Review of Quantum Chaology and Structural Complexity Approaches to Characterising Global Behaviour with Application to Primes

By Noel Patson

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1 Introduction

This paper reports the results of my study and research for the subjects 84333 special topic A mathematics and 85333 special topic A computing. The theme was proposed by the supervisor, Victor Korotkich and concentrates on descriptions of approaches dealing with how the local behaviour of individual components influences the global behaviour of a whole complex system. Among many others, there is the question of how the interplay of co-operation and sharing information among the components gives rise to behaviour that is inaccessible or unattainable for the components.

It is well known that complex systems consisting of individual components can exhibit new qualitative, sophisticated collective behaviour that emerges from their interactions (for example [1, 2, 3]).

Some examples can be found in:

- Chemistry, where the minimisation of energy potential mimics the ability of atoms to self organise into molecules
- Data communications, where independent WWW servers are interconnected to form the Internet which has stimulated a new global culture
- Economics, where buyers, sellers, brokers, and shares, interrelate according to their different rules to form the stock market where understanding of global behaviour is very desirable
- Biology, where individual cells are arranged to form organisms.
- Robotics, where independent robots can co-operate together as a team towards a global outcome far beyond the scope of the individual robots.

These types of systems are prevalent in many disciplines such as physics, chemistry, biology, economics, social and behavioural sciences. Currently these types of problems are under intensive investigation and many different approaches are used to understand how the local behaviour of individual components and how the co-operation and communication between components influence the global behaviour of the whole system (see [1, 2, 3, 4]). The aim of this study was to identify common threads in these approaches in an attempt to develop a more holistic viewpoint. The choices of approaches considered in this research reflect the personal preferences of the supervisor.

These problems are described in many different forms. For example, they can be characterised in terms of complex adaptive systems (CAS) Gell-Mann [1]. Research in CAS arises from investigating how rules used by individuals, (agents) to make decisions, combine together to form large-scale and global patterns. For example, drivers deciding which route to take, what speed to drive and whom they will overtake, etc. form a global pattern of traffic congestion. CAS are characterised by having a medium number of agents that are adaptive or intelligent, and may only have access to local information, i.e. no agent has access to information about what all the other agents are doing. As reported in [5], current mathematical frameworks fail to describe or analyse the properties of CAS. This is what motivates research in this area, to find approaches that will bring about solutions to these kinds of problems.

Specific areas of science are already very experienced in dealing with these sorts of problems, eg. quantum physics is very advanced in describing the interactions and self organising capabilities of the atomic realm [2]. It is known that in quantum phenomena, non-local effects persist where constituents are interrelated together globally. All parts are interconnected, integrated globally and inseparable. This type of global behaviour is being investigated in this research.

Quantum chaology [6, 7, 8], for example, has been found to be very useful for describing complex problems. The previous success of this approach has been a determinant in the decision to consider it. Quantum chaology considers complex systems that are not possible to characterise using standard techniques. In particular this approach considers the distribution of spacings between eigenvalues of the energy operator for a classical chaotic system. These values, called the energy spectrum of the system, are often extracted from experimental data and investigated statistically. Different statistical measures have been proposed in order to characterise global behaviour as a kind of signature of a system. For example, the result of applying a trace formula gives a value that encapsulates global information about a matrix.

Quantum physics has provided a successful framework for modelling the interactions, relationships and interconnections of atomic phenomena. The work of quantum chaology in using the power and tools of quantum physics to characterise classically chaotic systems, provides optimism for the use of these tools to characterise other complex systems that exhibit chaotic behaviour.

The quantum chaology approach has some very powerful tools, and effective mechanisms for encapsulating global behaviour. However, it lacks a foundation from which a methodology can be developed to increase the scope of its analytical power.

A second approach considered, structural complexity attempts to describe a non-local order in which objects are viewed as interrelated parts of one unbroken whole. David Bohm [2] suggested the notion of 'unbroken wholeness' where non-local connections are an essential aspect of this wholeness. He sees non-local connections as the source of the statistical formulation of the laws of quantum physics. These interconnections of the whole have nothing to do with locality in space and time but exhibit an entirely different quality of enfoldment, which he terms the 'implicate' order. A mathematical formulation of Bohm's point of view has been suggested in [9, 10, 11]. In the formulation to model wholeness, the proposed structure is seen as a web of integer relations and integer patterns.

The bridging of the quantum chaology approach and the structural complexity approach may be invaluable for understanding complex systems and similar broad fields of investigation. The work of this report, provides a setting for investigating this bridge, and suggests directions in which further work can be made.

Recently there have been promising developments in the characterisations of complex systems. The classical systems, which exhibit chaotic behaviour, are reflected in the statistics of the infinite set of energy levels, i.e. spectrum, of their quantum mechanical counterparts. Current research has proved that these spectra are connected with the spectrum of Riemann zeta function zeros and the distribution of prime numbers.

The Riemann zeta function can be expressed in several different forms. Gutzwiller [6] writes "Scientifically minded people from grade school on up are fascinated by the prime numbers, positive integers p whose only divisors are 1 and itself. They are the exceptions among the integers and show up at irregular intervals." The prime numbers seem to dance randomly among the integers yet it is well known that they exhibit well-defined global behaviour, the prime number theorem and Riemann's hypothesis connected with the zeta function being two outstanding examples of the global behaviour of primes.

As an introduction to this research area, the nature of the interactions of primes that give rise to this global behaviour was investigated. This was motivated also by the very strong evidence from quantum chaology of a connection between the distribution of spacings of energy levels of quantised chaotic systems and the distribution of Riemann zeta function zeros (see Figure 5) [7].

Computer experimentation was used in an attempt to determine scaling properties of the distribution of prime numbers in order to find some patterns that may be connected with non-local order. Also the connections of the zeta function from number theory to quantum chaology were researched.

In looking at the Riemann zeta function and the prime numbers we are touching on areas that present massive problems for the mathematics and scientific communities. Books on number theory are filled with conjectures, theorems, and postulates that begin with statements like "if the Riemann Hypothesis is true then…". The recent discoveries in quantum physics of the connection of the zeta function to the spectrum of spacings of energy levels also show that research in this area is needed.

2 Complex adaptive systems

Complex adaptive systems (CAS) are prevalent in many disciplines such as physics, chemistry, biology, economics, social and behavioural sciences. These systems form a basis for the investigation of co-operation and negotiation with respect to artificial intelligence. Research in CAS arises from investigating how rules used by individuals (agents) to make their own decisions, combine together to form large-scale, global patterns. For example, drivers deciding which route to take, what speed to drive and whom they will overtake, etc. form a global pattern of traffic congestion [5]. CAS are characterised by having a medium number of agents that are intelligent and adaptive, and may only have access to local information, i.e. no agent has access to information about what all the other agents are doing.

John Casti [5] presents an example proposed by W. B. Arthur, called the El Farol problem, which contains all the elements that characterise a complex system. One hundred Irish music fans (intelligent adaptive agents) want to visit a bar that hosts an Irish band every Thursday but has only 60 seats. Using some predictive algorithm or other decision making process, each agent makes a choice of whether to attend the next week and risk being turned away or overcrowded. This algorithm is based on only the local information of what the agent knows he or she is going to do, and the published attendance of previous week's performances. Note that the number of agents is large enough to be too complicated for intuition or hand calculation and too small for statistical methods to be useful.

A small team of robots playing against another team of robots in a game of soccer would be another example of a complex adaptive system. Here the desired behaviour of the robots is to have some kind of intuition or sense of the global arena in much the same way as electrons seem to “know” where other electrons are, so they don't crash into them.

In this game of robot soccer, it is also desired that the robots have intimate knowledge of what the other robots on their team are doing or about to do so that they can operate as a team to accomplish tasks that are beyond the scope of individual robots. Here the robots could be seen to be like the ants of an ant colony. Each ant seems on one scale to be an individual wandering about aimlessly on its daily tasks. However the global outcome of all the ants performing their tasks comes together through teamwork to give the colony a life of its own on a level that seems inaccessible to the individual ants, yet provides the environment for their existence.

The motivation for research in the area of CAS is to find approaches that will bring about solutions to difficult problems like robots co-operating together towards outcomes far beyond the scope of the individual robots. Current mathematical paradigms fail to analyse or describe the properties of CAS and much work is needed in this area.

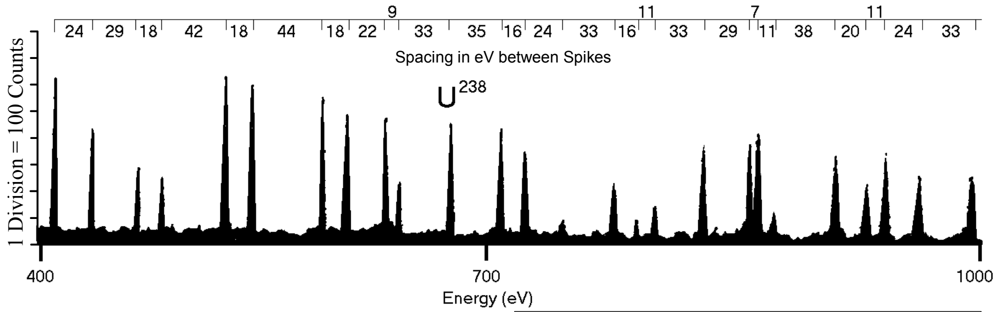
3 Quantum Chaology Approach

The interest in this study is not in atomic physics itself but rather in applying the results of global behaviour found in atomic phenomena to the large-scale phenomena that we experience in everyday life. It is proposed that systems such as robots playing soccer could be programmed to behave just like electrons; each robot would have special knowledge of the other robots.

The approach is to examine statistically certain values derived from a matrix that describes or encapsulates the behaviour of the system. This is usually viewed from an energy point of view, i.e. how does the system behave as the energy is changed.

Figure 1 (top) illustrates the concept of examining spacings on a sample of Uranium. Here we observe how for certain energy values, the number of neutrons emitted by a piece of U^{238} is very small while for other values the number of neutrons emitted is large (shown as spikes). The spacings have been measured and displayed above the spikes. The frequency of these spacings to be within certain categories has been represented in the frequency distribution table (left). The graph (right) is a plot of these distributions.

It is known that in quantum phenomena, non-local effects persist where constituents are globally interrelated. All parts are interconnected, integrated globally and inseparable. Quantum chaology has been found to be useful in describing very complex problems. This approach can be applied to systems that cannot be characterised using standard techniques.



spacing in electron volts	number of spacings in sample
0 to 9	2
9 to 18	8
18 to 27	7
27 to 36	5
36 to 45	3

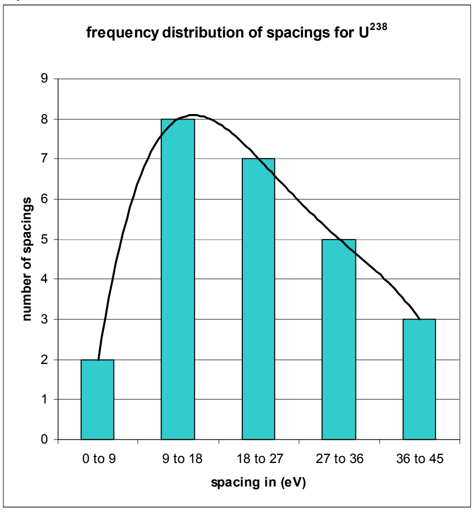


Figure 1 Spectrum for U ^{238} showing the spacings in eV between spikes (top). These spacings are counted according to categories shown in the frequency distribution table (left). The frequency distribution of the spacings is displayed in the graph (right) with a curve that suggests what the probability distribution is.

In particular this approach considers the distribution of spacings between eigenvalues of the energy operator for classical chaotic systems. These values, called the energy spectrum of the system, are often extracted from experimental data and investigated statistically. Different statistical measures are proposed to characterise global behaviour as a sort of signature for the system. The probability distribution of eigenvalues is one example that has been considered in this report.

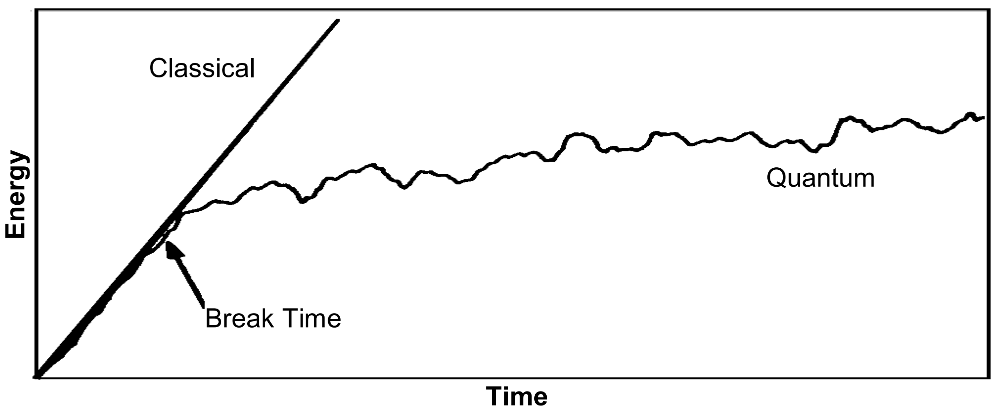


Figure 2 After the 'break time', the quantum rotator absorbs energy more slowly than the chaotic classical rotator [7].

Quantum chaology is an area of study that is largely unexplored on the boundary between quantum and classical mechanics. Niels Bohr, one of the first quantum theorists, realised that quantum mechanics must agree with classical mechanics when applied to large or heavy systems where wave effects can be ignored. This was termed the "correspondence principle" [12].

Theoretical investigations on electrons illuminated with waves of radiation showed this correspondence up until a certain break-time after which the quantum system absorbs energy more slowly than the chaotic classical system. However by making the particle heavier, the break-time, which signals the onset of quantum effects, gets later and later (see Figure 2).

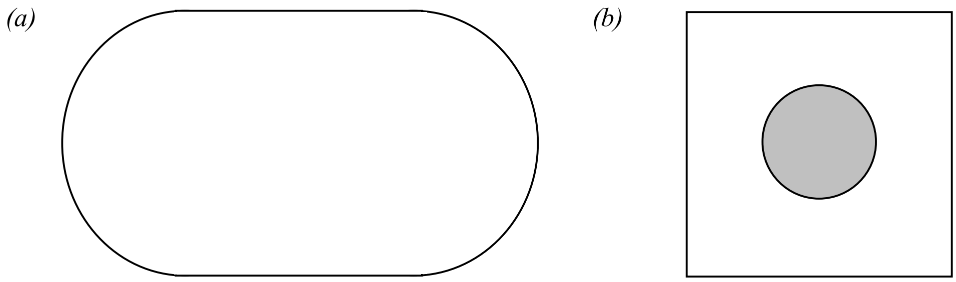


Figure 3 (a) Stadium of Bunimovich and (b) Billiard of Sinai are examples of chaotic systems that have some symmetry [6, 7, 8].

An example of a classical system that has been studied is an idealised billiard ball rebounding within the confines of certain shapes. With some shapes such as a circle or a rectangle, (classically regular systems), the behaviour of the billiard is very predictable (see Figure 3). With other shapes like a stadium of Bunimovich or billiard of Sinai (a square with a circular obstacle at its centre), which are classically chaotic systems, the behaviour is chaotic (see Figure 3 and Figure 4).

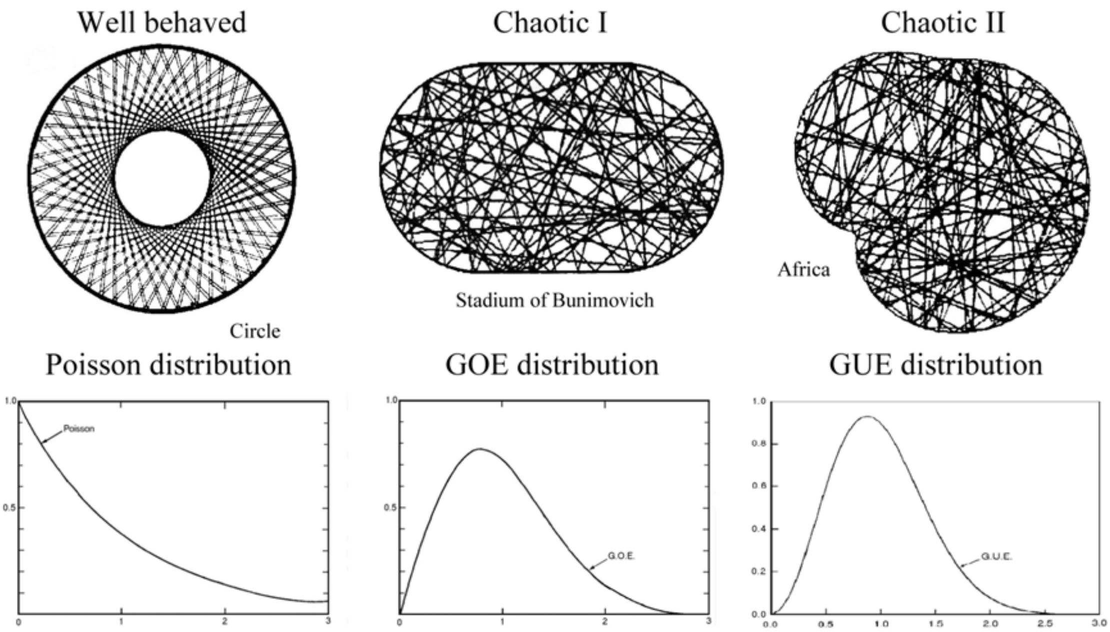


Figure 4 Quantum billiards in Symmetric (Circle), Semi-Symmetric (Stadium), and Asymmetric Systems (Africa) and their corresponding level spacing probability distributions [7].

3.1 Spectrum of Energy Levels

Recently, promising developments have revealed that characterisations of complex systems, which exhibit chaotic behaviour, are reflected in the statistics of the infinite set of energy levels. In particular this approach considers the distribution of spacings between eigenvalues of the energy operator for a classical chaotic system. These values are often extracted from experimental data and are investigated statistically in order to characterise global behaviour.

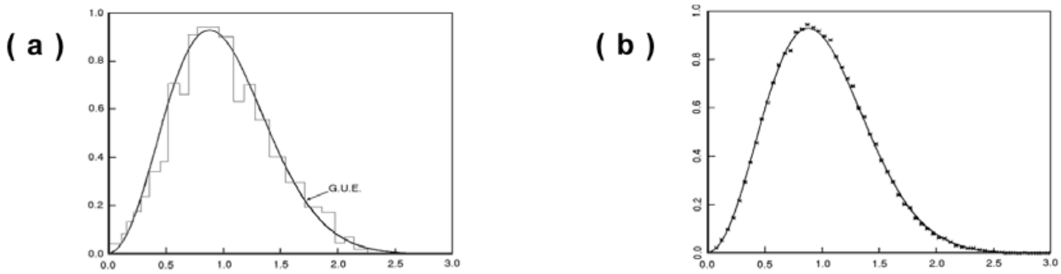


Figure 5 The distribution of spacings where there is no symmetry: (a) a representative image of several hundred energy levels of the 'Africa' game of quantum billiards in a magnetic field; (b) 100000 zeros of the Riemann's zeta function [7].

One statistical measure used to examine the spectrum is the *spectral distribution* $Q(x)$, which is the probability of finding an energy level in the vicinity of an interval, $\alpha + x \leq E < \alpha + x + dx$ relative to an existing energy level $E = \alpha$. A second related statistical measure used to examine the spectrum, the *level spacings probability* (LSP) distribution $P(x)$, is the probability of finding two consecutive energy levels a certain distance apart. Figure 5 shows an example of an LSP distribution for a classically chaotic system called the Africa game of quantum billiards [7].

The relationship between the two functions $Q(x)$ and $P(x)$ is demonstrated by the following [6]: take a segment of integer length L and divide it into small intervals of length ε. Place L markers at random, independent of each other, with probability $Q(x)$ into the small intervals. Let ξ be a coordinate inside any small interval being considered. The first marker above 0, will hit any particular interval ξ with probability $\varepsilon Q(\xi)/L$ and miss any other interval with probability $(1-\varepsilon Q(\xi)/L)$. Consider an interval of length x starting at α. The probability that a marker is not in the interval chosen but is in the interval $\alpha + x \leq E < \alpha + x + dx$ is given by the product of $Q(\xi)/L$ and all the $(1-\varepsilon Q(\xi)/L)$ for $0 < \xi < x$. In the limiting process for choice of small ε or large L the two functions $Q(x)$ and $P(x)$ are related in that

$$P(x) = Q(x)\exp\left(-\int Q(\xi)\, d\xi\right).$$

In particular if $Q(x)=1$, representing a situation of a completely random distribution of energy levels, the *Poisson distribution* $P(x)=\exp(-x)$ is formed.

The connection between $Q(x)$ and $P(x)$ is illustrated by the following example. Consider a 12-sided regular die that has **1** on one face, **3** on two faces, **6** on four faces and **8** on five faces. $Q(x)$ is the probability of getting 1, 3, 6, 8 assuming the die has equal probability of landing with any face up (see Figure 6 top). Take L=10 which is equivalent to taking ten throws of the die. The result of the 10 throws is 3, 8, 6, 8, 8, 1, 6, 3, 8, and 6. Arrange the result in sequential order 1, 3, 3, 6, 6, 6, 8, 8, 8, and 8. The space between each consecutive number is 2, 0, 3, 0, 0, 2, 0, 0, and 0 (see Figure 6 bottom). $P(x)$ is the probability distribution of these spacings in the limit $L \to \infty$.

It is remarkable that when the LSP distribution for the stadium of Bunimovich is superimposed over the LSP distribution for the billiard of Sinai system, universality is revealed between the two entirely different classically chaotic systems. The distribution $P(x)$ is a Gaussian Orthogonal Ensemble (GOE) see sect 3.2. Universality is also revealed in the LSP distributions of the predictable classically regular systems, such as the circle and the rectangle. Here the distribution $P(x)$ is *Poisson.* There is a third universality class corresponding to quantised chaotic systems such as the Africa Billiard (see Figure 5), for which the LSP distribution follows Gaussian Unitary Ensemble (GUE) statistics.

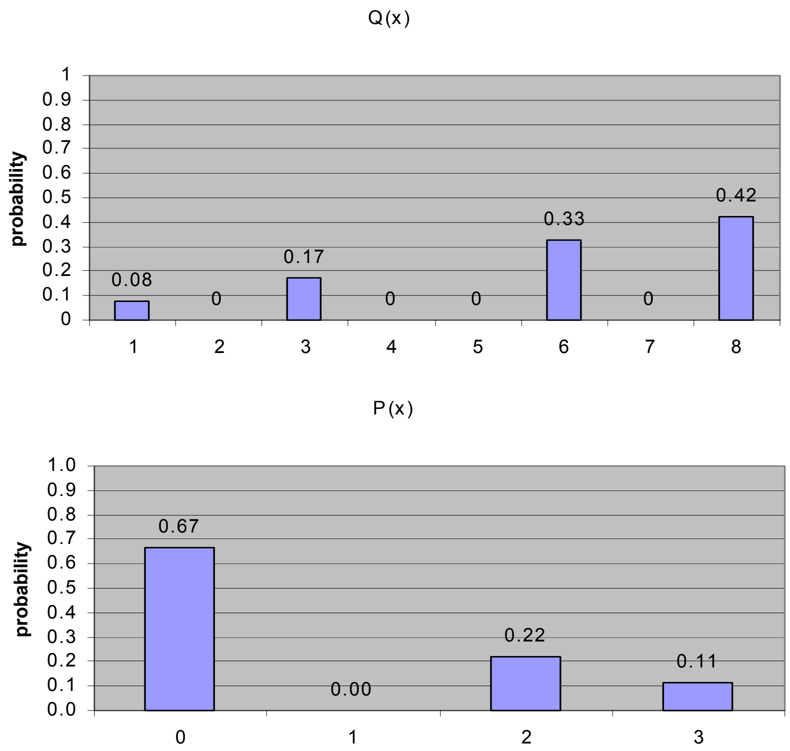


Figure 6 The relationship between Q(x) (top) and P(x) (bottom) based on a 12 sided die that has ***1*** *on one face,* ***3*** *on two faces,* ***6*** *on four faces and* ***8*** *on five faces, thrown 10 times.*

There is a remarkable connection between the distribution of zeros of the zeta function and the distribution of spacings between neighbouring energy levels in certain quantum phenomena [13]. In the early 1970's Hugh Montgomery developed a formula that described the statistics of the spacings between consecutive zeros of the zeta function. When physicist Freeman Dyson saw Montgomery's work, he recognised the formula as a function that arises in quantum mechanics. It seemed as if the zeros of the zeta function were behaving precisely like the solutions to the complex mathematical models in statistical mechanics that physicists were using to calculate energy levels in large nuclei. Berry and other physicists have seen a relationship between the distribution of zeta zeros and the distribution of level spacings in quantised chaotic systems such as the Africa Billiard mentioned above (see Figure 4). It seems that the distribution of zeta zeros follows the GUE distribution.

3.2 Statistics of Random Matrices

If a large matrix H contains random elements from the set of complex numbers in such a way so that it is Hermitian, ($H=H^*$ i.e. it is equal to its complex conjugate) then the eigenvalues will be real numbers. In this way about half the matrix elements are fixed by the values of the other half in order to generate the Hermitian property. For example, if the elements above the main diagonal were first chosen, then the elements below the diagonal would be determined from these chosen elements. Any elements not determined by this procedure are taken to be independent random variables [14].

If the real and imaginary parts of the matrix elements of matrix H that are above the main diagonal are distributed as normal random, these elements have a joint probability density proportional to $\exp(-\mathrm{tr}\ H^2)$. This probability distribution is called the Gaussian Unitary Ensemble (GUE), because it is invariant under unitary transformations of H. The GUE probability distribution corresponds to the universality demonstrated by the second class of classically chaotic systems that are asymmetric in shape such as the Africa billiard table (see Figure 4). It also corresponds to the distribution of spaces between zeros of the zeta function (see Figure 5).

If H is symmetric, ($H=H^T$ i.e. it is equal to its transpose) so that its elements must be real numbers, it has been found that the elements above the main diagonal are normal random. These elements have a joint probability density proportional to $\exp(-\mathrm{tr}\ H^2)$. This probability is invariant under real orthogonal transformations of H, which leads to its name, the Gaussian Orthogonal Ensemble (GOE). This ensemble describes the distribution of spacings in the quantum game of billiards played on the first class of classically chaotic systems that show some symmetry, eg. Stadium and Sinai (see Figure 4).

The very strong agreement of GOE statistics with data on atomic spectra of rare-earth metals and with large scale calculations of energy levels for the first class of classically chaotic systems, has led to the speculation that the GOE is a universal characteristic of energy levels in classically chaotic systems [6]. However, exceptions to this rule were soon found in systems with the "hard form of classical chaos", such as a compact smooth surface of constant negative curvature and the Anisotropic Kepler Problem [6]. The search for reasons explaining the departures from GOE is hoped to shed light on recognising different types of quantum chaos.

3.3 Riemann's Zeta Function

The Riemann zeta function can be expressed in several different forms. Gutzwiller [6] writes "Scientifically minded people from grade school on up are fascinated by the prime numbers, positive integers p whose only divisors are 1 and itself. They are the exceptions among the integers and show up at irregular intervals." The prime numbers seem to dance randomly among the integers yet it is well known that they exhibit well-defined global behaviour, the prime number theorem and Riemann's hypothesis connected with the zeta function being two outstanding examples of global behaviour of the primes.

Edwards [15] in his monograph on the subject of Riemann’s Zeta Function, includes the historical background of the zeta function as well as recent numerical results.

Euler’s theorem states that the sum of the reciprocals of the prime numbers is a divergent series. Euler may have been the first to notice that as x approached infinity the sum of the x terms of this series approaches log(log(x)). Gauss was looking from a different point of view when he stated in 1849 that the density of prime numbers appears on the average to be 1/log(x). This observation was based solely on empirical observations, which were taken from slightly inaccurate counts of primes below three million in intervals of five hundred thousand. When compared with the actual counts, Gauss' formula was more accurate than the available data he was analysing.

X	Count of primes < x	Gauss's estimate	Difference
500000	41538	41606	68
1000000	78498	78628	130
1500000	114155	114263	108
2000000	148933	149055	122
2500000	183072	183245	173
3000000	216816	216971	155

Figure 7 Table showing Lehmer's count of primes and comparing the density of primes against Gauss's estimate based on $\int_2^x \frac{dt}{\log(t)}$ [15].

Legendre was the first to attempt to prove this (unsuccessfully) with Chebyshev making a significant advance by proving the result that the number of primes less than x, denoted $\pi(x)$, is:

$$0.89\int_2^x \frac{dt}{\log(t)} < \pi(x) < 1.11\int_2^x \frac{dt}{\log(t)}.$$

Riemann contribution came in 1859 when he expressed a formula for $\pi(x)$ as the sum of an infinite series in which, $\int_2^x \frac{dt}{\log(t)}$ is the largest term. Although Riemann did not prove his results, his methods brought great insight to the prime number density question, which took over 30 years for the mathematics world to absorb.

In 1896 Hadamard and de la Vallée Poussin independently proved what is known as the prime number theorem, working on the foundation that Riemann established.

$$\pi(x) \to \int_2^x \frac{dt}{\log(t)} \quad as \quad x \to \infty.$$

The zeta function can be expressed very simply in two ways as shown below. The second expression is an infinite product over the sequence of prime numbers p.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{and} \quad \zeta(s) = \prod_p \frac{1}{1-p^{-s}}$$

In his work towards finding an analytic formula for $\pi(x)$, Riemann considered the complex zeros of the zeta function. By using the techniques of Fourier inversion, and Möbius inversion he came up with another way of expressing the zeta function in terms of the complex zeros ρ i.e. values of ρ where $\zeta(\rho) = 0$. His formula is:

$$\zeta(s) = f(s)\prod_i (1 - s/\rho_i),$$

where *f(s)* is a relatively simple fudge factor [13].

Riemann's motivation in analysing the zeta function was connected with his interest in the prime numbers. He showed that the complex zeros of the zeta function determine the fluctuations in the density of primes. What remains to be proved, and is “the single most desirable achievement for a mathematician” [6] is Riemann’s hypothesis, that the non-trivial complex zeros of the zeta function all have a real part equal to ½.

Certain theorems concerning the prime numbers could be proved if the Riemann’s hypothesis is true. The first 1.5 billion non-trivial zeros have been tested using computers and they all lie on the line $x = ½$ of the complex plane, but nobody has been able to prove that all the zeros lie on this line. The distribution of spacings between neighbouring zeros of the zeta function corresponds to the Gaussian Unitary Ensemble as can be seen in Figure 5.

The quantum chaology approach has some very powerful tools, and effective mechanisms for encapsulating global behaviour. Quantum physics provides a framework for the interactions, relationships and interconnections of atomic phenomena. However, it lacks a foundation from which a methodology can be developed to increase the scope of its analytical power. The work of quantum chaology in using the power and tools of quantum physics to characterise classically chaotic systems gives optimism towards the use of these tools to characterise other complex systems that exhibit chaotic behaviour.

4 Structural Complexity Approach

The prominent paradigm in research over the centuries has been to describe complex processes of the universe in terms and analysis of component parts in the hope that the sum of the parts makes the whole. Current views, especially in Physics, have been changing so that the whole is being seen as being much greater than the sum of the parts.

Structural Complexity is based on non-local order, order that is not apparent in the individual parts that make up a system, but is revealed in the cooperation, negotiation or interaction between the parts to form a global relationship. The Structural Complexity approach attempts to describe objects in mathematical terms as interrelated parts of one undivided whole [9], [10], [11].

The physicist David Bohm [2] has suggested a notion of 'unbroken wholeness' where non-local connections are an essential aspect of this wholeness. He saw non-local connections as the source of the statistical formulation of the laws of quantum physics. These interconnections of the whole have nothing to do with locality in space and time but exhibit an entirely different quality of enfoldment, which he terms the 'implicate', or 'enfolded' order.

Bohm saw complex systems in terms of a web of relations. In this view objects are interrelated parts of one undivided whole. Elements of the web of relations are like organs adapted to their position and function in the whole. Imagine the universe as a batch of dough being kneaded (see Figure 8). The raisins, sesame seeds and yeast, flow in the enfolding process just as stars, planets and atoms flow in the universe. Our limited minds can only appreciate a small slice of the dough at one time, just as an ultrasound can only show a section of the uterus and the baby inside can move through this section.

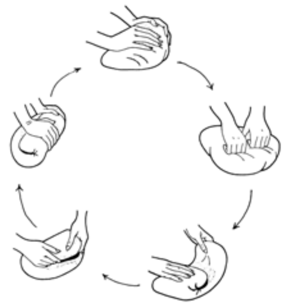


Figure 8 The universe as a batch of dough being folded over and over, the raisins, sesame seeds and yeast flow in the enfolding process just as stars, planets and atoms flow in the universe.

A mathematical formulation of this view [9], [10], [11], manifests the ideas of Bohm. The model exposes an interconnected web of integer relations that can be visualised to appreciate the underlying patterns, complexity and relationships among components of a complex system. Components that only have access to local information form together with other components to exhibit global phenomena and behaviour that was not available to the components.

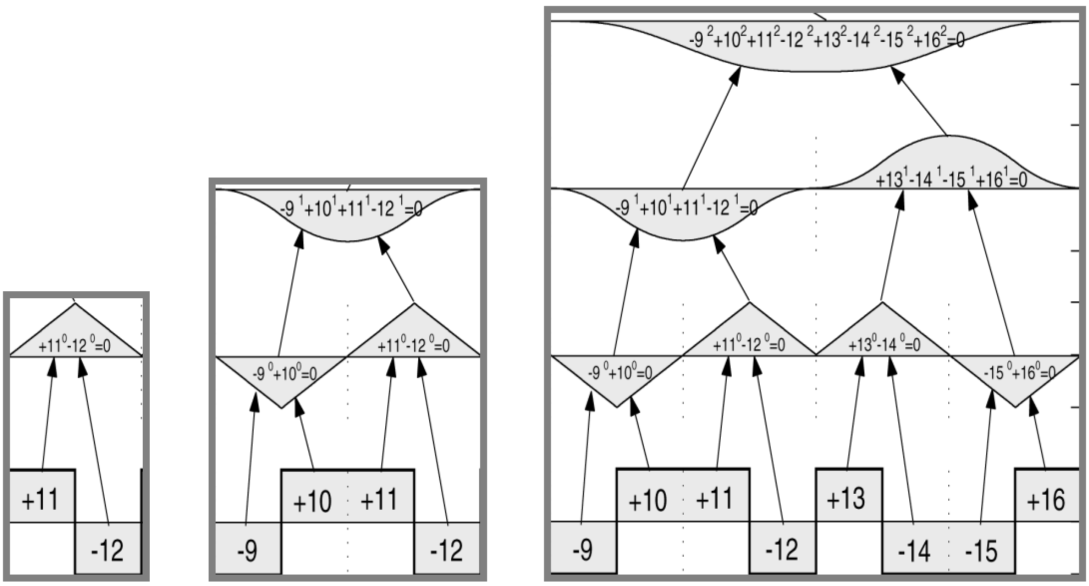


Figure 9 Three snapshots of windows looking into a small portion of structural space based on the Thue-Morse sequence [9, 10, 11].

In this view "the dough" is a space of integer relations and integer patterns called structural space. To illustrate, consider looking through some windows at a slice of this dough (Figure 9). It is not like a piece of toast that is rigid but still has the sense of movement like the ultrasound.

The integer relations are the patterns of integers that form together to equal zero on each successive level with a corresponding increment of exponents. The sign (positive or negative) of each integer is determined by the integer's corresponding position in the sequence (see bottom line of Figure 10). Here zero in the sequence determines a positive sign and one in the sequence gives a negative sign. The second bottom line of Figure 10 shows the translated sequence

The integer patterns are the boxes with integers in them along the bottom of each snapshot in Figure 9. They are areas under a step function that is specified by the translated sequence shown as the second last line in Figure 10. Under the process of integration the patterns form together to form new patterns on the next level. The area of the integer pattern in some way represents the energy of the pattern.

The arrows show how the patterns and relations are linked together in a kind of enfolding process. The sequence that is considered in each successive window is twice as long as in the preceding window. Also each integer relation in each successive level is twice as long as in the preceding level.

Each successive window also reveals a new level of formation. The number of levels that can be constructed from a given sequence is the called the structural complexity of the sequence. Figure 10 portrays the next successive window constructed from 16 elements of the Thue-Morse sequence and exhibits five hierarchical levels. So this subsequence has a structural complexity of 5.

The example shown in Figure 9 and Figure 10 is a small part of the infinite structure that is revealed by the Thue-Morse sequence. It is believed that this sequence is the most compact example chosen from an infinite number of sequences that possess the qualities demonstrated by structural space.

Certain sequences when regarded separately exhibit no visible properties but when depicted together reveal visible patterns that demonstrate structural complexity. These sequences are therefore related in the web of relationships. Visualisation plays an important part appreciating the interconnectedness of these sequences.

Complexity is viewed from the point of view of understanding the world as a dynamic web of relations. The process of formalisation is realised by the encoding of phenomena as a mapping onto the set of integers in a structural space where complexity is measured by the isomorphism between exponentiation and integration. The example with the Thue-Morse (TM) sequence over 16 integers as shown in Figure 9 and Figure 10 visually demonstrates this process.

There is a kind of symmetry in the TM sequence that captures the notion of enfoldment. Just as DNA folds certain molecules together into enzymes and amino acids. Enzymes fold amino acids together into proteins. Proteins are folded into cells. Cells into organs and systems. Which are folded into creatures and creatures into ecosystems, which are folded into the biosphere.

It is suggested that certain sequences and their corresponding structural complexity, encapsulate the interconnectedness of complex systems. Consider this simplified illustration from information technology: for some particular sequence that “enfolds” the interrelatedness of the Internet, the bottom level, (structural level 1) would represent the component building blocks, individual computers. These are linked up to a local network, LAN, (structural level 2). Various LANs are connected to WEB servers (structural level 3), and the whole thing is called the Internet (structural level 4).

Under the structural complexity approach a measure of complexity is proposed in order to formalise a system for describing the universe in global terms as a web of relations rather than in reductionist terms as the interaction of independent and separable parts. This measure provides a formal framework for investigating the problems exhibiting complex adaptive phenomena such as human decision making, robots cooperating as teams, data communications, national economies, and ecosystems, etc.

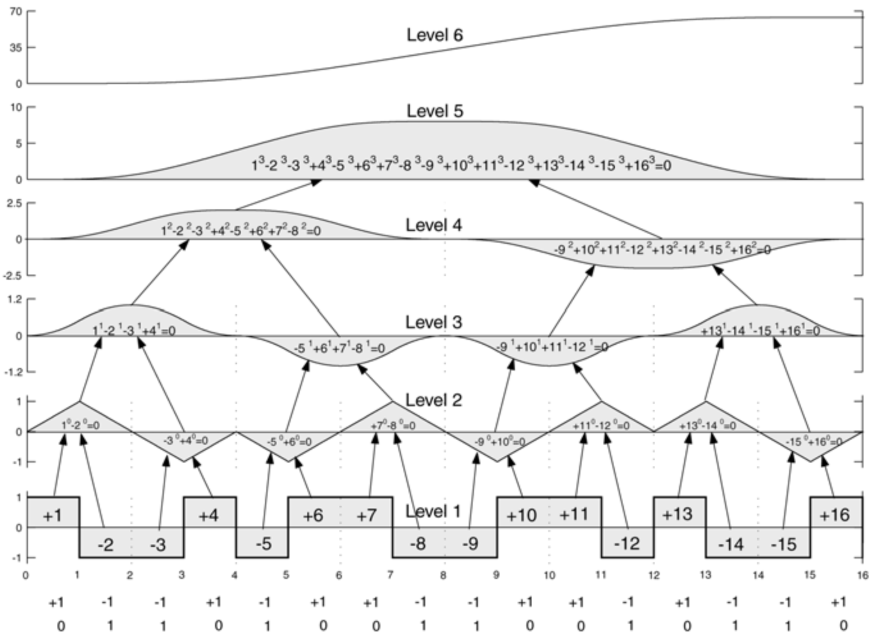


Figure 10 Structural complexity of the TM sequence over 16 integers. Components form together on ascending structural levels under exponentiation and integration [9, 10, 11].

4.1 Thue-Morse Sequence

Schroeder [16] describes the TM sequence, a binary aperiodic sequence with amazing properties. It can be constructed in several ways:

1) Taking the parity of the binary representation of each element of the non-negative integers.
2) Recursively by appending to each subsequence the complemented subsequence.
3) Self generating by starting with 0 and having each term have a complimentary baby

0
0 1
0 1 1 0
0 1 1 0 1 0 0 1

4) It also has a self-similar property in that striking out every second term reproduces the sequence:

0 ~~1~~ 1 ~~0~~ 1 ~~0~~ 0 ~~1~~ 1 ~~0~~ 0 ~~1~~ 0 ~~1~~ 1 ~~0~~
0 1 1 0 1 0 0 1

Self-generating sequences mimic the self-organisation of matter. For example the signs of the Rudin-Shapiro polynomial, which form a periodic sequence, mimic in one dimension the growth of crystals [16]. The TM sequence, which is aperiodic, imitates a newly discovered solidification process. The Fibonnacci sequence, another self-similar aperiodic sequence, is related to the ten-fold symmetry found in electron diffraction patterns of an aluminium manganese alloy [16]. This evidence gives impetus to the theory that some sequence or union of sequences may depict the interconnectedness of complex systems.

4.2 Fourier Amplitude Spectrum (FAS)

The Fourier Amplitude Spectrum (FAS) is a tool for investigating periodic behaviour of sequences. It is found by taking the modulus of the Fourier transform of a portion of a sequence. To create these graphs, the author used the macro and data analysis features of Microsoft excel. In Figure 11 the FAS has been taken for 4 simple sequences. It is seen that as the length of the repeating section of the sequence is increased, the number of spikes increase.

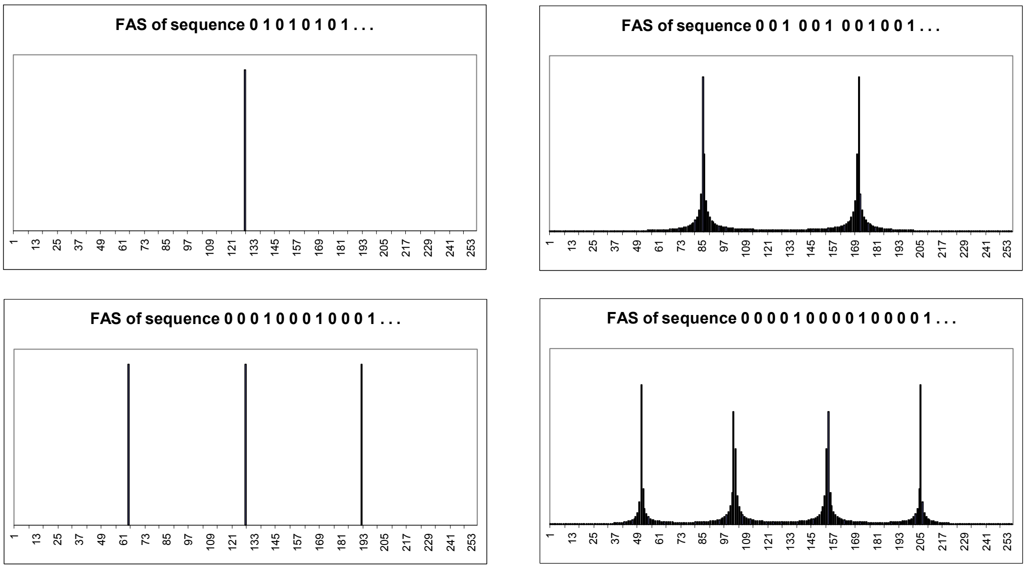


Figure 11 Fourier Amplitude Spectrum (FAS) for 4 simple sequences. This illustrates how the FAS is a measure of the periodic behaviour of a sequence.

Figure 12 shows the Fourier amplitude spectrum of the self-similar TM sequence (see section 4.1) for the first 256 elements as shown in [16]. The Thue-Morse sequence is an aperiodic sequence. That means it never repeats itself. What is interesting is that the FAS of the Thue-Morse sequence shows regular high spikes. This indicates there is some long-range order in the Thue-Morse sequence. Each element of the sequence seems to know its place much like electrons know their position so they don't crash into each other.

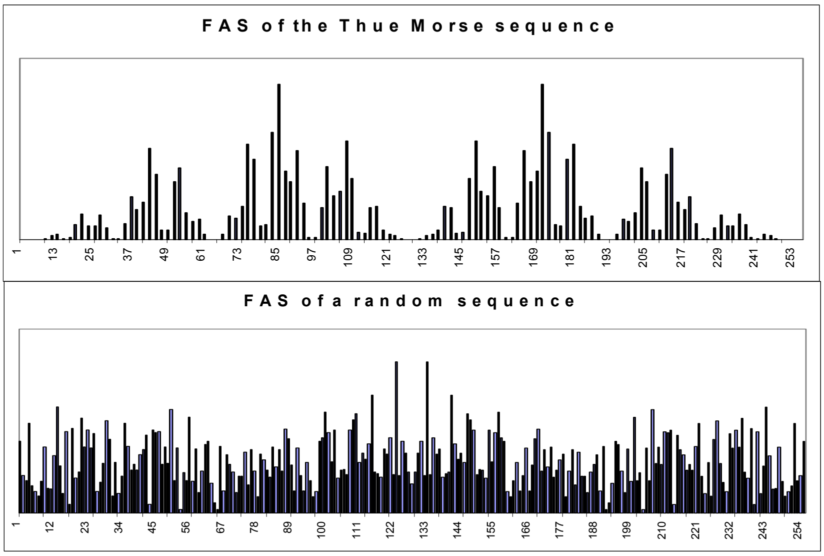


Figure 12 Fourier Amplitude Spectrum for the Thue-Morse sequence (top) (graph taken from Schroeder [16]*) and of a random sequence (bottom). The TM sequence is aperiodic (it never repeats itself), however the evidence of distinct high peaks seen in the FAS of the TM sequence, indicates that there is long range order that is not local and displays an interconnectedness and interrelatedness within the sequence.*

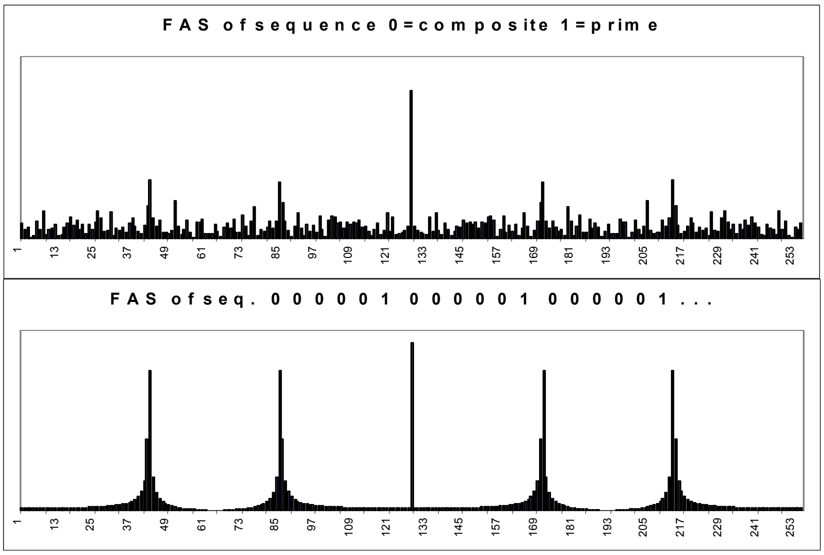


Figure 13 Fourier Amplitude sequence for the sequence constructed by turning every composite (non prime) number into 0 and every prime number into 1 (top). FAS of simple periodic sequence (bottom). The coincidence of spikes between the two spectrums indicates that primes have underlying long-range order and interconnectedness.

The prime numbers also form an aperiodic sequence. Allocating zero to a non-prime and one to a prime number creates a related aperiodic sequence. This sequence encapsulates the sense of space between primes. The FAS taken on this sequence derived from the prime numbers (see Figure 14) also indicates long range order and interconnectedness among the primes. Figure 14 also shows a comparison to a FAS of the simple periodic sequence 0,0,0,0,0,1 repeated, to highlight the correspondence between the two sequences.

Because the Fourier amplitude spectrum is used to study cyclic behaviour it is one way of uncovering non-local order.

5 Global Characteristics of Prime Numbers

As an introduction to this research area, the nature of the interactions of primes that gives rise to this global behaviour was investigated. This was also motivated by the very strong evidence from quantum chaology of a connection between the distribution of spacings of energy levels and the distribution of Riemann zeta function zeros (see Figure 5).

The search for a pattern in the prime numbers has been an elusive undertaking that has fascinated mathematicians for centuries. The connection between the prime numbers and other areas of study is now being seen in many areas, quantum chaology, [8] cryptography [17] and acoustics [7] are a few examples.

5.1 Universal Scaling Property in the Prime Numbers

Most scientific endeavours throughout time have concentrated on understanding phenomena that is simple and orderly. However, most natural phenomena are complex and disorderly. For example, weather patterns of the atmosphere, turbulence in a fluid, noise in electronic signals all demonstrate qualities of unpredictability, nonlinearity and complexity. Feigenbaum [18, 19], along with many other researchers, has discovered that underlying the outward disorderly behaviour of these kind of systems there appears a universality that is common to all of them.

These investigations are part of the cutting edge of research known as the study of chaos. In this field dynamical systems such as the ones mentioned above, are being investigated to discover if there are any patterns and order underlying the disorderly and complex behaviour that is usually observed.

Some very simple systems, which produce erratic behaviour, have features in common with complex systems that also behave erratically. It is the universality that they exhibit on their way to chaos. Feigenbaum suggests that what science has not understood in the past and called random behaviour, may not be random at all. He gives as evidence simple systems such as iteration of the logistic equation, which has the same behaviour as the transition from smooth to turbulent flow in fluids. Turbulence and other natural phenomena have traditionally been considered random but remarkably this universal behaviour indicates there is some order beneath the randomness which contradicts the whole notion of randomness.

The common feature of systems that exhibit this type of universality is that as some external parameter is varied the systems vary between organised predictable behaviour and disorganised unpredictable behaviour. Some examples of this parameter are speed of rotation in a fluid chamber or λ in the logistic equation $\mathbf{f(x) = \lambda x(1-x)}$

For these types of systems, under certain values of the parameter, very orderly and periodic behaviour occurs. This means that after some period of time the system will reproduce itself, i.e. repeat the behaviour of the previous period of time. If the value of the parameter is increased the system will continue to have this periodic behaviour until a certain critical value Δ_1 is reached. When the parameter is increased past the critical value Δ_1, a "period doubling" will occur, i.e. two periods are required for the system to reproduce itself. With a further increase of the parameter the next critical value Δ_2 will result in another period doubling so that 4 periods are required for the system to reproduce itself.

This process continues with the critical values coming closer together as the periods approach 2^n until a certain value for the parameter has been reached when it has "doubled ad infinitum" and the behaviour is no longer periodic. What is surprising is the convergence of the ratios of the values of Δ to a constant for all systems that have this period doubling route to chaos.

$$4.6692016 \cdots = \frac{\Delta_{n+1} - \Delta_n}{\Delta_{n+2} - \Delta_{n+1}}$$

This is known as Feigenbaum's constant.

In the search for patterns among prime numbers, it is understood that any universal properties that can be uncovered would be very revealing in characterising their global behaviour.

5.2 Is there Structural Complexity in the Primes?

From the structural complexity point of view, visualisation is an important tool in understanding global behaviour. An example from Korotkich [9], [10], [11], of a sequence connected with the Fibonacci numbers and a complimentary sequence shows how when considered separately they exhibit no visible properties but when overlayed a structural complexity of 4 is revealed. This heuristic approach has been taken as a technique for searching for structures and patterns in this project. In particular the prime numbers have been compared with themselves in reverse order to see if some pattern is revealed.

It was proposed by the supervisor in an attempt to uncover non-local behaviour, to search for a pattern among the primes by combining the primes with another sequence and investigating the combination as a whole. The only other sequence complex enough to be worthy of combining with the primes is the sequence of primes in reverse order. A pattern is revealed when the two sequences are considered as a whole. As an analogy when what can be considered from one point of view to be two identical objects, particle and anti-particle, are brought together the result is violently more complex than the individual particles by themselves.

The investigation was experimental rather than formal. The first 500 primes were entered in ascending order into column A of an Excel spreadsheet. The column was copied to the adjacent column B and the order was reversed (500 “anti-primes”) so it was in descending order. A third column C was created by adding corresponding numbers in the first two columns ($C_i = A_i + B_i$.). This column was graphed and the process was repeated using the first 1000 primes and 2000 primes (see Figure 14). On the graph the horizontal axis represents the index of the number in the sequence and the vertical axis represents the value of the calculation in column C. The scaling properties of the graphing software revealed a similarity among the three graphs.

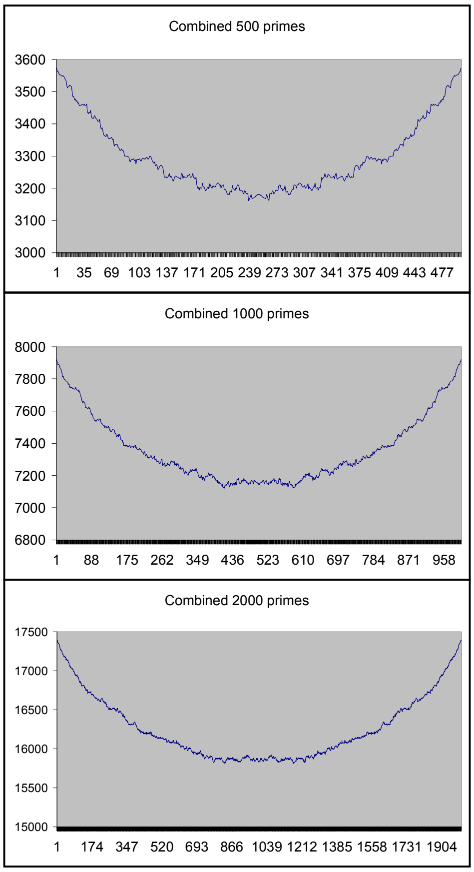


Figure 14 Representation of prime number sequence added to itself in reverse order

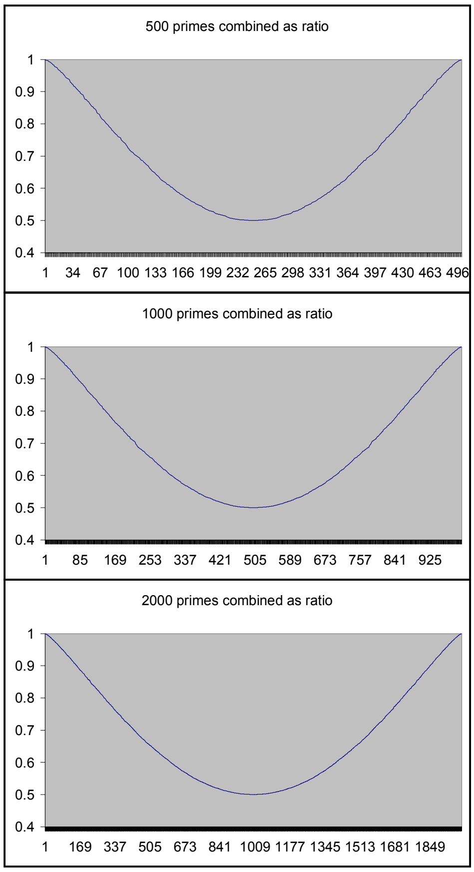


Figure 15 Smoothed representations of sequences from Figure 14

This scaling feature became more conspicuous when the two sequences, primes and "anti-primes", were combined as the ratio $R_i = \dfrac{A_i^2 + B_i^2}{(A_i + B_i)^2}$.

Figure 15 shows the graphs for this process applied to the first 500, 1000 and 2000 primes

5.3 Regularity in the prime numbers

Guiasu [20] gives evidence to suggest that there is regularity at the beginning of the sequence of primes. The question of regularity is approached from a statistical point of view. A formula using elementary functions for finding the number of primes less than 4×10^{16} was devised with the aid of the statistical program SAS based on the least squares method. This formula compared very favourably with Riemann's function for the number of primes less than a given integer $\pi(x)$.

The conclusion that $\pi(x)$ can be approximated quite accurately using a formula of elementary functions implies a hidden order or pattern below the apparent random distribution of primes. Goldbach's famous conjecture concerning the primes also hints at a hidden order. Goldbach's conjecture states that every even number greater than two can be expressed as the sum of two prime numbers.

Also the Fourier amplitude spectrum of the sequence constructed by turning every composite (non prime) number into 0 and every prime number into 1 (see Figure 13) suggests some long range underlying order which encourages investigation in this area.

Because the binary parity of the even numbers form the TM sequence, and Goldbach's conjecture links the primes to the even numbers, it seems reasonable to investigate the relationship between primes and even numbers.

The TM sequence is believed to be the ultimate in structural complexity. It seems to be the most compact set that exhibits the highest order of structural complexity. Figure 10 shows that the first 16 elements of the sequence have a structural complexity of 5.

As the prime numbers form a very complex set one approach taken was to consider component subsets of the prime numbers and how they "perform" in Goldbach's conjecture. This is also following the structural complexity approach in seeing how simpler building blocks form together in the next level of the web of relations.

The following terms are introduced in a general sense to describe properties of even numbers in relation to some set of numbers. A certain even number greater than 2 is a "Goldbach" number if any two numbers from the set can be added together to form it. A certain even number is a " Non-Goldbach" number if no two numbers from the set can be added together to form it. Using this terminology, Goldbach's conjecture over the set of prime numbers proposes that all even numbers except two and four are Goldbach numbers.

Because the spacings between prime numbers are related to the distribution of primes and also play a part in how they may combine to form even numbers, this was used as the criterion in determining the subsets.

In computer testing the first 16383 primes were considered. That is all the primes less than 180500. Of these primes the primes that are distance two, denoted sp(2), from another prime, (eg 3,5,7,11,13,17,19,29,31,41,43,59,61,…) were determined, which contained 3961 elements.

The set sp(2) has some remarkable properties. All of the even numbers greater than 4208 are Goldbach numbers for sp(2). This set which contains only 24% of primes less than 180500, missed 34 even numbers greater than 2, out of the 90144 even numbers tested. That means it missed only 0.039% of even numbers.

sp(x)	**Size of set**	**Proportion of the primes**	**Number of Non Goldbach**	**Percentage Goldbach out of the even no.**	**Highest Non-Goldbach**	**Property of Clustering in triples**	**Spacing between triples**
2	3961	24.18%	34	99.96%	4208	Yes	2
4	3920	23.93%	43	99.95%	2956	Yes	4
6	5839	35.64%	61	99.93%	4916	Some	6
8	2558	15.61%	1455	98.39%	40261	Consistent clusters of triples from 4528, 40% of the non-Goldbach	8
10	3110	18.98%	1025	98.86%	39274	Eventually consistent clusters	10
Random 10	3110	18.98%	1359	98.49%	40746	No	N/A

Figure 16 Summary of testing prime subsets for goldbach properties.

If we ignore 4 at the beginning and consider the other 33 non-Goldbach numbers for sp(2), even more surprising patterns are revealed. These are the other 33 numbers:

94	400	514	784	904	1114	1144	1264	1354	3244	4204
96	402	516	786	906	1116	1146	1266	1356	3246	4206
98	404	518	788	908	1118	1148	1268	1358	3248	4208

They are all in clusters of three numbers, distance two apart!

The same tests were conducted on sp(4), sp(6), sp(8), sp(10) with similar remarkable results that are summarised in Figure 16. These results lead to the questions:

1. What are the minimum requirements for a set of numbers to satisfy the property of "Goldbachness"?
2. How is the minimum even number after which all evens will be Goldbach numbers, determined?
3. What are the minimum requirements to determine that a set of numbers will eventually cause the evens to be Goldbach?
4. A set of odd numbers with the same distribution and properties as sp(10) was tested to compare with the prime subsets which also seemed to indicate that all evens became Goldbach after a certain number but the property of clustering in groups of three was not apparent. Could the result that all evens after 40746 were goldbach with this set, be determined analytically?

Figure 17 shows the frequency distribution in logarithmic format of spacings between 65636 (bottom) and 118396 (top) consecutive primes. Note how the shape of the graph is clustered in groups of three and also in groups of fifteen spacings. Note also how similar the graphs are for the two samples. This frequency distribution of spacings shows very definite shape that could possibly be described analytically. An analytic formula, if it exists, would describe the frequency of the gaps between consecutive primes up to a large number n. It could be used to estimate the prime closest to n, by multiplying each spacing by its frequency and adding the results together.

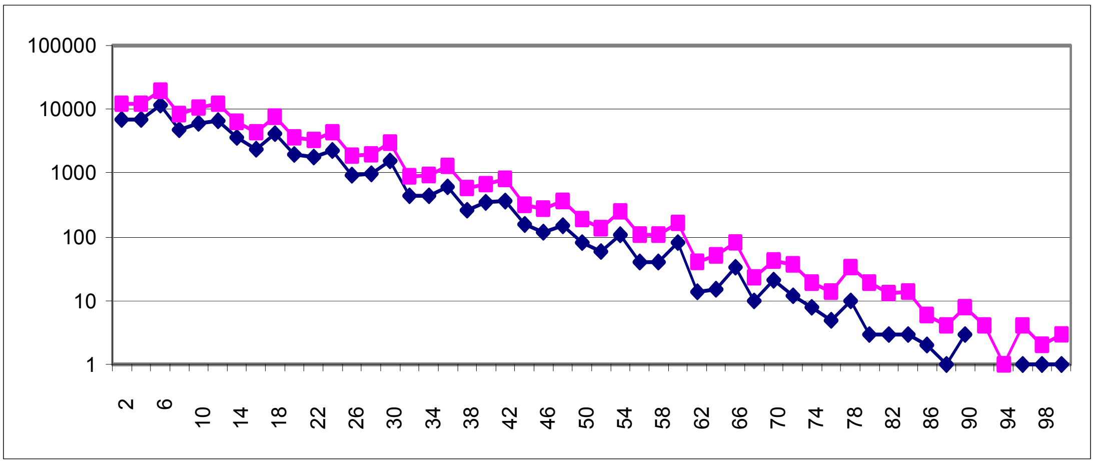


Figure 17 Frequency distribution of spacings between 65636 (bottom) and 118396 (top) consecutive prime numbers. Note the cyclic behaviour every 3 and every 15 points and the close correspondence between the two graphs.

The evidence from these tests suggests the following conjecture: that every even number is spaced between consecutive primes. Therefore every even number can be expressed as the difference of two prime numbers perhaps in an infinite number of ways.

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