Proposed Proof of Riemann Hypothesis

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Abstract

Proposed proof of the Riemann hypothesis showing that positive decreasing continuous function which tends to zero as t goes to infinity can't have zeros of its Laplace transform in the right half plane, extending the result to the two sided Laplace transform and then showing that there is a representation of the Zeta function which meets these criteria

Key words: Riemann Hypothesis, Zeta, real positive decreasing functions

1 Analogue of the Pölya theory

Pölya produces a similar theory [1] except his is for positive increasing functions and for this the integral can't extend to infinity.
Given a real continuous positive decreasing function $f(t)$ where
$\lim_{t\to\infty} f(t) = 0$, $f(t) \geq 0$ for $t \geq 0$ and $\frac{d\,f(t)}{dt} \leq 0$ for $t \geq 0$

for which the following integral converges for some $k \geq 0$

$$\int_0^\infty e^{-k\,t} f(t)dt = \tau$$

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Then we have

$$s\int_0^\infty e^{-s\,t}\,f(t)\,dt = f(0) - e^{-s\,b}\,f(b) + \int_b^\infty e^{-s\,t}\,f(t)\,dt + \int_0^b e^{-s\,t}\,f'(t)\,dt$$

$$= s\,\mathcal{L}(f(t),s)$$

giving

$$s\,\mathcal{L}(f(t),s) = (1 - e^{-b\,t})\,f(b) + \int_b^\infty e^{-s\,t}\,f(t)\,dt + \int_0^b (e^{-s\,t} - 1)f'(t)\,dt$$

Taking limit as $b \to \infty$ gives

$$s\,\mathcal{L}(f(t),s) = \int_0^\infty (e^{-s\,t} - 1)\,f'(t)\,dt$$

If we let $s \to c + \imath\,d$ where $c \geq 0$ then we get

$$s\,\mathcal{L}(f(t),s) = \int_0^\infty (e^{-c\,t}\,\cos(d\,t) - 1)\,f'(t)\,dt - \imath\int_0^\infty (e^{-c\,t}\,\sin(d\,t))\,f'(t)\,dt$$

For $c \geq 0$, $e^{-c\,t}\,\cos(d\,t) - 1 \leq 0$ with equality only when $t = 0$ or ($c = 0$ and $d\,t = 2\,n\,\pi$) where n is an integer.

Thus $s\,\mathcal{L}(f(t),s)$ has no zeros for $\Re(s) \geq 0, (s \neq 0)$ because $f'(t)$ is generally ≤ 0. We have just shown that any positive real decreasing function which tends to zero as $t \to \infty$ has no zeros in the right half plane. This result also clearly extends to any function whose any derivative satisfies these criteria.

2 Extension to two sided transform

We can now extend this result by considering

$$\int_{-\infty}^\infty e^{-s\,t}\,f(t)\,dt = \mathcal{L}(f(t),s) + \mathcal{L}(f(t),-s) = \mathcal{L}2(f(t),s)$$

with the added conditions that $f(t)$ is even and that the integral converges everywhere and that $\int_0^\infty f(t)\,dt \neq 0$

If we represent $\mathcal{L}(f(t), s) = Even(s) + Odd(s)$ which we can do without loss of generality since the full integral is obviously even. ($Even(s)$ is an even function, $Odd(s)$ is an odd function) then

$$\mathcal{L}(f(t), s) + \mathcal{L}(f(t), -s) = 2\,Even(s)$$

$$\mathcal{L}(f(t), s) - \mathcal{L}(f(t), -s) = 2\,Odd(s)$$

Since $\int_0^\infty f(t)\,dt \neq 0$, if the one sided transform is a converging polynomial then we can represent it by

$$\mathcal{L}(f(t), s) = \mathcal{L}(f(t), 0) \prod_{i=1}^{\infty}(1 - s/\lambda_i)$$

where λ_i are the zeros of the polynomial. So

$$\mathcal{L}(f(t), s) = k \prod_{i=1}^{\infty}(1 - s/\lambda_i)$$

where $k = \int_0^\infty f(t)\,dt$

From our earlier proof we know that $\Re(\lambda_i) \leq 0$

When the two sided integral is zero we have

$$\mathcal{L}2(f(t), s) = 0 \Rightarrow Even(s) = 0$$

and so if one zeros is at $c + \imath\,d, c \geq 0$

$$\mathcal{L}(f(t), c + \imath\,d) = Odd(c + \imath\,d),\ and$$

$$\mathcal{L}(f(t), -c - \imath\,d) = Odd(-c - \imath\,d) = -Odd(c + \imath\,d)$$

i.e. $\mathcal{L}(f(t), c + \imath\,d) = -\mathcal{L}(f(t), -c - \imath\,d)$, and

$$|\mathcal{L}(f(t), c + \imath\,d)| = |\mathcal{L}(f(t), -c - \imath\,d)|$$

But from our product formula $|\mathcal{L}(f(t), s)| = |k| \prod_{i=1}^{\infty} |(1 - s/\lambda_i)|$ and since we have no positive zeros the only way that the two moduli can be equal is if $\Re(\lambda_i) = 0$ for all i. If this is the case then our polynomial has no zeros $\Re(s) \neq 0$ and thus $Even(s)$ cannot be zero off the imaginary axis and similarily the two sided Laplace transform cannot be zero off the imaginary axis. We have thus shown that for a real positive even decreasing function which tends to 0 as t

$\to \infty$ whose two sided transform is a converging polynomial over the whole plane and whose integral is non zero has no zeros off the imaginary axis of its two sided Laplace transform.

3 Towards Riemann

If we have $h(e^x) + e^{-x/2} = h(e^{-x}) + e^{x/2}$ then we also have

$$-1/4\, h(e^{-x}) + e^{-x}\, h'(e^{-x}) + e^{-2\,x}\, h''(e^{-x})$$

$$= -1/4\, h(e^{x}) + e^{x}\, h'(e^{x}) + e^{2\,x}\, h''(e^{-x})$$

and therefore if

$$g(x) = -1/4\, h(e^{-x}) + e^{-x}\, h'(e^{-x}) + e^{-2\,x}\, h''(e^{-x})$$

we get g(x) = g(-x) If we now choose

$$h(x) = \sum_{i=1}^{\infty} 2\, e^{-\frac{i^2\,\pi}{x^2} - \frac{\log(x)}{2}} = \frac{-1 + \Theta(e^{-\frac{\pi}{x^2}})}{\sqrt{x}}$$

where $\Theta(e^{-x}) = 1 + 2\,\sum_{i=1}^{\infty} e^{-i^2\,x}$, then

$$h(\frac{1}{x}) = \sum_{i=1}^{\infty} 2\, e^{-i^2\,\pi\,x^2 - \frac{1}{2}\,\log(\frac{1}{x})} = \frac{-1 + \Theta(e^{-\pi\,x^2})}{\sqrt{\frac{1}{x}}}$$

Therefore from the theory of elliptic functions

$$\frac{1}{\sqrt{x}} + h(x) = \frac{\Theta(e^{-\frac{\pi}{x^2}})}{\sqrt{x}} = \sqrt{x}\,\Theta(e^{-\pi\,x^2}) = \sqrt{x} + h(\frac{1}{x})$$

and thus

$$\frac{d^2}{dt^2} h(e^t) - \frac{h(e^t)}{4} = \frac{d^2}{dt^2} h(e^{-t}) - \frac{h(e^{-t})}{4}$$

which we will choose to make $j(t)$. So $j(t) = j(-t)$ and

$$j(t) = \frac{d^2}{dt^2} h(e^t) - \frac{h(e^t)}{4} = \sum_{i=1}^{\infty} 4\, e^{-e^{-2\,t}\,i^2\,\pi - \frac{9\,t}{2}}\, i^2\,\pi\, \left(-3\, e^{2\,t} + 2\, i^2\,\pi\right)$$

So we know that $j(t)$ is even. If we now look at the derivative by chosing a function

$$y(t) = -4\, e^{-\left(\frac{i^2\,\pi}{e^{2\,t}}\right) - \frac{9\,t}{2}}\, i^2\,\pi\,\left(3\,e^{2\,t} - 2\,i^2\,\pi\right)$$

Then as $t \to -\infty$, $y(t) \approx 8\,i^4\,\pi^2\,e^{-e^{-2\,t}\,i^2\,\pi}$

which is a very small positive number and also

$$\frac{d\,y(t)}{dt} = 2\,e^{-\left(\frac{i^2\,\pi}{e^{2\,t}}\right) - \frac{13\,t}{2}}\, i^2\,\pi\,\left(15\,e^{4\,t} - 30\,e^{2\,t}\,i^2\,\pi + 8\,i^4\,\pi^2\right)$$

which is zero when

$$t = \log(-\sqrt{i^2\,\pi - \sqrt{\tfrac{7}{15}}\,i^2\,\pi}),\ t = \log(\sqrt{i^2\,\pi - \sqrt{\tfrac{7}{15}}\,i^2\,\pi}),$$
$$t = \log(-\sqrt{i^2\,\pi + \sqrt{\tfrac{7}{15}}\,i^2\,\pi}),\ t = \log(\sqrt{i^2\,\pi + \sqrt{\tfrac{7}{15}}\,i^2\,\pi})$$

Only two of these are real and the real ones only occur at $t \le 0$ when $i^2\,\pi - \sqrt{\frac{7}{15}}\,i^2\,\pi < 1$ or $i < \frac{1}{\sqrt{\pi - \sqrt{\frac{7}{15}}\,\pi}}$ whose value is approximately equal to 1.00227. This means that the only term which can be decreasing for $t \le 0$ is the i = 1 term. Since we know that the function is even we must have that the derivative at t = 0 is 0. Thus at zero the negative gradient of the first term is equal to the positive gradients of all the others. Since all the other terms gradients are increasing and the first terms is becoming more negative this shows that there can be no other point where $j'(t)$ is zero for $t \le 0$. Incidentally we know that j(0) is finite since it is convergently

$$\frac{2\,\pi\,\left(-3 + 2\,\pi\right)\,\Theta'(e^{-\pi})}{e^{\pi}} + \frac{4\,\pi^2\,\Theta''(e^{-\pi})}{e^{2\,\pi}} \approx 1.78679$$

Therefore $j(t)$ is an even decreasing positive function which tends to zero as $t \to -\infty$ and therefore as $t \to \infty$

Now we have

$$j(t) = \sum_{i=1}^{\infty} 4\,e^{-e^{-2\,t}\,i^2\,\pi - \frac{9\,t}{2}}\,i^2\,\pi\,\left(-3\,e^{2\,t} + 2\,i^2\,\pi\right)$$

and therefore

$$\int_{-\infty}^{\infty} e^{-s\,t} j(t)\,dt = \pi^{\frac{-1-2\,s}{4}}\,\left(-1 + 2\,s\right)\Gamma(1/4(2\,s+5))\zeta(s+1/2)$$

But we also have

$$\pi^{\frac{-1-2\,s}{4}}\,(-1+2\,s)\,\Gamma(1/4(2\,s+5))\zeta(s+1/2) =$$

$$-\Gamma(1/4)\zeta(1/2)/(4\,\pi^{\frac{1}{4}})\prod_{i=1}^{\infty}(1-s/\rho_i)$$

from the Hadamard product expression form of the Zeta function where ρ_i are the non-trivial zeros of $\zeta(1/2+s)$ so we get

$$\int_{-\infty}^{\infty} e^{-s\,t} j(t)\,dt = -\Gamma(1/4)\zeta(1/2)/(4\,\pi^{\frac{1}{4}})\prod_{i=1}^{\infty}(1-s/\rho_i)$$

We are nearly there now because all we need to show is that

$$\mathcal{L}(j(t),s) = \int_{0}^{\infty} e^{-s\,t} j(t)\,dt = Even(s) + Odd(s)$$

can be expressed as a fully convergent polynomial.

We already know that $Even(s)$ has this property so we need to show that $Odd(s)$ also has this property. If this were not the case, $Odd(s)$ would have poles. These would be mirrored around the imaginary axis (since $Odd(s) = -Odd(-s)$) and since we can show that there are no poles for $\Re(s) \geq 0$ of

$$\mathcal{L}(j(t),s) = Even(s) + Odd(s) = \int_{0}^{\infty} e^{-s\,t}\,j(t)\,dt$$

from the decay of $j(t)$, we know that $Odd(s)$ can be represented as a full convergent polynomial.

Now since $j(t)$ satisfies all our criteria its two sided transform has no zero's off the imaginary axis which means that our Zeta function can have no zero's off the imaginary axis which proves the hypothesis.

References

[1] Pölya, G.,"Über die Nullstellen gewisser ganzer Funktionen", 1918