon the cosmological model selected. Its meaning is that all measured proper distances $D(t)$ between co-moving points increase proportionally to $R$. In other words:

$$
\begin{equation*}
\frac{D(t)}{D\left(t_{0}\right)}=\frac{R(t)}{R\left(t_{0}\right)}, \tag{A36}
\end{equation*}
$$

where $t_{0}$ is some reference time. If light is emitted from a galaxy at time $t_{e}$ and received by us at $t_{0}$, it is red shifted due to the expansion of space, and this redshift $z$ is simply:

$$
\begin{equation*}
z=\frac{R\left(t_{0}\right)}{R\left(t_{e}\right)}-1 . \tag{A37}
\end{equation*}
$$

Suppose a galaxy is at distance $D$, and this distance changes with time at a rate $d t D$. We call this rate of recession the "recession velocity" $v_{r}$ :

$$
\begin{equation*}
v_{r}=d_{t} D=\frac{d_{t} R}{R} D \tag{A38}
\end{equation*}
$$

We now define the Hubble constant as

$$
H \equiv \frac{d_{t} R}{R}
$$

that insert in eq. (A38) leads to the Hubble law:

$$
\begin{equation*}
v_{r}=H D . \tag{A39}
\end{equation*}
$$

From the Hubble's law, it follows that

$$
\begin{align*}
& V_{\max }=\tilde{H} l_{\max }=\tilde{H}\left(\frac{d_{\max }}{2}\right) \\
& V_{\min }=\tilde{H} l_{\min }=\tilde{H}\left(\frac{d_{\min }}{2}\right) \tag{A40}
\end{align*}
$$

whence

$$
\begin{equation*}
\frac{V_{\max }}{V_{\min }}=\frac{d_{\max }}{d_{\min }} \tag{A41}
\end{equation*}
$$

Equations (A28) tell us that $d_{\text {max }} / d_{\text {min }}=n_{\text {max }}$. Thus the equation above gives

$$
\begin{equation*}
V_{\min }=\frac{V_{\max }}{n_{\max }} \tag{A42}
\end{equation*}
$$

which leads to the following conclusion

$$
\begin{equation*}
V=\frac{V_{\max }}{n} \tag{A43}
\end{equation*}
$$

this equation shows that velocity is also quantized.
Furthermore, for the eqs. (A40), we note that

$$
V_{\max }=\tilde{H} l_{\max }=\tilde{H}\left(\frac{d_{\max }}{2}\right)
$$

and that $d=l \sqrt{3}$ (see. Eqs. (A27), (A28) and (A17b)) i.e. $d_{\max }=l_{\max } \sqrt{3}$.
Thence, we can rewrite the eq. (A43) also as follows:

$$
\begin{equation*}
V=\frac{V_{\max }}{n}=\tilde{H}\left(\frac{l_{\max } \sqrt{3}}{2}\right) \frac{1}{n}=\frac{\tilde{H} l_{\max }}{n} \cdot \frac{\sqrt{3}}{2} . \tag{A44}
\end{equation*}
$$

The temporal coordinate $x^{0}$ of space-time is now ( $x^{0}=c t$ is obtained when $V_{\max } \rightarrow c$ )

$$
x^{0}=V_{\max } x . \quad \text { (A45) }
$$

Substitution, from (A43) and (A40), of

$$
\begin{equation*}
V_{\max }=n V=n(\tilde{H} l) \tag{A46}
\end{equation*}
$$

into eq. (A45) yields

$$
\begin{equation*}
t=\frac{x^{0}}{V_{\max }}=\left(\frac{1}{n \widetilde{H}}\right)\left(\frac{x^{0}}{l}\right) . \tag{A47}
\end{equation*}
$$

Since $V=\tilde{H} l$ and $V=V_{\max } / n$ we can write that $l=V_{\max } \tilde{H}^{-1} / n=\frac{V_{\max }}{n \widetilde{H}}$. Thus, we have that (remember that $n t=t_{\text {max }}$ )

$$
\begin{equation*}
\left(\frac{x^{0}}{l}\right)=\tilde{H}(n t)=\tilde{H} t_{\max } \tag{A48}
\end{equation*}
$$

Therefore, we can finally write

$$
\begin{equation*}
t=\left(\frac{1}{n \widetilde{H}}\right)\left(\frac{x^{0}}{l}\right)=\frac{1}{n \widetilde{H}} \cdot \tilde{H} t_{\max }=t_{\max } / n . \tag{A49}
\end{equation*}
$$

Note that the eq. (A49), remember that $d=l \sqrt{3}$, can be also write as follows:

$$
\begin{equation*}
t=\left(\frac{1}{n \widetilde{H}}\right)\left(\frac{x^{0}}{l}\right)=\frac{1}{n \widetilde{H}} \cdot x^{0} \frac{\sqrt{3}}{d}=\sqrt{3}\left(\frac{x^{0}}{n \widetilde{H} d}\right) \tag{A49b}
\end{equation*}
$$

From this equation we can to observe that $t_{\max }=\sqrt{3}\left(\frac{x^{0}}{\tilde{H} d}\right)$.
From eqs. (A26) and (A49) we can easily conclude that the spacetime is not continuous it is quantized:

$$
\begin{align*}
L=n L_{\min } & \rightarrow \frac{d}{\sqrt{3}}=n \frac{d_{\min }}{\sqrt{3}}, \\
t=\left(\frac{1}{n \widetilde{H}}\right)\left(\frac{x^{0}}{l}\right)=\frac{1}{n \widetilde{H}} \cdot \tilde{H} t_{\max } & =t_{\max } / n \rightarrow \frac{1}{n \widetilde{H}} \cdot x^{0} \frac{\sqrt{3}}{d}=\sqrt{3}\left(\frac{x^{0}}{n \widetilde{H} d}\right) . \tag{A50}
\end{align*}
$$

Now we take the eq. (A44):

$$
V=\frac{V_{\max }}{n}=\tilde{H}\left(\frac{l_{\max } \sqrt{3}}{2}\right) \frac{1}{n}=\frac{\tilde{H} l_{\max }}{n} \cdot \frac{\sqrt{3}}{2} .
$$

For $\frac{\tilde{H} l_{\text {max }}}{n}=i$, i.e. the imaginary unit $i=\sqrt{-1}$, we have that

$$
\begin{equation*}
V=\frac{V_{\max }}{n}=\frac{i \sqrt{3}}{2} . \tag{A51}
\end{equation*}
$$

This expression show that the quantized velocity is related to the imaginary unit multiplied the irrational number 0,866025 .

We remember that the equilateral triangle ABC is divided into two congruent rectangular triangles whose catheti are the height and half of the side of the equilateral triangle and the hypotenuse is the side of the triangle ABC . Indicating with $\left(h, \frac{a}{2}, a\right)$ respectively, the measures of catheti and of the hypotenuse of the rectangular triangle HBC , and applying the theorem of Pythagoras, we have: the
measure of the height of an equilateral triangle is obtained by multiplying the half of his side for the square root of 3 .


Indeed, we have the following expression:

$$
h=\sqrt{a^{2}-\left(\frac{a}{2}\right)^{2}}=\sqrt{a^{2}-\frac{a^{2}}{4}}=\sqrt{\frac{3 a^{2}}{4}}=\frac{a}{2} \sqrt{3}=a \frac{\sqrt{3}}{2}=a \cdot 0,866
$$

In this picture, we note that the hexagon inscribed in the circle is formed from the six equilateral triangles and the side $a$, in this case, is equal to the radius of the circle inscribed. Thence, also here come out the number 0,866


We note that the number $0,866 \cong \frac{\sqrt{3}}{2}$ is the fix number of the regular hexagon. The regular hexagon can be divided also in 6 equal isosceles triangles whose height is equal to the apothem of the hexagon, namely, to the radius of the circle inscribed. In each hexagon there is always the same ratio between apothem and side, i.e.

$$
\begin{equation*}
a: l=0,866 \text {, for which } a=l \times 0,866 \text { and } l=\frac{a}{0,866} . \tag{A52}
\end{equation*}
$$

Now, we take an hexagon with side $l=4$. We have that:
$a=l \times 0,866=4 \times 0,866=3,464$;
$p=l \times 6=4 \times 6=24$;
$A=\frac{p \times a}{2}=\frac{24 \times 3,464}{2}=41,568$
$\varphi=\frac{A}{l^{2}}=\frac{41,568}{16}=2,598 \ldots$
For each regular hexagon, the ratio between the area and the side, is always equal to the 2,598 . Thence, the number 2,598 is the "constant" or fix number for the calculus of the area of a regular hexagon.
Now, from the formulae $A=\frac{p \times a}{2}, \varphi=\frac{A}{l^{2}}$, we have:

$$
\begin{gather*}
A=\frac{p \cdot \frac{l \sqrt{3}}{2}}{2} ; \quad(\mathrm{A} 53)  \tag{A53}\\
\varphi=\frac{p \cdot \frac{l \sqrt{3}}{2}}{2 l^{2}}=\frac{24 \cdot \frac{4 \sqrt{3}}{2}}{2 \cdot 16}=\frac{24 \cdot 4 \cdot 0,866}{32}=\frac{83,136}{32}=2,598 . \tag{A54}
\end{gather*}
$$

But, we note that $2,598 \cong 2,618=\left(\frac{\sqrt{5}+1}{2}\right)^{2}$. Indeed, the value of $A=l^{2} \times \varphi=16 \times 2,598=41,568$ is very near to $A=l^{2}\left(\frac{\sqrt{5}+1}{2}\right)^{2}=16 \times 2,618=41,888=41,8 \cong 41,56$
From (54), we have also that:

$$
\begin{equation*}
\frac{\sqrt{3}}{2}=\frac{2 \varphi l}{p} \tag{A55}
\end{equation*}
$$

Thence:

$$
\begin{equation*}
\frac{\sqrt{3}}{2} \cong \frac{2(\Phi)^{2} l}{p} \tag{A56}
\end{equation*}
$$

where $\Phi=\frac{\sqrt{5}+1}{2}=1,61803398$ is the aurea ratio.
In conclusion, the eq. (A51) $V=\frac{V_{\max }}{n}=\frac{i \sqrt{3}}{2}$, can be rewritten also as follows:

$$
\begin{equation*}
V=\frac{V_{\max }}{n}=\frac{i \sqrt{3}}{2} \cong \frac{i 2(\Phi)^{2} l}{p} \tag{A57}
\end{equation*}
$$

This expression show that the quantized velocity is related to the imaginary unit and to the aurea ratio.

## Appendix B

PROOF OF FERMAT'S LAST THEOREM FOR THE CUBIC EQUATION CASE N = 3

The equation

$$
\begin{equation*}
x^{3}+y^{3}=z^{3} \tag{B1}
\end{equation*}
$$

don't admits positive integer solutions when $\mathrm{xyz} \neq 0$.
Proof
Suppose $\mathrm{x} ; \mathrm{y}$; z are relatively prime, in fact, if they were not, any factor divides two of these, for the equation (1), it should also divide the third. Consequently, one and only one between $x ; y ; z$ can be even, so we can distinguish the two cases:
(1) $z$ is even and $x ; y$ are odd

Since x and y are odd, $\mathrm{x}-\mathrm{y}$ and $\mathrm{x}+\mathrm{y}$ are even, we put
$x+y=2 p$
$x-y=2 q$
from these we can obtain $x$ and $y$ as a function of $p$ and $q$
$\mathrm{x}=\mathrm{p}+\mathrm{q}$
$\mathrm{y}=\mathrm{p}-\mathrm{q}$
where p and q are co-prime, in fact, if they had one factor in common, this should also divides x and $y$, which is impossible as we said that they are relatively prime; They have opposite parity as x and y are odd, so p and/or q are one even and one odd.
Finally we can assume $p$ and $q$ positive, as we could possibly exchange the role between $x$ and $y$. Besides if $\mathrm{x}=\mathrm{y}$ then $\mathrm{z}^{3}=2 \mathrm{x}^{3}$ and being x and 2 prime among them, the only way for it to be a cube is that even 2 is a cube, but this is impossible.

We rewrite the (B1) in terms of p and q in the following way

$$
\begin{equation*}
z^{3}=x^{3}+y^{3}=(x+y)\left(x^{2}-x y+y^{2}\right)=2 p\left(p^{2}+3 q^{2}\right) \tag{B2}
\end{equation*}
$$

where the product $2 p\left(p^{2}+3 q^{2}\right)$ is a cube.
So we were able to transform a sum of cubes in a product of cubes which is much more simple to manage.

In fact we have, for example that $2^{3} * 3^{3}=6^{3}$
(2) x is even and z and y are odd (the case in which y is even is obtained by swapping the role between x and y )

For the same reasons as the previous case, we put
$\mathrm{z}-\mathrm{y}=2 \mathrm{p}$
$z+y=2 q$
thus obtaining
$y=q-p$
$\mathrm{z}=\mathrm{p}+\mathrm{q}$
where p and q are positive, in fact, $\mathrm{q}>0($ since $\mathrm{q}=(\mathrm{z}+\mathrm{y}) / 2$ and $\mathrm{p}=(\mathrm{z}-\mathrm{y}) / 2)$ and so $\mathrm{q}>\mathrm{p}$ (because $\mathrm{z}>\mathrm{y}$ ).
Besides for the same reasons as the previous case, they are prime between them and in opposite parity so p and/or q are one even and one odd.
Substituting into (B1) with an immediate calculation

$$
\begin{equation*}
x^{3}=z^{3}-y^{3}=(z-y)\left(y^{2}+z y+z^{2}\right)=2 p\left(p^{2}+3 q^{2}\right) \tag{B3}
\end{equation*}
$$

where the product $2 p\left(p^{2}+3 q^{2}\right)$ is a cube.

So we were able to transform a difference of cubes in a product of cubes which is much more simple to manage.

At this point we just have to prove that $2 p$ and $\left(p^{2}+3 q^{2}\right)$ are coprime, and that the only way for their product is a cube is that they are themselves cubes.

2 p must be a cube
$p^{2}+3 q^{2}$ must be a cube
we know that if $p^{2}+3 q^{2}$ is a cube only when we have
$p=a^{3}-9 a b^{2}=a\left(a^{2}-9 b^{2}\right)=a(a-3 b)(a+3 b)$
$\mathrm{q}=3 \mathrm{a}^{2} \mathrm{~b}-3 \mathrm{~b}^{3}=3 \mathrm{~b}\left(\mathrm{a}^{2}-\mathrm{b}^{2}\right)=3 \mathrm{~b}(\mathrm{a}-\mathrm{b})(\mathrm{a}+\mathrm{b})$
with $a$ and $b$ (not necessarily positive) co-prime and of opposite parity.
so that $p^{2}+3 q^{2}=\left(a^{2}+3 b^{2}\right)^{3}$.
We have so that
$2 p=2 a(a-3 b)(a+3 b)$
But this is impossible to be a cube because for the same reason we must have that:
2a must be a cube
$a-3 b$ must be a cube
$a+3 b$ must be a cube
if we put
$a-3 b=c^{3}$
$a+3 b=h^{3}$
$2 \mathrm{a}=\mathrm{w}^{3}$
So

$$
\begin{equation*}
c^{3}+h^{3}=\mathrm{w}^{3} \tag{B4}
\end{equation*}
$$

This seems to be a solution of (B1) with integers smaller than the solution original. Indeed
$c^{3} h^{3} \mathrm{w}^{3}=2 a(a+3 b)(a-3 b)=2 p$
by equations (B2) and (B3) we get that
$\mathrm{z}^{3}=2 \mathrm{p}\left(\mathrm{p}^{2}+3 \mathrm{q}^{2}\right) ; \quad \mathrm{z}^{3}=c^{3} h^{3} \mathrm{w}^{3}\left(\mathrm{p}^{2}+3 \mathrm{q}^{2}\right)$
$\mathrm{x}^{3}=2 \mathrm{p}\left(\mathrm{p}^{2}+3 \mathrm{q}^{2}\right) ; \quad x^{3}=c^{3} h^{3} \mathrm{w}^{3}\left(\mathrm{p}^{2}+3 \mathrm{q}^{2}\right)$
$z>x$
and then we have that
$c h w<z$
and then we have
$\mathrm{c}<\mathrm{Z}$
$\mathrm{h}<\mathrm{z}$
w $<$ Z
since $\mathrm{c}<\mathrm{z}, \mathrm{h}<\mathrm{z}$ and $\mathrm{w}<\mathrm{z}$, the equation (B4) is satisfied and is in contrast with Lemma Infinite Descent because we obtain an infinite descent of positive integers
This theorem proofs a property that, if satisfied by a positive integer, it can be satisfied by a positive integer smaller. The method of infinite descent shows that some properties or relations are impossible, if applied to positive integers, in fact, if we try these apply to any number, they must also apply when considering numbers smaller; but the latter, in turn, for the same reasons earlier, hold for some numbers even smaller, so to infinity. This process is impossible, as a sequence of integers cannot decrease indefinitely.

Then $\mathrm{x} * 3+\mathrm{y} * 3=\mathrm{z}^{3}$ has no positive $\mathrm{x}, \mathrm{y}$ and z integer solutions.

## Observations

The golden section is found in quantum physics (electron orbitals, masses of quarks, strings), chemistry (atomic numbers of the elements most stable), visible macrocosm (flowers, pine cones, etc.), astronomy (Bode's law on the orbits of the planets, etc.).
It could be defined, together $\pi$, a kind of glue between the two worlds (relativistic and quantum), then an indication of the truth of a possible future TOE.

We note that the golden ratio, as well as in this paper, is present in all four fundamental forces of nature, and this could be one of the possible unifying factors for future Theory of Everything that unifies, in fact, all four the forces referred to above, possibly thanks to string theory and its connections with some sectors of the Theory of Numbers, Fibonacci in particular.

In previous work we have discovered the presence of the golden section in

- Strong force, by the masses of the quarks,
- Gravity, in the orbits of the planets and in the rings of Saturn
- Electromagnetism;
- Weak force


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[^0]:    1 "Mathematical Foundations of the Relativistic Theory of Quantum Gravity" - Fran De Aquino arXiv:physics/0212033

