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# Möbius, Mellin, and mathematical physics

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We examine some results and techniques of analytic number theory which have application, or potential application, in mathematical physics. We consider inversion formulae for lattice sums, various transformations of infinite series and products, functional equations and scaling relations, with selected applications in electrostatics and statistical mechanics. In the analysis, the Mellin transform and the Riemann zeta function play a key role.

## 1. Introduction

This paper was stimulated by a brief note of Chen [1], which attracted some interest. Chen showed how to effect the asymptotic solution of several standard inverse problems in statistical physics by invoking the Möbius inversion formula, an apparently obscure result of algebraic number theory. The cornerstone of Chen's analysis is equivalent to the assertion that, under modest hypotheses on the functions $\alpha$ and $\beta$, if

$$\alpha(x) = \sum_{n=1}^{\infty} \beta(nx) \qquad \text{for all } x > 0\,, \tag{1.1}$$

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then

$$\beta(x) = \sum_{n=1}^{\infty} \mu(n)\, \alpha(nx) \qquad \text{for all } x > 0\,, \tag{1.2}$$

where the Möbius function $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} 1\,, & n = 1\,, \\ (-1)^r\,, & \text{if } n \text{ has } r \text{ distinct prime factors}\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{1.3}$$

To number theorists this key result in Chen is utterly trivial and well known, and Chen [2] subsequently noted that the rather circuitous original derivation of eq. (2) can be replaced by appeal to theorem 270 in ref. [3], yet to physicists not familiar with analysis buried in the classics like Titchmarsh [4] or Hardy and Wright [3] some new magical tools seemed to have been invented. Indeed the Editor of Nature suggested [5] that by so calling in the treasure-trove of the old world some new insights into physics might become accessible.

In fact the resurgence of interest in classical analysis and in number theoretic formulae is of fairly long standing [6–8]. It is represented by the work of Andrews and of Askey on $q$-series and generalised hypergeometric functions [9], in Rogers–Ramanujan relations [10], in the popularity of Ramanujan's notebooks, which have finally become available [11], in work of Glasser and Zucker [12] on applications of number theory to the calculation of lattice sums, in a re-evaluation of R.B. Dingle's contribution to asymptotics [13], in Coxeter's astonishing results on sphere packing (discussed by Dyson [14]), in random walks with infinite variance [15], conferences on analytic number theory in physics, and the very beautiful work of the School associated with Dingle's student Berry at Bristol [16–19].

A paper of Dyson [14] is a classic and deep exposition of the kind of tantalising insights that might be coming into sight. To the average or garden variety physicist or chemist unfamiliar with what Dyson in his paper calls "unfashionable pursuits", these matters are a deep mystery. The language of mathematicians is almost deliberately obscure in its pursuit of exactness, and designed to admit only the anointed to its hallowed halls of revealed truth. In fact all mathematics is a tautology, and all physics uses mathematics to look in different ways at a fundamental problem of philosophy – how to bridge the discrete and continuous. Much of the work discussed that skirts around the role of number theory in physics pays lip service to motivation through calling in physical problems, e.g. Brownian motion, distribution of eigenvalues of random matrices, whatever, but really represents mathematicians in search of an application. Although an exception can be found in the work of Berry, this

paper is no exception. The unfamiliar results reported may or may not be new, and whether or not they are useful remains to be seen.

That number theory, arithmetic, is the Queen of Mathematics is self-evident to number theorists, and a matter of aesthetics. It can be established or bolstered by appeal to Gauss [20] who said so[#1]. Indeed it can be argued that the analytic structure of the Riemann zeta function, defined for $\mathrm{Re}(s) > 1$ by the series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{1.4}$$

is about all there is to ponder[#2]. But the proposition is not so evident to others, so we shall make some observations that render it more palatable.

In section 2, we revisit the inversion formulae (1.1) and (1.2). Viewed from an appropriate perspective, these formulae and Chen's applications of them lose their mystique. Indeed, as we show, from this perspective it is easy to discover some attractive generalizations of the inversion formulae. In section 3, we examine some tautologies associated with the Riemann zeta function, which lead to (among other things) a strange integral representation of $\zeta(s)$ involving Euler's infinite product from the theory of theta functions. The point of view taken here is beautifully espoused in the charmingly instinctive classic on theta functions by Bellman [21], which the uninitiated reader is urged to consult. Section 4 contains an excursion into statistical mechanical systems with a maximum permissible temperature. Sections 5, 6 and 7 address the transformation of infinite series, with applications to the solution of nonlinear difference equations, the theory of Euler's product and the derivation of new infinite product representations of theta functions. We conclude in section 8 with a discussion of some series related to the electrostatic properties of crystalline and quasicrystalline solids. Some results from the theory of theta functions are summarized in the appendix.

Our discussion makes frequent use of the Mellin transform. Although this transform continually finds new and elegant applications [22, 23], it is not often useful in the solution of differential equation problems and is therefore unfortunately often omitted from undergraduate mathematics and physics courses. The Mellin transform $\tilde{f}(s)$ of a function $f(x)$ is defined by the integral

$$\tilde{f}(s) = \int_0^{\infty} x^{s-1} f(x)\,\mathrm{d}x \tag{1.5}$$

[#1] "Mathematics is the Queen of the Sciences and Arithmetic is the Queen of Mathematics".

[#2] The Riemann zeta function is central at least to *multiplicative* number theory. The present paper makes no significant incursions into *additive* number theory.

for those values of $s$ for which the integral converges, and by analytic continuation elsewhere. The Mellin inversion formula [24] establishes sufficient conditions for the inversion of the transform (1.5) by the contour integral

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \tilde{f}(s)\, ds\,, \tag{1.6}$$

where the contour of integration is the line $\mathrm{Re}(s) = c$, with $c$ chosen so that the original integral defining the Mellin transform converges adequately for all points on the contour. In particular, the usual gamma function is the Mellin transform of the function $e^{-x}$, i.e.,

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx \qquad (\mathrm{Re}(s) > 0) \tag{1.7}$$

and so for any real, positive $c$,

$$e^{-x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \Gamma(s)\, ds\,. \tag{1.8}$$

The Riemann zeta function can also be expressed in terms of a Mellin transform. If we multiply eq. (1.7) by $n^{-s}$, take $t = x/n$ as a new integration variable and sum over $n$ from 1 to $\infty$, we find that

$$\zeta(s)\,\Gamma(s) = \int_0^\infty \frac{t^{s-1} e^{-t}\, dt}{1 - e^{-t}} = -(s-1) \int_0^\infty t^{s-2} \log(1 - e^{-t})\, dt \qquad (\mathrm{Re}(s) > 1)\,. \tag{1.9}$$

The Riemann zeta function is the classic example of a *Dirichlet series* $\sum_{n=1}^{\infty} a_n n^{-s}$. If such a series converges at $s = s_0$, it converges to a holomorphic function for $\mathrm{Re}(s - s_0) > 0$.

## 2. Inversion formulae

Taking the Mellin transforms of both sides of eqs (1.1) and (1.2), we find that these equations underlying Chen's analysis reduce to

$$\tilde{\alpha}(s) = \zeta(s)\, \tilde{\beta}(s) \tag{2.1}$$

and

$$\tilde{\beta}(s) = \tilde{\alpha}(s) \sum_{n=1}^{\infty} \mu(n)\, n^{-s}, \tag{2.2}$$

for appropriately behaved $\alpha$ and $\beta$ and $\mathrm{Re}(s) > 1$. Chen's result is equivalent to the assertion that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n)\, n^{-s}. \tag{2.3}$$

Eq. (2.3) is very well known, but we indulge ourselves in proving it here. If we subtract $2^{-s}\zeta(s)$ from the series (1.4), we remove all values of $n$ containing a factor of 2 from the summation. We can next apply a similar trick to remove all $n$ values containing a factor of 3, and so on. If $p$ denotes a generic prime number and $P$ is any fixed prime, recalling the convention that 1 is not a prime number, we have

$$\zeta(s) \prod_{p \le P} (1 - p^{-s}) = 1 + \sum_{n \ge P}^{*} n^{-s}, \tag{2.4}$$

where in the starred summation, all $n$ values containing a prime factor less than or equal to $P$ are omitted. We now let $P \to \infty$ (that the number of primes is infinite was known to Euclid!) and deduce that

$$\frac{1}{\zeta(s)} = \prod_{p} (1 - p^{-s}). \tag{2.5}$$

Eq. (2.3) follows from expanding the infinite product.

The reader may consult Hughes et al. [25] for a discussion of how Chen's applications of the inversion formulae (1.1) and (1.2) to several physical problems are clarified by the Mellin transform perspective from which they have been derived above. Let us use these ideas to discover higher-dimensional inversion formulae. Consider the $d$-dimensional hypercubic lattice $\mathbb{Z}^d$, the sites of which correspond to vectors $\boldsymbol{m} = (m_1, m_2, \ldots, m_d)$, where $m_1, m_2, \ldots, m_d$ are integers. The natural measure of distance is $\sqrt{m_1^2 + m_2^2 + \cdots m_d^2}$. The analogue of the inversion problem (1.1) is the determination of $\beta(x)$, given that

$$\alpha(x) = \sum_{\boldsymbol{m} \in \Omega} \beta(x\sqrt{m_1^2 + m_2^2 + \cdots m_d^2}). \tag{2.6}$$

Here $\Omega$ is a subset of $\mathbb{Z}^d$, with the point $\boldsymbol{m} = \boldsymbol{0}$ excluded. Since the argument of

$\beta$ in eq. (2.6) is $x\sqrt{\text{integer}}$, by analogy with eqs (1.1) and (1.2), we seek an inversion formula of the form

$$\beta(x) = \sum_{k=1}^{\infty} \omega(k)\, \alpha(x\sqrt{k}) \qquad \text{for all } x > 0\,. \tag{2.7}$$

Taking Mellin transforms of eqs (2.6) and (2.7), we find them to be equivalent provided that

$$\sum_{k=1}^{\infty} \omega(k)\, k^{-s/2} = \left( \sum_{m \in \Omega} (m_1^2 + m_2^2 + \cdots + m_d^2)^{s/2} \right)^{-1}. \tag{2.8}$$

We can now expand the right-hand side in powers of $k^{-s/2}$, and equate coefficients of $k^{-s/2}$ to deduce the value of $\omega(k)$ for $k$ as large as our patience permits.

We illustrate the technique for two examples where the summation variables are drawn from sets of points on the two-dimensional square lattice.

*Example (i). A quadrant of the square lattice.* Let

$$\alpha(x) = \sum_{m,n=0}^{\infty}{}' \beta(x\sqrt{m^2 + n^2}) \qquad \text{for all } x > 0\,, \tag{2.9}$$

where the prime denotes the omission of the term corresponding to $m = n = 0$ from the sum. The coefficients $\omega(k)$ in the inversion formula (2.7) are given by

$$\sum_{k=1}^{\infty} \omega(k)\, k^{-s/2} = \left( \sum_{m,n=0}^{\infty}{}' (m^2 + n^2)^{-s/2} \right)^{-1}. \tag{2.10}$$

In table I, we show the values of $\omega(k)$ for $k \leq 30$. These can be generated on paper by brute force in a few minutes. In table II, we illustrate the inversion formula for the case $\beta(x) = e^{-x^2}$. For this choice of $\beta(x)$, the function $\alpha(x)$ is easily evaluated numerically. We now attempt to recover $\beta(x)$ using eq. (2.10). The results are shown in table II. It will be seen from table II that unless $x$ is very small, only a few terms need to be taken to ensure an accurate approximation to $\beta(x)$.

Inspection of table I reveals some striking properties of the sequence $\omega(k)$. After a fairly innocuous beginning $[\omega(1) = \frac{1}{2},\ \omega(2) = -\frac{1}{4},\ \omega(3) = 0, \ldots]$, the values 0 and $-\frac{1}{2}$ are taken rather often, while

$$\omega(2^n) = (-1)^{n+1} 3/2^{n+1}\,, \qquad n = 2, 3, 4, \ldots\,. \tag{2.11}$$

We seek an explanation for this. If we write $A(z) = \alpha(\sqrt{z})$ and $B(z) = \beta(\sqrt{z})$,

Table I
Coefficients $\omega(k)$ for the inversion of the two-dimensional lattice sum in example (i).

| $k$ | $\omega(k)$ | $k$ | $\omega(k)$ |
|---|---|---|---|
| 1 | $\frac{1}{2}$ | 16 | $-\frac{3}{32}$ |
| 2 | $-\frac{1}{4}$ | 17 | $-\frac{1}{2}$ |
| 3 | 0 | 18 | $\frac{1}{4}$ |
| 4 | $-\frac{3}{8}$ | 19 | 0 |
| 5 | $-\frac{1}{2}$ | 20 | $\frac{5}{8}$ |
| 6 | 0 | 21 | 0 |
| 7 | 0 | 22 | 0 |
| 8 | $\frac{5}{16}$ | 23 | 0 |
| 9 | $-1$ | 24 | 0 |
| 10 | 0 | 25 | $-\frac{3}{4}$ |
| 11 | 0 | 26 | 0 |
| 12 | 0 | 27 | 0 |
| 13 | $-\frac{3}{4}$ | 28 | 0 |
| 14 | 0 | 29 | $\frac{1}{2}$ |
| 15 | 0 | 30 | 0 |

eqs. (2.9) and (2.10) are equivalent to the assertion that

$$A(x) = \sum_{m,n=0}^{\infty}{}' \sum_{k=1}^{\infty} \omega(k)\, A[k(m^2+n^2)x]\,. \tag{2.12}$$

If we denote by $r_2^-(n)$ the number of representations of $n$ in the form $a^2 + b^2$,

Table II
A test of the inversion formula for two-dimensional lattice sums in example (i), using the function $\beta(x) = \exp(-x^2)$. Here the series in the inversion formula is truncated for $k > N$. Values of $N$ for which $\omega(N) = 0$ are not shown.

| $N$ | $x = 0.2$ | $x = 0.4$ | $x = 0.6$ | $x = 0.8$ |
|---|---|---|---|---|
| 1 | 11.59509258 | 3.18715291 | 1.45435322 | 0.79248486 |
| 2 | 8.54503635 | 2.36939992 | 1.10803662 | 0.63030593 |
| 4 | 6.15467167 | 1.77503628 | 0.90636108 | 0.57005751 |
| 5 | 3.57534444 | 1.16373356 | 0.72652995 | 0.52846165 |
| 8 | 4.18865917 | 1.28536775 | 0.74817525 | 0.53070936 |
| 9 | 2.73430595 | 1.01646703 | 0.70824200 | 0.52755328 |
| 13 | 1.73962757 | 0.88345898 | 0.69891993 | 0.52730966 |
| 16 | 1.59103666 | 0.86839687 | 0.69832817 | 0.52730296 |
| 17 | 0.85118233 | 0.80033230 | 0.69612729 | 0.52728413 |
| 18 | 1.19749893 | 0.82919271 | 0.69689479 | 0.52728909 |
| 20 | 1.96162734 | 0.88118753 | 0.69782837 | 0.52729255 |
| 25 | 1.50068770 | 0.86270404 | 0.69770495 | 0.52729243 |
| 29 | 1.12529414 | 0.85299970 | 0.69767571 | 0.52729242 |
| Exact answer | 0.96078944 | 0.85214379 | 0.69767633 | 0.52729242 |

with $a \geqslant 0$ and $b \geqslant 0$, then eq. (2.12) becomes

$$A(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} r_2^+(n)\, \omega(k)\, A(knx)\,. \tag{2.13}$$

and if by $d \mid m$ we mean that $d$ is a divisor of $m$ (where 1 and $m$ are regarded as divisors of $m$), then we may write

$$A(x) = \sum_{m=1}^{\infty} \sum_{d \mid m} r_2^+(m/d)\, \omega(d)\, A(mx)\,. \tag{2.14}$$

We now see that the coefficients $\omega(k)$ must be chosen such that

$$r_2^+(1)\, \omega(1) = 1\,, \tag{2.15}$$

and for $m \geqslant 2$,

$$\sum_{d \mid m} r_2^+(m/d)\, \omega(d) = 0\,. \tag{2.16}$$

Since $r_2^+(1) = 2$, we infer at once that $\omega(1) = \frac{1}{2}$, while eq. (2.16) furnishes us with a recurrence relation, from which the entries in table 1 are easily calculated, as an alternative to the brute force calculation described in section 2. Moreover, the values of $\omega(k)$ for many values of $k$ can be written down at once. Consider the case in which $p$ is an odd prime number. Setting $m = p$ in eq. (2.16), since the divisors are $d = 1$ and $d = p$, we have $r_2^+(p)\, \omega(1) + r_2^+(1)\, \omega(p) = 0$, and since $\omega(1) = \frac{1}{2}$ and $r_2^+(1) = 2$,

$$\omega(p) = -\tfrac{1}{4} r_2^+(p)\,. \tag{2.17}$$

It is trivial to verify that a prime of the form $4m + 3$ cannot be expressed as a sum of two integral squares. It can be shown (ref. [3], pp. 218–219) that a prime number of the form $4m + 1$ can be expressed as a sum of two integral squares in only one way, apart from the sign of the integers and their order in the sum, so that if $p = 4m + 1$ is prime, then $r_2^+(p) = 2$. Hence for any odd prime $p$,

$$\omega(p) = \begin{cases} -\frac{1}{2}\,, & p = 4m + 1\,, \\ 0\,, & p = 4m + 3\,. \end{cases} \tag{2.18}$$

We can determine $\omega(k)$ for many other values of $k$. For example, if $p$ and $p'$ are distinct odd primes, then

$$\omega(2p) = 0\,, \qquad \omega(p^2) = -\tfrac{1}{2}\,, \qquad \omega(pp') = 0\,, \tag{2.19}$$

and $\omega(2pp')$ takes only two values ($\frac{1}{2}$ if $p = 4m+1$ and $p' = 4m'+1$; 0 otherwise). The analysis is simplified if one notes the classical result of Jacobi from the theory of elliptic theta functions (see e.g., ref. [26], or ref. [3], theorem 278) that the number of integral solutions of

$$x_1^2 + x_2^2 = n \tag{2.20}$$

is

$$r_2(n) = 4[d_1(n) - d_3(n)]\,, \tag{2.21}$$

where $d_1(n)$ and $d_3(n)$ are the number of divisors of $n$ of the form $4m+1$ and $4m+3$ respectively. In the definition of $r_2(n)$, the order of the decomposition and the signs of the integers are both counted, so that for example, $r_2(1) = 4$. If $n$ is neither a perfect square nor twice a perfect square, all decompositions of the form (2.20) involve unequal, nonzero integers and it is clear that $r_2(n) = 4r_2^1(n)$.

Eq. (2.18) and (2.19) explain the frequent occurrence of 0 and $-\frac{1}{2}$ in table I. The reader is invited to derive eq. (2.11) from eq. (2.16).

*Example (ii). The full square lattice.* Let

$$\alpha(x) = \sum_{m,n=-\infty}^{\infty}{}' \beta(x\sqrt{m^2+n^2}) \qquad \text{for all } x > 0\,, \tag{2.22}$$

where the prime denotes the omission of the term corresponding to $m = n = 0$ from the sum. Here the inversion formula can be discovered exactly as in example (i) above, but we can in fact proceed somewhat further. If we take the Mellin transform of both sides of eq. (2.22), we deduce that

$$\tilde{\alpha}(s) = \left(\sum_{m,n=-\infty}^{\infty}{}' (m^2+n^2)^{-s/2}\right)\tilde{\beta}(s)\,. \tag{2.23}$$

Titchmarsh [4] (p. 35), Hardy and Wright [3] (theorem 306) and Glasser [27] have shown that for $\mathrm{Re}(\zeta) > 1$,

$$\sum_{m,n=-\infty}^{\infty}{}' (m^2+n^2)^{-z} = 4L(z)\,\zeta(z)\,, \tag{2.24}$$

where

$$L(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^z}\,. \tag{2.25}$$

Hence

$$\tilde{\beta}(s) = \frac{\tilde{\alpha}(s)}{4L(\frac{1}{2}s)\,\zeta(\frac{1}{2}s)} \,. \tag{2.26}$$

The function $L(z)$ (denoted by $\beta(z)$ in some texts) has been extensively studied [27]. It governs the distribution of prime numbers in the progression $4m+1$ and $4m+3$. Its values are known for the odd positive integers, for example $L(1) = \frac{1}{4}\pi$ and $L(3) = \frac{1}{32}\pi^3$, while $L(2) \approx 0.916$ defines Catalan's constant.

To effect the inversion of eq. (2.22) as a double sum, we write

$$\frac{1}{L(s)} = \sum_{m=1}^{\infty} \bar{\omega}(m)\, m^{-s} \tag{2.27}$$

and use eq. (2.3) to deduce that

$$\tilde{\beta}(s) = \frac{1}{4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\bar{\omega}(m)\,\mu(n)\,\tilde{\alpha}(s)}{m^{s/2} n^{s/2}} \,. \tag{2.28}$$

Inverting the Mellin transform, we deduce that

$$\beta(x) = \frac{1}{4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{\omega}(m)\,\mu(n)\,\alpha(x\sqrt{mn}) \,. \tag{2.29}$$

We now need a recipe for computing the coefficients $\bar{\omega}(m)$. If $\varphi(m)$ is a completely multiplicative function of positive integers $m$ and $n$, in the sense that

$$\varphi(m)\,\varphi(n) = \varphi(mn) \,, \tag{2.30}$$

then, where the product over $p$ is taken over all prime numbers,

$$\sum_{m=1}^{\infty} \varphi(m)\, m^{-s} = \prod_{p} [1 - \varphi(p)\, p^{-s}]^{-1} \,, \tag{2.31}$$

provided that the series on the left-hand side converges absolutely (ref. [28], p. 168). The proof of this is a simple generalization of the proof of eq. (2.5). If we write

$$L(s) = \sum_{m=1}^{\infty} \varphi(m)\, m^{-s} \,, \tag{2.32}$$

then $\varphi(2k) = 0$ and $\varphi(2k-1) = (-1)^{k-1}$. It is easily verified that in this case eq.

(2.30) holds, so that

$$\frac{1}{L(s)} = \prod_{p} [1 - \varphi(p)p^{-s}] = \sum_{m=1}^{\infty} \bar{\omega}(m)\, m^{-s}, \tag{2.33}$$

where $\bar{\omega}(1) = 1$ and $\bar{\omega}(m)$ is zero for $m \geq 2$ unless $m$ is the product of distinct, odd prime numbers (so that if $m$ is even, or if $\mu(m) = 0$, then $\bar{\omega}(m) = 0$). For those $m$ where $\bar{\omega}(m)$ is nonzero, if we write

$$m = (2k_1 - 1)(2k_2 - 1)\cdots(2k_M - 1), \tag{2.34}$$

where $m$ has exactly $M$ distinct prime factors, then

$$\bar{\omega}(m) = (-1)^{\kappa(m)}, \tag{2.35}$$

where

$$\kappa(m) = k_1 + k_2 + \cdots + k_M. \tag{2.36}$$

If, as above, $d_3(n)$ denotes the number of divisors of $n$ of the form $4m + 3$, we may rewrite eq. (2.35) as

$$\bar{\omega}(m) = |\mu(m)|(-1)^{d_3(m)} \qquad (m \text{ odd}). \tag{2.37}$$

If we change the summation indices in eq. (2.29), we may write the inversion formula in the form (2.7), where

$$\omega(k) = \frac{1}{4} \sum_{d|k} \bar{\omega}(d)\, \mu(k/d) = \frac{1}{4} \sum_{\substack{d|k \\ d \text{ odd}}} |\mu(d)|\, \mu(k/d)\, (-1)^{d_3(m)}. \tag{2.38}$$

Even if we had not seen how to find the explicit formula (2.38) for $\omega(k)$, imitating the analysis which led to eq. (2.14), we have

$$A(x) = \sum_{m=1}^{\infty} \sum_{d|m} r_2(m/d)\, \omega(d)\, A(mx), \tag{2.39}$$

where $r_2(n)$ is given by eq. (2.21). We now see that the coefficients $\omega(k)$ must be chosen such that

$$r_2(1)\, \omega(1) = 1, \tag{2.40}$$

and for $m \geq 2$,

$$\sum_{d \mid m} r_2(m/d)\, \omega(d) = 0\,, \tag{2.41}$$

so that the determination of extravagantly many coefficients is straightforward.

We have not discussed here the generalization of our analysis to higher-dimensional problems. The basic Mellin transform procedure and subsequent brute-force determination of the coefficients analogous to $\omega(k)$ in examples (i) and (ii) is evidently applicable. For hypercubic lattices, we can also exploit some number-theoretic results, since the number $r_k(n)$ of integral solutions of $x_1^2 + x_2^2 + \cdots + x_k^2 = n$ has been extensively studied, with a classical solution of Jacobi available for the cases $k = 2$, 4, 6 and 8, and solutions of Ramanujan and others covering even $k$ for $k \leqslant 24$ (ref. [26], ch. IX). For odd $k$, the analysis is more difficult, but some references can be found in pp. 157–158 of ref. [26].

## 3. Tautologies

Consider now the generalised zeta function, defined for $\mathrm{Re}(s) > 1$ and $0 < a \leqslant 1$ by the series

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s}\,. \tag{3.1}$$

With this definition,

$$\zeta(s, 1) = \zeta(s)\,. \tag{3.2}$$

We claim that the definition of $\zeta(s, a)$ is equivalent to the theory of integration [29] and therefore the calculus. To see this, note that from the binomial theorem

$$(n + a)^{-s} = n^{-s} \sum_{m=0}^{\infty} \frac{(-1)^m s(s+1)\cdots(s+m-1)a^m}{m!n^m}\,, \tag{3.3}$$

so that inserting this into the summand in eq. (3.1) and interchanging the orders of summation, we have the identity

$$\zeta(s, a) = a^{-s} + \zeta(s) - sa\zeta(s+1) + \frac{s(s+1)a^2}{2!}\,\zeta(s+2) - \frac{s(s+1)(s+2)a^3}{3!}\,\zeta(s+3) + \cdots\,, \tag{3.4}$$

which we establish originally for those values of $s$ for which the series manipulations are valid, but which hold more generally by analytic continuation.

Setting $a = 1$ we have the remarkable identity

$$1 - s\zeta(s+1) + \frac{s(s+1)}{2!}\zeta(s+2) - \frac{s(s+1)(s+2)}{3!}\zeta(s+3) + \cdots = 0 . \tag{3.5}$$

By assuming this identity holds for all $s$, we have

$$\lim_{s\to 0} s\zeta(1+s) = 1 , \qquad \zeta(0) = -\tfrac{1}{2} , \qquad \zeta(-1) = -\tfrac{1}{12} , \qquad \zeta(-2) = 0 , \ldots . \tag{3.6}$$

From eq. (3.4) we can obtain the standard expression for $\zeta(-n, a)$ in terms of Bernoulli polynomials (ref. [30], p. 267). We refer the reader to ref. [4], p. 33, for some similar formulae.

The apparently trivial identities (3.1)–(3.5) are in reality quite profound. Indeed a theory of integration (and differentiation and interpolation) can be derived from them [29]. The theory so obtained gives an extension of the usual Riemann integral as a sum over an arbitrary finite partition together with a remainder series which is an asymptotic expansion. (The Riemann integral is the special case of an infinite partition.) This more general definition of an integral gives directly analytic continuation and includes integrals over the delta function and other generalized functions automatically.

Which is the more fundamental way to view matters is immaterial. What is important that one sees through such considerations is that the zeta function of Riemann is fundamental and central to all analysis.

An interesting result can be obtained from eq. (3.4). If we write for brevity

$$(s)_n = s(s+1)\cdots(s+n-1) = \Gamma(s+n)/\Gamma(s) , \tag{3.7}$$

we have

$$\zeta(s, a) - a^{-s} = \sum_{m=0}^{\infty} \frac{(-1)^m \zeta(s+m)\,(s)_m a^m}{m!} \tag{3.8}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \zeta(s+m)\,\zeta(s+m+1)\,(s)_m a^m}{\zeta(s+m+1)\,m!} . \tag{3.9}$$

Replacing $1/\zeta(s+m+1)$ with the series $\sum_{n=1}^{\infty} \mu(n)\, n^{-s-m-1}$ and interchanging orders of summation, we have

$$\zeta(s, a) - a^{-s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{m=0}^{\infty} \frac{(-1)^m \zeta(s+m)\, \zeta(s+m+1)\, (s)_m a^m}{n^{s+m} m!} . \tag{3.10}$$

If we replace $s$ by $s+1$ in eq. (1.9), we obtain

$$\zeta(s+1) = -\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \log(1 - e^{-t})\, dt . \tag{3.11}$$

Now letting $t = kx$, and summing over $k$, we produce the identity

$$\zeta(s)\, \zeta(s+1) = -\sum_{k=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \log(1 - e^{-kx})\, dx \tag{3.12}$$

$$= -\frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \log\Big( \prod_{k=1}^{\infty} (1 - e^{-kx}) \Big)\, dx \tag{3.13}$$

$$= -\frac{n^s}{\Gamma(s)} \int_0^{\infty} x^{s-1} \log\Big( \prod_{k=1}^{\infty} (1 - e^{-nkx}) \Big)\, dx . \tag{3.14}$$

This identity (with $s$ replaced by $s+m$) can now be used, together with eq. (3.7), to eliminate $\zeta(s+m)\, \zeta(s+m+1)\, (s)_m$ from eq. (3.9). When this is done, the sum over $m$ can be recognized as an exponential, giving

$$\Gamma(s)\, [\zeta(s, a) - a^{-s}] = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \int_0^{\infty} x^{s-1} e^{-ax} \log\Big( \prod_{k=1}^{\infty} (1 - e^{-nkx}) \Big)\, dx . \tag{3.15}$$

This formula illustrates another connection that will recur below, namely the intimate connection between the generalised zeta function and Euler's product

$$F_E(x) = \prod_{k=1}^{\infty} (1 - x^k) , \tag{3.16}$$

which is central to the theory of partitions and the theory of elliptic modular functions [26]. We may view eq. (3.15) as a "totem pole" for the higher transcendental functions, just as the wonderful formula $e^{i\pi} = -1$ may be regarded as a "totem pole" for the elementary transcendental functions.

By manipulating the inversion formulae (2.1) and (2.2), which reduce to the tautology

$$\alpha(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mu(n)\, \alpha(mnx) \qquad \text{for all } x > 0\,, \tag{3.17}$$

we can produce additional bizarre formulae involving the Möbius function, for example,

$$e^{-x} = \sum_{n=1}^{\infty} \frac{\mu(n)\, e^{-nx}}{1 - e^{-nx}}\,. \tag{3.18}$$

Neither of the results (3.15) and (3.18) would be deemed obvious by the casual reader, yet the proofs of them have required only some elementary manipulations of series and integrals. Eq. (3.18), when rewritten in the form

$$z = \sum_{n=1}^{\infty} \frac{\mu(n)\, z^n}{1 - z^n}\,, \tag{3.19}$$

is actually a special case of a known result for *Lambert series* [3]. The reader is invited to set $z = e^{-x}$ and take Mellin transforms to deduce that

$$\sum_{n=1}^{\infty} \frac{a_n z^n}{1 - z^n} = \sum_{n=1}^{\infty} b_n z^n \quad \text{if and only if} \quad \zeta(s) \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s}\,. \tag{3.20}$$

The formulae produced in this section are examples of "fruitful tautologies", obtained by playing around with series and integrals. Any inversion formula, such as the familiar Fourier integral theorem or the Mellin inversion theorem (1.6), can be a source of similar tautologies. Many superficially unrelated tautologies can be proved to be precisely equivalent. The canonical example of this is furnished by the functional equation for the Riemann zeta function

$$\zeta(s) = 2^s \pi^{s-1} \sin(\tfrac{1}{2}\pi s)\, \Gamma(1 - s)\, \zeta(1 - s) \tag{3.21}$$

(more succinctly described as the *Riemann relation*) and a transformation from the theory of theta functions,

$$\phi(x) = x^{-1/2}[\phi(1/x) + \tfrac{1}{2}] - \tfrac{1}{2} \qquad (x > 0)\,, \tag{3.22}$$

where

$$\phi(x) = \sum_{m=1}^{\infty} e^{-m^2 \pi x}\,. \tag{3.23}$$

If we represent $\phi(x)$ as an inverse Mellin transform and translate the contour of integration across two poles, use of the Riemann relation and a change of

integration variable produces an inverse Mellin transform which can be recognized as $x^{-1/2}\phi(1/x)$, so that the Riemann relation (3.21) implies eq. (3.22). There are eight different proofs of the Riemann relation in ref. [4]. One of these commences with eq. (3.22), so that eq. (3.22) implies the Riemann relation (3.21). It is purely a matter of taste which of the two formulae is the more fundamental, and each can be derived in other ways apparently without reference to the other.

In section 8, we uncover some further tautologies, arising from the equivalence of certain sums of delta functions with divergent sums of cosines, interpreted within the sense of Lighthill's presentation of generalized functions. The simplest prototype of these results is the identity (ref. [31], pp. 67–68)

$$\sum_{m=-\infty}^{\infty} \delta(x-m) = \sum_{m=-\infty}^{\infty} e^{2\pi m x i} . \tag{3.24}$$

This is the generalized function statement of the Poisson summation formula

$$\sum_{m=-\infty}^{\infty} P(m) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} P(x)\, e^{2\pi m x i}\, dx , \tag{3.25}$$

and is equivalent to the classical formula for the determination of the coefficients in a Fourier series. Either of eqs. (3.24) or (3.25) can be used to derive the Riemann relation (3.21) and the theta function transformation (3.22). Bellman [21], pp. 34–36, has obtained the Poisson summation formula as a consequence of the Riemann relation, remarking that this "shows what is so often true in analysis, namely that it is not easy to distinguish between the general and the particular". The Riemann relation, the theta function transformation, Poisson's summation formula and the key results of the theory of Fourier series are all manifestations of the same tautology.

## 4. Prime and natural gases

In the preceding discussion, we have been preoccupied with the Riemann zeta function (1.4), Euler's infinite product representation of it, (2.5), and some consequences, producing connections with Euler's product (3.16) and other fruitful mathematical tautologies. Here we wish to make contact with physics, with examples drawn from statistical mechanics. The Riemann zeta function is paramount in the statistical mechanics of Bose–Einstein gases [32] and in the examples which follow it is also pre-eminent. Some of these considerations have been anticipated by Mackey [33] and by Julia [34, 35], and

no doubt other persons unknown to us, but in the context of the present paper, they become especially illuminating.

The canonical partition function for a classical gas of $N$ non-interacting particles with single-particle excitation spectrum $\varepsilon_n$ is [32]

$$Q_N(T) = \frac{1}{N!}\left(\sum_n \exp(-\beta\varepsilon_n)\right)^N , \tag{4.1}$$

where $\beta = (kT)^{-1}$, with $T$ the absolute temperature and $k$ Boltzmann's constant. We consider the following two single-particle spectra:

the "natural gas"

$$\varepsilon_n = \begin{cases} \varepsilon_0 \log n , & n \text{ a natural number } (1, 2, 3, \ldots) , \\ \infty , & \text{otherwise} , \end{cases} \tag{4.2}$$

the "prime gas"

$$\varepsilon_n = \begin{cases} \varepsilon_0 \log n , & n \text{ prime } (2, 3, 5, 7, 11, \ldots) , \\ \infty , & \text{otherwise} , \end{cases} \tag{4.3}$$

where $\varepsilon_0$ is a positive parameter with the dimensions of energy.

For the natural gas (4.2), we find readily that

$$\sum_n \exp(-\beta\varepsilon_n) = \zeta(\beta\varepsilon_0) \qquad \text{when } \beta\varepsilon_0 > 1 . \tag{4.4}$$

The partition function diverges for all $T \geqslant T_{\max}$, where

$$T_{\max} = \varepsilon_0/k \tag{4.5}$$

appears as the *maximum allowable temperature*. All the thermodynamic properties, such as the internal energy and heat capacity, can be obtained from eqs (4.1) and (4.4) in the usual way. They will all become singular as $T \to T_{\max}^-$. This singular behaviour, discussed further below, results because the density of single-particle states grows exponentially in the energy.

For the prime gas (4.3), where $p$ now denotes a generic prime number,

$$\sum_n \exp(-\beta\varepsilon_n) = \sum_p \frac{1}{p^{\beta\varepsilon_0}} . \tag{4.6}$$

Using the infinite product representation of $\zeta(s)$ one readily obtains (ref. [4], p. 12, ref. [36], p. 5)

$$\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} = \sum_p \frac{1}{p^s} + R_1(s) , \tag{4.7}$$

where

$$R_1(s) = \sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{ms}} \tag{4.8}$$

is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}$. Indeed (ref. [36], p. 5), for $\mathrm{Re}(s) \geq 1$,

$$|R_1(s)| \leq \frac{\zeta(2s)}{2(1-2^{-s})} , \tag{4.9}$$

so that $|R_1(s)| \leq \zeta(2) = \frac{1}{6}\pi^2$ for $\mathrm{Re}(s) \geq 1$. It is interesting to note that, in the spirit of the totem pole (3.15),

$$\sum_p \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns) \tag{4.10}$$

(ref. [4], p. 12). Thus we find that as $\beta\varepsilon_0 \to 1^+$,

$$\sum_n \exp(-\beta\varepsilon_n) = \sum_p \frac{1}{p^{\beta\varepsilon_0}} \sim \log \zeta(\beta\varepsilon_0) \sim \log[(\beta\varepsilon_0 - 1)^{-1}] . \tag{4.11}$$

Once again there is a maximum allowable temperature $T_{\max}$, given by eq. (4.5) above. Since the singular behaviour at $T_{\max}$ is weaker than that for the natural gas, the corresponding thermodynamic properties will have weaker (though more exotic) singular behaviour. The singularity in the partition function is weaker for the prime gas as a direct consequence of the asymptotic density of the prime numbers. Since the number $\pi(x)$ of prime numbers less than or equal to $x$ grows as $x/\log x$, from the prime number theorem [4, 36], the single particle density of states for the prime gas is effectively $1/\log x$, while that for the natural gas is unity.

At this point it is of more than passing interest to ask if any physical system can exhibit a maximum allowable temperature. Hagedorn [37] has developed a fascinating statistical-thermodynamic "fireball" model of hadron (elementary particle) physics and a maximum allowable temperature arises for comparable reasons as in the prime and integer gases. In particle physics, the maximum allowable temperature has become known as the Hagedorn temperature, and it appears in all fundamental microscopic theories of hadron physics. We briefly re-encounter hadron physics in a different context in section 6 below.

Proceeding now to quantum statistical mechanics, the canonical partition function for a gas of non-interacting, non-number-conserved particles with single particle spectrum $\varepsilon_n$ is

for bosons

$$Q_{\mathrm{B}}(T) = \prod_{n} [1 - \exp(-\beta\varepsilon_n)]^{-1}, \tag{4.12}$$

for fermions

$$Q_{\mathrm{F}}(T) = \prod_{n} [1 + \exp(-\beta\varepsilon_n)]. \tag{4.13}$$

Such non-number-conserved particles are to be considered as quantized excitations of some appropriate Bose or Fermi fields, such as the photons and neutrinos in conventional quantum field theory.

For the prime gas (4.3), we readily verify that

$$Q_{\mathrm{B}}(T) = \zeta(\beta\varepsilon_0), \qquad \text{for } \beta\varepsilon_0 > 1, \tag{4.14}$$

and (cf. eq. (7.2) below)

$$Q_{\mathrm{F}}(T) = \frac{\zeta(\beta\varepsilon_0)}{\zeta(2\beta\varepsilon_0)}, \qquad \text{for } \beta\varepsilon_0 > 1. \tag{4.15}$$

Once again there is the Hagedorn phenomenon, with $T_{\max}$ given by eq. (4.5), and all thermodynamic properties (which are correspondingly singular at $T = T_{\max}$) follow by the usual arguments.

For the boson case of the natural gas, the ground state $n = 1$ must be excluded. With this modification, by a similar procedure to that which produced eq. (4.7), we have

$$\log Q_{\mathrm{B}}(T) = \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mn^{m\beta\varepsilon_0}} = \zeta(\beta\varepsilon_0) - 1 + R_2(\beta\varepsilon_0), \tag{4.16}$$

where

$$R_2(s) = \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{mn^{ms}} \tag{4.17}$$

is holomorphic for $\mathrm{Re}(s) \geq 1$, and $|R_2(s)| \leq [\zeta(2s) - 1]/[2(1 - 2^{-s})] \leq \zeta(2) - 1 = \frac{1}{6}\pi^2 - 1$ for $\mathrm{Re}(s) \geq 1$.

For the fermion case of the natural gas, we again exclude the ground state $n = 1$, though this is not essential here, as the inclusion of the ground state only contributes a constant to $\log Q_{\mathrm{F}}(T)$. We have

$$\log Q_F(T) = \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mn^{m\beta\varepsilon_0}} - \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mn^{2m\beta\varepsilon_0}} \tag{4.18}$$

$$= \zeta(\beta\varepsilon_0) - \zeta(2\beta\varepsilon_0) + R_3(\beta\varepsilon_0)\,, \tag{4.19}$$

where $R_3(s)$ is holomorphic for $\mathrm{Re}(s) \geqslant 1$ and readily bounded. It will come as no surprise to the reader that for both the boson and fermion natural gases, the Hagedorn phenomenon again appears. We remark that to obtain precise results for the thermodynamic functions near the Hagedorn temperature, we need an expansion of $\zeta(s)$ in the vicinity of $s = 1$, viz.

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} \frac{(-1)^k c_k}{k!} (s-1)^k\,, \tag{4.20}$$

where $\gamma \approx 0.5772$ is Euler's constant. The expansion (4.20) was first published in 1885 by Stieltjes. It is the Laurent series for the Riemann zeta function, and so is convergent for all $s \neq 1$. The reader is referred to p. 164 of the first volume of Ramanujan's notebooks, edited by Berndt [11], for some intriguing material and accurate numerical values for the first few coefficients $c_k$.

In closing this section, we note that if we imitate Planck by neglecting zero point energy and writing the single-particle spectrum for black body radiation as

$$\varepsilon_n = n\hbar\omega\,, \tag{4.21}$$

then omitting the ground state $n = 0$, we have for the Bose and Fermi cases respectively

$$Q_B(T) = \prod_{n=1}^{\infty} (1 - x^n)^{-1} = \frac{1}{F_E(x)} \tag{4.22}$$

and

$$Q_F(T) = \prod_{n=1}^{\infty} (1 + x^n) = \frac{F_E(x^2)}{F_E(x)}\,, \tag{4.23}$$

where $x = \exp(-\beta\hbar\omega)$ and $F_E(x)$ is Euler's infinite product (3.16), which we have already encountered in the totem pole (3.15), which is the cornerstone of the theory of partitions [26] and which figures prominently in section 6 below. The study of quantum field theory is sufficiently well advanced that one could easily approach these problems without explicit reference to the classical analysis which underlies them, but we feel that one is the poorer if deprived of the rich mathematical antecedents.

## 5. Functional equations and scaling relations

Consider the function $g(x)$, defined for $x>0$ by the series

$$g(x)=\sum_{n=-\infty}^{\infty}[h(x\,\mathrm{e}^{n})-h(\mathrm{e}^{n})]\,, \tag{5.1}$$

where $h$ is a continuous function for $0\leqslant x<\infty$ and for some $\varepsilon>0$,

$$h(x)=h(0^{+})+\mathrm{O}(x^{\varepsilon})\quad \text{as } x\to 0^{+}\quad\text{and}\quad h(x)=\mathrm{O}(1/x^{\varepsilon})\quad \text{as } x\to\infty\,. \tag{5.2}$$

The series converges uniformly for all finite closed subintervals of $(0,\infty)$, so that $g(x)$ is continuous for $x>0$. Obviously $g(x)$ satisfies the functional equation

$$g(\mathrm{e}x)=g(x)+g(\mathrm{e})\,, \tag{5.3}$$

the same functional equation as $\log x$, and $g(1)=0$. Perhaps, then, our function is identically equal to $\log x$ up to a constant factor? To refute this plausible conjecture, we use the relation (1.6) to deduce that

$$g(x)=\sum_{n=-\infty}^{\infty}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{e}^{-ns}(x^{-s}-1)\tilde{h}(s)\,\mathrm{d}s\quad(0<c=\mathrm{Re}(s)<\varepsilon)\,, \tag{5.4}$$

where $\tilde{h}(s)$ is the Mellin transform of $h$, which is holomorphic in the strip $0<\mathrm{Re}(s)<\varepsilon$. The analytic continuation of $\tilde{h}(s)$ is easily seen from eq. (5.2) to be

$$\tilde{h}(s)=\frac{h(0^{+})}{s}+\text{a function holomorphic in } -\varepsilon<\mathrm{Re}(s)<\varepsilon\,. \tag{5.5}$$

If we write

$$g_{1}(x)=\sum_{n=0}^{\infty}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{e}^{-ns}(x^{-s}-1)\tilde{h}(s)\,\mathrm{d}s \tag{5.6}$$

and

$$g_{2}(x)=\sum_{n=1}^{\infty}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{e}^{ns}(x^{-s}-1)\tilde{h}(s)\,\mathrm{d}s\,, \tag{5.7}$$

then

$$g(x) = g_1(x) + g_2(x)\,. \tag{5.8}$$

Since the series defining $g_1(x)$ converges uniformly with respect to $s$ on the integration contour, we can interchange orders of integration and summation to obtain

$$g_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{(x^{-s}-1)\tilde{h}(s)\,ds}{1-e^{-s}} \qquad (c = \mathrm{Re}(s) > 0)\,. \tag{5.9}$$

The interchange is not permissible in the series defining $g_2(x)$, but we can overcome the problem, by observing that the integrand has no pole at $s = 0$, the pole of $\tilde{h}(s)$ being cancelled by the zero of $(x^{s}-1)$. Therefore we may translate the contour to the left to write

$$g_2(x) = \sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} e^{ns}(x^{s}-1)\tilde{h}(s)\,ds \qquad (-1 < c' = \mathrm{Re}(s) < 0)\,, \tag{5.10}$$

and on the new contour our sum converges uniformly so that we can perform the interchange to produce

$$g_2(x) = -\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{(x^{-s}-1)\tilde{h}(s)\,ds}{1-e^{-s}} \qquad (-1 < c' = \mathrm{Re}(s) < 0)\,. \tag{5.11}$$

We thus deduce that

$$g(x) = \frac{1}{2\pi i}\int \frac{(x^{s}-1)\tilde{h}(s)\,ds}{1-e^{-s}}\,, \tag{5.12}$$

where the integration contour (traversed anticlockwise) encircles the imaginary axis. The singularities enclosed are simple poles at $s = 0$ and at $s = \pm 2n\pi i$ $(n = 1, 2, \ldots)$. Summing over the residues we obtain

$$g(x) = -h(0^{+})\log x + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \tilde{h}(2\pi i n)\,(x^{2\pi i n}-1)\,. \tag{5.13}$$

The reader offended by the informality of the argument here will easily supply the appropriate rigorous argument involving an expanding sequence of rectangles with corners at the points $c \pm (2n+1)\pi i$, $c' \pm (2n+1)\pi i$.

A hint into the structure of $g(x)$ can actually be obtained by much simpler arguments. If we write $g(x) = -h(0^+)\log x + Q(x)$, then

$$Q(\mathrm{e}x) = Q(x)\,, \tag{5.14}$$

so that $Q(x)$ is self-similar, and is either constant, or a periodic function of $\log x$ with period 1. Hughes et al. [38, 39] have encountered an analogous but more complicated problem than this in the context of the Fourier analysis of a random walk with self-similar clusters. Analogies between this random walk Fourier analysis problem and real-space renormalization have been discussed by Shlesinger and Hughes [40]. For some mathematically related problems in hierarchical models, see ref. [41].

Our function $g(x)$ does coincide with $-h(0^+)\log x$ at isolated points where $(x^{-2\pi i n} - 1) = 0$ for all $n$, i.e., $x = \mathrm{e}^m$ ($m = 0, \pm 1, \pm 2, \ldots$). In between $g(x)$ wobbles about the function $-h(0^+)\log x$. A similar problem led Ramanujan to his fallacious proof of the prime number theorem. He forgot the wobbles (ref. [26], pp. 38–39). Our problem is really much deeper than it appears. Consider the special case[#3]

$$h(x) = \mathrm{e}^{-x}\,, \tag{5.15}$$

for which $\tilde{h}(s) = \Gamma(s)$ and eq. (5.13) becomes

$$g(x) + \log x = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \Gamma(2\pi i n)\,(x^{2\pi i n} - 1)\,. \tag{5.16}$$

Computation [M.T. Batchelor, personal communication] shows that in addition to $x = \mathrm{e}^m$, there exists another infinite set of real zeros of $g(x) + \log x$. Although the function $g(x) + \log x$ is self-similar, it is not fractal in the sense of Mandelbrot, for it is everywhere continuous and everywhere infinitely differentiable for $x > 0$.

If we write

$$x = \mathrm{e}^t \quad \text{and} \quad g(x) = -1/N(t)\,, \tag{5.17}$$

we convert the linear functional equation (5.3) into the nonlinear functional equation

[#3] An unpublished derivation of eq. (5.16) due to one of the authors, which serves as the prototype for the present general discussion, was briefly reproduced in ref. [42]. The analysis can be extended to accommodate the transformation of the series $\Sigma_{n=-\infty}^{\infty} z^n[h(x\,\mathrm{e}^n) - h(\mathrm{e}^n)]$, for a mildly restricted class of functions $h$.

$$N(t+1) = \frac{N(t)}{1 - g(\mathrm{e})\, N(t)} \,. \tag{5.18}$$

For the special case (5.15), $g(\mathrm{e}) = -1$, while more generally $g(\mathrm{e}) = -h(0^+)$. Our analysis therefore produces solutions to the nonlinear difference equation which, although simple for integer values of $t$, behave somewhat irregularly for intermediate values of $t$. It is natural to speculate that some other nonlinear difference equations which produce chaotic solutions may be manifestations of tautologies in the spirit of eq. (5.13).

## 6. Euler's product

Consider now Euler's product $F_{\mathrm{E}}(x)$ defined by eq. (3.16) above and write

$$f(x) = -\log F_{\mathrm{E}}(x) = -\sum_{k=1}^{\infty} \log(1 - x^k) \,. \tag{6.1}$$

From eq. (3.13),

$$\Gamma(s)\, \zeta(s)\, \zeta(s+1) = -\sum_{k=1}^{\infty} \int_0^{\infty} x^{s-1} \log(1 - \mathrm{e}^{-kx})\, \mathrm{d}x \,, \tag{6.2}$$

so that

$$\Gamma(s)\, \zeta(s)\, \zeta(s+1) = \int_0^{\infty} x^{s-1} f(\mathrm{e}^{-x})\, \mathrm{d}x \,, \tag{6.3}$$

and so by the Mellin inversion formula

$$f(\mathrm{e}^{-x}) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s} \Gamma(s)\, \zeta(s)\, \zeta(s+1)\, \mathrm{d}s \qquad (c = \mathrm{Re}(s) > 1) \,. \tag{6.4}$$

The only pole of $\zeta(s)$ is at $s = 1$, near which

$$\zeta(s) = (s-1)^{-1} + \gamma + \cdots \,, \tag{6.5}$$

where $\gamma = -\Gamma'(1)$ is Euler's constant, while $\Gamma(s)$ has a simple pole of residue $(-1)^n/n!$ at $s = -n$ ($n = 0, 1, 2, \ldots$). The integral therefore has a double pole at $s = 0$ and simple poles at $s = \pm 1$. The remaining poles of the gamma function are cancelled by zeros of one or other of the zeta functions, since $\zeta(-2n) = 0$

for $n = 1, 2, 3, \ldots$. Evaluating the residues at the poles, we have

$$f(\mathrm{e}^{-x}) = -\zeta(0)\,\zeta(-1)\,x - \zeta(0)\log x + \zeta'(0) + \zeta(2)/x + I_{\mathrm{R}}\,, \tag{6.6}$$

where

$$I_{\mathrm{R}} = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s}\Gamma(s)\,\zeta(s)\,\zeta(s+1)\,\mathrm{d}s \qquad (c = \mathrm{Re}(s) < -1)\,. \tag{6.7}$$

At first sight the first three terms in eq. (6.6) give us our product explicitly. However the remainder integral $I_{\mathrm{R}}$ does not vanish in this case, a circumstance connected with the curious properties of the zeta function at infinity. The integral $I_{\mathrm{R}}$ can be evaluated by transforming to a new integration variable $z = -s$, using the Riemann relation (3.21) on both zeta functions and removing a pair of gamma functions using the reflection formula $\Gamma(s)\,\Gamma(1-s) = \pi/\sin(\pi s)$ to yield

$$I_{\mathrm{R}} = f(\mathrm{e}^{-4\pi^2/x})\,. \tag{6.8}$$

Using well-known values of the Riemann zeta function ($\zeta(-1) = -\frac{1}{12}$ and $\zeta(0) = -\frac{1}{2}$, briefly derived in section 2, $\zeta(2) = \frac{1}{6}\pi^2$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$) we deduce that

$$f(\mathrm{e}^{-x}) = -\frac{x}{24} + \frac{\pi^2}{6x} + \frac{1}{2}\log\left(\frac{x}{2\pi}\right) + f(\mathrm{e}^{-4\pi^2/x})\,. \tag{6.9}$$

Taking exponentials of both sides and recalling the definition of Euler's product $F_{\mathrm{E}}(x)$, we deduce that

$$F_{\mathrm{E}}(\mathrm{e}^{-x}) = \left(\frac{2\pi}{x}\right)^{1/2} \exp\left(\frac{x}{24} - \frac{\pi^2}{6x}\right) F_{\mathrm{E}}(\mathrm{e}^{-4\pi^2/x})\,. \tag{6.10}$$

We can rewrite this remarkable formula in a different form, which reduces to a symmetry under inversion. If we define

$$G(x) = x^{1/4}\,\mathrm{e}^{-\pi x/12} F_{\mathrm{E}}(\mathrm{e}^{-2\pi x})\,, \tag{6.11}$$

then we have

$$G(x) = G(1/x)\,. \tag{6.12}$$

It is fair to ask if the elegant mathematics leading to eq. (6.10) has any

applications. The significance of Euler's product in number theory is discussed by Hardy [26] (chs. VI and VIII). Let $p_U(n)$ denote the number of *unrestricted partitions* of the integer $n$, i.e. the number of ways in which $n$ can be expressed as a sum of positive integers (not necessarily distinct), without regard to order and let $p_D(n)$ denote the number of partitions of $n$ into *distinct* positive integers. Then it can be shown that

$$F_E(x)^{-1} = 1 + \sum_{n=1}^{\infty} p_U(n)\, x^n , \tag{6.13}$$

$$\frac{F_E(x^2)}{F_E(x)} = 1 + \sum_{n=1}^{\infty} p_D(n)\, x^n , \tag{6.14}$$

and also that

$$F_E(x) = 1 + \sum_{k=1}^{\infty} (-1)^k (x^{k(3k-1)/2} + x^{k(3k+1)/2}) . \tag{6.15}$$

The asymptotic forms of $p_U(n)$ and $p_D(n)$ have applications in the theory of the self-avoiding walk [43] and the spiral self-avoiding walk [44], and eq. (6.11) is used in the rigorous derivation of these forms [45]. Also, the function $x^{1/24} F_E(x)^{-1}$ and the coefficients of $x$ in the expansion of $F_E(x)^{-24}$ appear in relativistic string theory [46, 47]. Eqs. (6.13)–(6.15) are not far from the theory of the Rogers–Ramanujan identities, which have been found useful lately in the analysis of two-dimensional lattice statistical mechanics problems [10, 48]. The function $F_E(x)^{-1}$ is, to quote Hardy [26], "one of a well-known class, the elliptic modular functions, whose properties have been studied intensively and are very exactly known . . . These functions all have the same peculiarities . . . and exist only inside the [unit] circle . . .". Ramanujan's function $\tau(n)$ is defined by the equation (ref. [26], ch. X)

$$x F_E(x)^{-24} = \sum_{n=1}^{\infty} \tau(n)\, x^n . \tag{6.16}$$

and it is satisfying to see from the preceding references that this bizarre function, for which

$$\sum_{n=1}^{\infty} \tau(n)\, n^{-s} = \prod_p [1 - \tau(p)\, p^{-s} + p^{11-2s}]^{-1} \tag{6.17}$$

(where $p$ is a prime number), may have [14] more than recreational interest.

Other tautologies can be constructed for Euler's product. It can be shown (ref. [4], p. 8) that for $\mathrm{Re}(s) > \max[1, \mathrm{Re}(a) + 1]$,

$$\zeta(s)\,\zeta(s-a)=\sum_{m=1}^{\infty}\frac{\sigma_a(m)}{m^s}\,, \tag{6.18}$$

where $\sigma_a(m)$ is the sum of the $a$th powers of the divisors of $m$. Eq. (6.4) thus becomes

$$f(\mathrm{e}^{-x})=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}x^{-s}\Gamma(s)\sum_{m=1}^{\infty}\frac{\sigma_{-1}(m)}{m^s}\,\mathrm{d}s \qquad (c=\mathrm{Re}(s)>1) \tag{6.19}$$

and a term-by-term integration gives

$$f(\mathrm{e}^{-x})=\sum_{m=1}^{\infty}\sigma_{-1}(m)\,\mathrm{e}^{-mx}\,. \tag{6.20}$$

## 7. Assorted series and products

A variety of interesting results involving sums over the Möbius function can be derived from Mellin transform methods. We illustrate this first with the assertion that

$$\begin{aligned}\sum_{k=1}^{\infty}\exp(-kx)&=\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}|\mu(m)|\exp(-mk^2x)\\&=\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mu(m)\exp(-k^2m^2nx)\,.\end{aligned} \tag{7.1}$$

Eq. (7.1) is trivially established by taking Mellin transforms and evaluating the sums over $k$, $m$ and $n$ in terms of zeta functions or ratios of zeta functions using eqs. (1.4), (2.3) and the identity (ref. [4], p. 5)

$$\sum_{m=1}^{\infty}\frac{|\mu(m)|}{m^s}=\frac{\zeta(s)}{\zeta(2s)}\,, \qquad \mathrm{Re}(s)>1\,. \tag{7.2}$$

(Eq. (7.2) is a simple consequence of eq. (2.5.)) All three expressions in eq. (7.1) have Mellin transform $\Gamma(s)\,\zeta(s)$.

For a less obvious example, consider

$$T_{\pm}(x,z)=\pm\sum_{m=1}^{\infty}\log[1\pm|\mu(m)|z\exp(-mx)]\,. \tag{7.3}$$

Expanding the logarithm and noting that $|\mu(m)|^n=|\mu(m)|$, we have

$$T_{+}(x, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|\mu(m)|}{n} (\mp)^{n-1} z^{n} \exp(-mnx) . \tag{7.4}$$

We shall not address in detail the justification for the interchange of orders of summations in double sums, or for the interchange of orders of summation and integration. Adequate convergence to permit this is either clear, or secured by restricting $\mathrm{Re}(s)$ to sufficiently large values.

Taking the Mellin transform over $x$, we have

$$\int_{0}^{\infty} x^{s-1} T_{\pm}(x, z)\, \mathrm{d}x = \Gamma(s) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|\mu(m)|}{m^{s} n^{s+1}} (\mp)^{n-1} z^{n} . \tag{7.5}$$

Using eq. (7.2), we find that

$$\int_{0}^{\infty} x^{s-1} T_{+}(x, z)\, \mathrm{d}x = \frac{\Gamma(s)\, \zeta(s)}{\zeta(2s)} \sum_{n=1}^{\infty} \frac{(\mp)^{n-1} z^{n}}{n^{s+1}} . \tag{7.6}$$

Expanding $1/\zeta(2s)$ as $\sum_{m=1}^{\infty} \mu(m)\, m^{-2s}$, we find that

$$\int_{0}^{\infty} x^{s-1} T_{\pm}(x, z)\, \mathrm{d}x = \Gamma(s)\, \zeta(s) \sum_{m=1}^{\infty} \mu(m) \sum_{n=1}^{\infty} \frac{(\mp)^{n-1} z^{n}}{m^{2s} n^{s+1}} . \tag{7.7}$$

Recognizing $m^{-2s} n^{-s} \Gamma(s)\, \zeta(s)$ as the Mellin transform of $\sum_{k=1}^{\infty} \exp(-m^{2} nkx)$, we may now write

$$\int_{0}^{\infty} x^{s-1} T_{\pm}(x, z)\, \mathrm{d}x = \int_{0}^{\infty} x^{s-1} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \mu(m) \sum_{n=1}^{\infty} \frac{(\mp)^{n-1} z^{n}}{n} \times [\exp(-m^{2} kx)]^{n}\, \mathrm{d}x , \tag{7.8}$$

and the sum over $n$ can now be expressed as a logarithm. Inverting the Mellin transforms, we deduce that

$$T_{\pm}(x, z) = \pm \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \mu(m) \log[1 \pm z \exp(-m^{2} kx)] . \tag{7.9}$$

Recalling eq. (7.3) and exponentiating, we obtain

$$\prod_{m=1}^{\infty} [1 \pm |\mu(m)| z \exp(-mx)]^{\pm 1} = \prod_{k=1}^{\infty} \prod_{m=1}^{\infty} [1 \pm z \exp(-m^{2} kx)]^{\pm \mu(m)} . \tag{7.10}$$

In the special case $z = 1$, selecting the formula for $T_-(x, z)$, eq. (7.7) reduces to

$$\int_0^\infty x^{s-1} T_-(x, 1)\, dx = \Gamma(s)\, \zeta(s)\, \zeta(s+1) \sum_{m=1}^{\infty} \mu(m)\, m^{-2s} \tag{7.11}$$

$$= \int_0^\infty x^{s-1} \sum_{m=1}^{\infty} \mu(m)\, f(\exp(-m^2 x))\, dx \tag{7.12}$$

(the last equality flowing from eq. (6.3)). Inverting the Mellin transform and using eq. (6.20), we may deduce that

$$T_-(x, 1) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mu(m)\, \sigma_{-1}(n) \exp(-m^2 n x)\,. \tag{7.13}$$

For a slightly harder example, consider

$$U_\pm(x, z) = \pm \sum_{m=1}^{\infty} \log[1 \pm \mu(m)\, z \exp(-mx)]\,. \tag{7.14}$$

Expanding the logarithm and noting that

$$\mu(m)^n = \begin{cases} \mu(m)\,, & n \text{ odd}\,, \\ |\mu(m)|\,, & n \text{ even}\,, \end{cases}$$

we have

$$U_\pm(x, z) = \sum_{m=1}^{\infty} \Bigg( \sum_{\substack{n \geq 1 \\ \text{and odd}}} \frac{\mu(m)}{n}\, z^n \exp(-mnx) \mp \sum_{\substack{n \geq 1 \\ \text{and even}}} \frac{|\mu(m)|}{n}\, z^n \exp(-mnx) \Bigg)\,. \tag{7.15}$$

Taking Mellin transforms of both sides and evaluating the sums over $m$ using eqs. (2.3) and (7.2), we deduce that

$$\int_0^\infty x^{s-1} U_\pm(x, z)\, dx = \frac{\Gamma(s)}{\zeta(s)} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^{s+1}} \mp \frac{\Gamma(s)\zeta(s)}{\zeta(2s)} \sum_{n=1}^{\infty} \frac{z^{2n}}{(2n)^{s+1}}\,. \tag{7.16}$$

Expanding the denominators using $\zeta(s)^{-1} = \Sigma_{m=1}^{\infty} \mu(m)\, m^{-s}$, we obtain

$$\int_0^\infty x^{s-1} U_\pm(x, z)\,\mathrm{d}x$$

$$= \Gamma(s) \sum_{m=1}^{\infty} \mu(m) \sum_{n=0}^{\infty} \frac{z^{2n+1}}{m^s (2n+1)^{s+1}}$$

$$\mp \Gamma(s)\,\zeta(s) \sum_{m=1}^{\infty} \mu(m) \sum_{n=1}^{\infty} \frac{z^{2n}}{m^{2s}(2n)^{s+1}} \tag{7.17}$$

$$= \int_0^\infty x^{s-1} \sum_{m=1}^{\infty} \mu(m) \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1} \exp[-(2n+1)mx]\,\mathrm{d}x$$

$$\mp \int_0^\infty x^{s-1} \zeta(s) \sum_{m=1}^{\infty} \mu(m) \sum_{n=1}^{\infty} \frac{z^{2n}}{2n} \exp[-2m^2 nx]\,\mathrm{d}x\,, \tag{7.18}$$

where in passing from eq. (7.17) to eq. (7.18), we have replaced $\Gamma(s)\, m^{-s}(2n+1)^{-s}$ and $\Gamma(s)\, m^{-2s} n^{-s}$ respectively by the Mellin transforms of $\exp[-(2n+1)mx]$ and $\exp[-2m^2 nx]$. The sums over $n$ can now be evaluated in terms of logarithms, and if $\zeta(s)$ is expanded in its Dirichlet series, we can express the entire right-hand side as a Mellin transform. Inverting this transform, we obtain

$$U_\pm(x, z) = \frac{1}{2} \sum_{m=1}^{\infty} \mu(m) \log\left(\frac{1 + z\exp(-mx)}{1 - z\exp(-mx)}\right)$$

$$\pm \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mu(m) \log[1 - z^2 \exp(-2m^2 nx)]\,. \tag{7.19}$$

If we return to eqs. (7.3) and (7.6), we have (setting $z = 1$ and selecting the formula for $T(x, z)$)

$$\int_0^\infty x^{s-1} \sum_{n=1}^{\infty} \log\{[1 - |\mu(n)| \exp(-nx)]^{-1}\}\,\mathrm{d}x = \frac{\Gamma(s)\,\zeta(s)\,\zeta(s+1)}{\zeta(2s)}\,. \tag{7.20}$$

If we multiply both sides of this equation by $m^{-2s}$ and sum over $m$, we obtain

$$\int_0^\infty x^{s-1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \log\{[1 - |\mu(n)| \exp(-m^2 nx)]^{-1}\}\,\mathrm{d}x = \Gamma(s)\,\zeta(s)\,\zeta(s+1)\,. \tag{7.21}$$

Recognizing the right-hand side in terms of the Mellin transform of Euler's

product (via eq. (6.2)), we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\log\{[1-|\mu(n)|\exp(-m^2nx)]^{-1}\}=\sum_{m=1}^{\infty}\log\{[1-\exp(-mx)]^{-1}\}\,. \tag{7.22}$$

If we now write $z=\exp(-x)$ and differentiate with respect to $x$, we obtain the identity

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|\mu(n)|m^2nz^{m^2n}}{1-|\mu(n)|z^{m^2n}}=\sum_{m=1}^{\infty}\frac{mz^m}{1-z^m}\,. \tag{7.23}$$

The right-hand side is a Lambert series. Using eq. (3.20) we may express it as a power series. In this case, we determine $b_n$ by the requirement that $\sum_{n=1}^{\infty}b_nn^{-s}=\zeta(s)\,\zeta(s-1)$. From eq. (7.18), we see that $b_m=\sigma_1(m)$, the sum of the divisors of $m$, and we arrive at the far from self-evident result that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|\mu(n)|m^2nz^{m^2n}}{1-z^{m^2n}}=\sum_{m=1}^{\infty}\sigma_1(m)\,z^m\,. \tag{7.24}$$

We have removed the factor of $|\mu(n)|$ from the denominator of the left-hand side by noting that the summand is nonzero only when $|\mu(n)|=1$.

These are other interesting generalisations of these ideas, which take us into the theory of theta functions, some key results of which are collected in the appendix. If we replace $x$ by $n^2x$ in eq. (7.9), sum over $n$ and take the Mellin transform, we obtain

$$\begin{aligned}&\int_0^{\infty}x^{s-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\log[1\pm|\mu(m)|z\exp(-mn^2x)]\,\mathrm{d}x\\&\quad=\int_0^{\infty}x^{s-1}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mu(m)\log[1\pm z\exp(-m^2n^2kx)]\,\mathrm{d}x\,.\end{aligned} \tag{7.25}$$

If we make $m^2n^2x$ the new integration variable on the right-hand side, the sum over $n$ may be identified as $\zeta(2s)$, while the sum over $m$ is $1/\zeta(2s)$. Inverting the Mellin transform, we deduce that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\log[1\pm|\mu(m)|z\exp(-mn^2x)]=\sum_{k=1}^{\infty}\log[1\pm z\exp(-kx)]\,, \tag{7.26}$$

which is equivalent to the following identity for infinite products:

$$\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}[1\pm|\mu(m)|z\exp(-mn^2x)]=\prod_{k=1}^{\infty}[1\pm z\exp(-kx)]\,. \tag{7.27}$$

In the appendix, we summarize some key results from the theory of theta functions, one of which is the infinite product representation

$$\vartheta_3(z,q)=Q\prod_{n=1}^{\infty}[1+2q^{2n-1}\cos(2z)+q^{4n-2}]\,, \tag{7.28}$$

where

$$Q=\prod_{n=1}^{\infty}(1-q^{2n})\,. \tag{7.29}$$

Using eq. (7.27) we can rewrite the last equation as

$$Q=\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}[1-|\mu(m)|q^{2mn^2}]\,. \tag{7.30}$$

The multiplicand in the product (7.28) can be factored as $(1+q^{2n-1}e^{2iz})(1+q^{2n-1}e^{-2iz})$. In each product, we may now use the identity (7.27) with appropriate identifications of $z$ and $e^{-x}$, giving

$$\vartheta_3(z,q)=Q\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}(1+|\mu(m)|q^{2mn^2-1}e^{2iz})(1+|\mu(m)|q^{2mn^2-1}e^{-2iz})\,. \tag{7.31}$$

In a similar manner,

$$\vartheta_4(z,q)=Q\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}(1-|\mu(m)|q^{2mn^2-1}e^{2iz})(1-|\mu(m)|q^{2mn^2-1}e^{-2iz})\,. \tag{7.32}$$

Similar product representations exists for $\vartheta_1(z,q)$ and $\vartheta_2(z,q)$, so that in particular

$$\prod_{n=1}^{\infty}[1-2q^{2n}\cos(2z)+q^{4n}]=\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}[1-2|\mu(m)|q^{2mn^2}\cos(2z)+|\mu(m)|q^{4mn^2}]\,. \tag{7.33}$$

Differentiation of the logarithm of eq. (7.33) with respect to $z$ gives, on cancellation of a factor of $4\sin(2z)$ common to both sides,

$$\sum_{n=1}^{\infty} \frac{q^{2n}}{1-2q^{2n}\cos(2z)+q^{4n}} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{|\mu(m)|q^{2mn^2}}{1-2|\mu(m)|q^{2mn^2}\cos(2z)+|\mu(m)|q^{4mn^2}} . \tag{7.34}$$

The summand in both sides can be written as $\alpha[1-2\alpha\cos(2z)]^{-1}$, with $\alpha = q^{2n}/(1+q^{4n})$ and $\alpha = |\mu(m)|q^{2mn^2}/(1+|\mu(m)|q^{4mn^2})$ respectively, and expanded in powers of $\cos(2z)$. Equating the coefficients of equal powers of $\cos(2z)$, we obtain

$$\sum_{n=1}^{\infty}\left(\frac{q^{2n}}{1+q^{4n}}\right)^j = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{|\mu(m)|q^{2mn^2}}{1+|\mu(m)|q^{4mn^2}}\right)^j , \qquad j=1,2,3,\ldots . \tag{7.35}$$

Since $|\mu(m)|$ takes only the values 0 and 1, this result may be simplified to

$$\sum_{n=1}^{\infty}\left(\frac{q^{2n}}{1+q^{4n}}\right)^j = \sum_{m=1}^{\infty}|\mu(m)|\sum_{n=1}^{\infty}\left(\frac{q^{2mn^2}}{1+q^{4mn^2}}\right)^j , \qquad j=1,2,3,\ldots . \tag{7.36}$$

If we now set $q^2 = e^{\theta}$, we obtain

$$\sum_{n=1}^{\infty}\left(\frac{1}{\cosh(n\theta)}\right)^j = \sum_{m=1}^{\infty}|\mu(m)|\sum_{n=1}^{\infty}\left(\frac{1}{\cosh(mn^2\theta)}\right)^j , \qquad j=1,2,3,\ldots . \tag{7.37}$$

For $j=1$, the series on the left-hand side is a special case of a series for the elliptic function dn $u$ given by Whittaker and Watson [30] (p. 511, example 1), so that

$$\sum_{m=1}^{\infty}|\mu(m)|\sum_{n=1}^{\infty}\frac{q^{2mn^2}}{1+q^{4mn^2}} = \frac{1}{2}\sum_{m=1}^{\infty}|\mu(m)|\sum_{n=1}^{\infty}\operatorname{sech}(mn^2\theta) = \frac{\mathbf{K}(k)}{2\pi} - \frac{1}{\pi} , \tag{7.38}$$

where $\mathbf{K}(k)$ the complete elliptic integral of the first kind, $\mathbf{K}'(k) = \mathbf{K}(\sqrt{1-k^2})$ and $\theta = \pi\mathbf{K}'(k)/\mathbf{K}(k)$. For $j=2$, the series can be summed for $\theta = \pi$ using the identity [49]

$$\sum_{n=-\infty}^{\infty}\operatorname{sech}^2(n\pi) = \frac{4}{\pi^2}\mathbf{K}(2^{-1/2})\,\mathbf{E}(2^{-1/2}) = \frac{1}{\pi} + \frac{\Gamma^4(\frac{1}{4})}{8\pi^3} . \tag{7.39}$$

## 8. Ewald sums and quasi-crystals

We conclude our discussion of tautologies by noting some consequences of the identity

$$\sum_{m=1}^{\infty} |\mu(m)| Q(mx) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mu(m)\, Q(m^2 nx)\,. \tag{8.1}$$

This identity is easily established by taking Mellin transforms with respect to $x$ and evaluating the series in terms of zeta functions. In the special case $Q(x) = \exp(-\varepsilon x^2)\cos(x)$, we obtain

$$\sum_{m=1}^{\infty} |\mu(m)| \exp(-m^2\varepsilon x^2)\cos(mx) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mu(m) \exp(-m^4 n^2 \varepsilon x^2) \times \cos(m^2 nx)\,. \tag{8.2}$$

If we take $\varepsilon = 0$, we have, formally,

$$\sum_{m=1}^{\infty} |\mu(m)| \cos(mx) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mu(m) \cos(m^2 nx)\,. \tag{8.3}$$

The series do not converge, but may be interpreted in the sense of generalized functions [31]. From eq. (3.24), we have

$$\sum_{n=-\infty}^{\infty} \delta(y - n) = \sum_{n=-\infty}^{\infty} \cos(2\pi n y)\,, \tag{8.4}$$

and setting $2\pi y = m^2 x$, we have

$$\sum_{n=1}^{\infty} \cos(m^2 nx) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta\left(\frac{m^2 x}{2\pi} - n\right) - \frac{1}{2} = \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} \frac{2\pi}{m^2}\, \delta\left(x - \frac{2\pi n}{m^2}\right) - 1\right]. \tag{8.5}$$

Hence

$$\sum_{m=1}^{\infty} |\mu(m)| \cos(mx) = \frac{1}{2} \sum_{m=1}^{\infty} \mu(m)\left[\sum_{n=-\infty}^{\infty} \frac{2\pi}{m^2}\, \delta\left(x - \frac{2\pi n}{m^2}\right) - 1\right]. \tag{8.6}$$

This formula relates a sum of cosines, the Fourier transform of a set of delta functions at selected points where the Möbius function is non-zero, to a double sum of delta functions. This is precisely the kind of relation that is needed for a description of quasi-crystalline or other pseudo-ordered structures [50].

A motivation for consideration of such sums comes from the work of von Schnering and Nesper [51] and Andersson et al. [52] on minimal surfaces in inorganic chemistry, further explored by Barnes [53], Barnes et al. [54] and Ninham [55]. The problem is to find the electrostatic potential set up by an array of charges. The general problem can more easily be treated if we note that the potential $V$ has to satisfy Poisson's equation

$$\nabla^2 V = -4\pi\rho\,, \tag{8.7}$$

where $\rho$ is a periodic sum of products of delta functions with appropriate symmetry. Consider the potential defined by

$$V(x, y, z) = \int_0^\infty \vartheta_l(\alpha x, \mathrm{e}^{-t^2})\,\vartheta_m(\alpha y, \mathrm{e}^{-t^2})\,\vartheta_n(\alpha z, \mathrm{e}^{-t^2})\,t\,\mathrm{d}t\,, \tag{8.8}$$

where the theta functions $\vartheta_l(z, q)$ $(l = 1, 2, 3, 4)$ are defined and their relevant properties collected in the appendix. In particular, the theta functions satisfy the same partial differential equation (A.6). If we apply the Laplace operator $\nabla^2$ to eq. (8.8), take it under the integral and use the partial differential equation (A.6), we find that the integrand becomes a multiple of the $t$ derivative of $\vartheta_l(\alpha x, \mathrm{e}^{-t^2})\,\vartheta_m(\alpha y, \mathrm{e}^{-t^2})\,\vartheta_n(\alpha z, \mathrm{e}^{-t^2})$ and it follows that

$$\nabla^2 V = 2\alpha^2[\vartheta_l(\alpha x, \mathrm{e}^{-t^2})\,\vartheta_m(\alpha y, \mathrm{e}^{-t^2})\,\vartheta_n(\alpha z, \mathrm{e}^{-t^2})]_0^\infty\,. \tag{8.9}$$

As discussed in the appendix, when their second argument is unity, all four theta functions reduce to infinite series of periodically spaced delta functions. When their second argument is zero, $\vartheta_1$ and $\vartheta_2$ vanish, while $\vartheta_3$ and $\vartheta_4$ take the value 1. By choosing an appropriate combination of theta functions in integrals of the form (8.8), or in the difference of two integrals of the form (8.8), we can write down a potential for all standard lattices, and easily construct numerical solutions and a formula for the Fourier transform of the potential [53, 54]. For example, setting $l = m = n = 2$ and $\alpha = \pi$ in eq. (8.9) and using eq. (A.21), we obtain the potential corresponding to the charge density,

$$\rho(x, y, z) = \frac{\pi}{2} \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{l+m+n} \delta(x - l)\,\delta(y - m)\,\delta(z - n)\,, \tag{8.10}$$

which is the product of periodic delta functions for the NaCl lattice.

Zero potential surfaces for all known crystals have the symmetry of minimal surfaces. For example, the zero potential surface of the CsCl lattice has the

topology of the Schwarz P minimal surface [56]. The electrostatic potential is also needed in the derivation of the zero point energy of the system and in crystallographic analysis. From eqs. (8.2) and (8.3) we can construct a whole series of new pseudolattices from the "Möbius theta function"

$$\Theta(z, q) = \sum_{m=1}^{\infty} |\mu(m)| q^{m^2} \cos(2mz) \tag{8.11}$$

and related series. Such lattices may also satisfy charge neutrality, since over long-$n$-intervals, the number of positive $\mu(n)$ and the number of negative $\mu(n)$ encountered are closely balanced[#4]. Because of the relation of these pseudolattices to ordinary close-packed Bravais lattices revealed through the infinite products of section 7, we can expect to find a rich field of applications through such considerations.

That part of classical analysis which celebrates the properties of individual higher transcendental functions has fallen out of favour in recent years, as it has become more fashionable to speak of the general in mathematics, rather than the particular. For this reason, Ramanujan, this century's greatest master of the particular, has not always been ranked as highly as he deserves. Even G.H. Hardy has remarked that his work would be greater if it were less strange. The Greek poet Archilochus wrote "The fox knows many things, but the hedgehog knows one big thing". Several authors, most notably Sir Isaiah Berlin in an essay on Tolstoy [59], have taken this cryptic statement as the contrasting of two different styles of thinkers, those who "relate everything to a single central vision" (hedgehogs) and those who "pursue many ends, often unrelated and even contradictory" (foxes). It is tempting to classify Ramanujan as a fox.

In this paper, we have pursued a number of mathematical tautologies. As a consequence of the evolution of accepted mathematical notation, many of the equations appear at first glance to have little in common. However, using the Mellin transform and the Riemann zeta function as our Rosetta stone, the simplicity and naturalness of the interrelations becomes apparent. We offer our analysis as modest support to the conjecture that Ramanujan may have been a hedgehog after all.

[#4] Whether they balance exactly in the limit of an infinitely long interval is a delicate question. Let $M(N) = \Sigma_{n=1}^{N} \mu(n)$. The assertion that $M(N) = O(N^{1/2+\varepsilon})$ for all $\varepsilon > 0$ is precisely equivalent (ref. [4], p. 315) to the Riemann hypothesis that all complex zeros of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. Edwards [57] (p. 6) has described the proof or refutation of the Riemann hypothesis as "unquestionably the most celebrated problem in mathematics". If the nonzero values of $\mu(n)$ were a sequence of independent random variables, taking the values 0 and 1 with equal probability, then by the law of the iterated logarithm (ref. [58], p. 205), we would have $\limsup_{N\to\infty} M(N)/\sqrt{N \log\log N} = \text{constant}$.

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**Appendix**

The theta functions [21, 30] are defined by the series

$$\vartheta_1(z, q) = 2\sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin[(2n+1)z]\,, \tag{A.1}$$

$$\vartheta_2(z, q) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos[(2n+1)z]\,, \tag{A.2}$$

$$\vartheta_3(z, q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2nz)\,, \tag{A.3}$$

$$\vartheta_4(z, q) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nz)\,. \tag{A.4}$$

It is assumed here that $|q| < 1$, and we write $q = e^{i\pi\tau}$, with $\mathrm{Im}(\tau) > 0$. To indicate the dependence of the theta functions on $\tau$, we define

$$\vartheta_k(z, q) = \vartheta_k(z|\tau)\,. \tag{A.5}$$

All four theta functions are solutions of the partial differential equation

$$\frac{i\pi}{4}\frac{\partial^2 y}{\partial z^2} + \frac{\partial y}{\partial \tau} = 0 \tag{A.6}$$

and have infinite product representations,

$$\vartheta_1(z, q) = 2Gq^{1/4} \sin z \prod_{n=1}^{\infty} [1 - 2q^{2n}\cos(2z) + q^{4n}]\,, \tag{A.7}$$

$$\vartheta_2(z, q) = 2Gq^{1/4} \cos z \prod_{n=1}^{\infty} [1 + 2q^{2n}\cos(2z) + q^{4n}]\,, \tag{A.8}$$

$$\vartheta_3(z, q) = G \prod_{n=1}^{\infty} [1 + 2q^{2n-1}\cos(2z) + q^{4n-2}]\,, \tag{A.9}$$

$$\vartheta_4(z, q) = G \prod_{n=1}^{\infty} [1 - 2q^{2n-1}\cos(2z) + q^{4n-2}], \tag{A.10}$$

where

$$G = \prod_{n=1}^{\infty} (1 - q^{2n}). \tag{A.11}$$

The theta functions are intimately related to the elliptic functions and elliptic integrals, and in view of the preceding infinite product representations, are also relevant to problems in the theory of partitions. (In the notation of section 6, $G = F_E(q^2)$.) The many interrelations between the theta functions and the plethora of transformation formulae have given the topic something of a mystique. However, these apparent subtleties are neither more nor less deep than the tautologies pursued in the present paper. A case in point is the theta function transformation ("Jacobi's imaginary transformation")

$$\vartheta_3(z \mid \tau) = (-i\tau)^{-1/2} \exp\left(\frac{z^2}{i\pi\tau}\right) \vartheta_3(z/\tau \mid -1/\tau). \tag{A.12}$$

This formula is equivalent to the assertion that

$$\sum_{m=-\infty}^{\infty} \exp(m^2 i\pi\tau + 2miz) = (-i\tau)^{-1/2} \sum_{m=-\infty}^{\infty} \exp[(z - m\pi)^2/i\pi\tau], \tag{A.13}$$

a special case of which is eq. (3.22). If we write $z = \pi x$ and $\tau = i\varepsilon$, where $\varepsilon > 0$, eq. (A.13) becomes

$$\sum_{m=-\infty}^{\infty} \exp(-m^2\pi\varepsilon + 2\pi mix) = \varepsilon^{-1/2} \sum_{m=-\infty}^{\infty} \exp[-(\pi/\varepsilon)(x - m)^2]. \tag{A.14}$$

It is easily verified that in the limit $\varepsilon \to 0$,

$$\varepsilon^{-1/2} \exp[-(\pi/\varepsilon)(x - m)^2] \to \delta(x - n\pi), \tag{A.15}$$

so that the generalized function identity

$$\sum_{m=-\infty}^{\infty} \delta(x - m) = \sum_{m=-\infty}^{\infty} \exp(2\pi mxi) \tag{A.16}$$

is a special limiting case of the theta-function transformation (A.12). Although it is more usual to derive the theta function transformation (A.12) from eq. (A.16) (cf. ref. [31], p. 70), implicitly regarding the general function identity as the more fundamental formula, we would be equally justified in taking the theta function identity as fundamental.

Having obtained eq. (A.16), we may now assign meanings to the theta functions for $\tau = 0$, for example,

$$\vartheta_3(z\,|0) = \vartheta_3(z, 1) = 1 + 2\sum_{n=1}^{\infty} \cos(2nz) = \sum_{m=-\infty}^{\infty} \exp(2miz)$$

$$= \sum_{m=-\infty}^{\infty} \delta(z/\pi - m) \tag{A.17}$$

the series being interpreted in the sense of the theory of generalized functions [31]. Expressions for $\vartheta_1(z\,|0) = \vartheta_1(z, 1)$, $\vartheta_2(z\,|0) = \vartheta_2(z, 1)$ and $\vartheta_4(z\,|0) = \vartheta_4(z, 1)$ can be deduced from eq. (A.17) via the identities (ref. [30], p. 464)

$$\vartheta_1(z, q) = -\vartheta_2(z + \tfrac{1}{2}\pi, q),\ \vartheta_4(z, q) = \vartheta_3(z + \tfrac{1}{2}\pi, q) \tag{A.18}$$

and

$$\vartheta_3(z, q) = \vartheta_3(2z, q^4) + \vartheta_2(2z, q^4)\,. \tag{A.19}$$

In particular, we have

$$\vartheta_2(\pi x, 1) = \vartheta_3(\tfrac{1}{2}\pi x, 1) - \vartheta_3(\pi x, 1) = \sum_{m=-\infty}^{\infty} [\delta(\tfrac{1}{2}x - m) - \delta(x - m)] \tag{A.20}$$

$$= \sum_{m=-\infty}^{\infty} [2\delta(x - 2m) - \delta(x - m)] = \sum_{m=-\infty}^{\infty} (-1)^m \delta(x - m)\,. \tag{A.21}$$

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