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Periodic orbits and zeta functions

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0. Introduction

The study of periodic orbits for dynamical systems dates back to the very origins of the subject. In this survey we shall try to give an overview of some of the main results, without any claims to being exhaustive. For other perspectives, we refer the reader to [**Ba**], [**DP**], [**PP1**] and [**Ru5**]. We begin by mentioning two results which illustrate an important theme in the subject.

Given a diffeomorphism $f : M \to M$ of a compact manifold it is a classical problem to count the number of periodic orbits $\{x, fx, \dots, f^{n-1}x\}$, with $f^n x = x$. For each $n \geq 1$, we denote by $N_f(n)$ the number of periodic points of period n.

An important approach to studying the number $N_f(n)$ of isolated periodic points was proposed by Artin and Mazur [**AM**]. They defined a function of a single complex variable z as follows.

DEFINITION. The *Artin-Mazur zeta function* is defined by

$$\zeta_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} N_f(n)\right). \tag{0.1}$$

Although, a priori, there is no guarantee that this series converges, Artin and Mazur established the following elegant result.

THEOREM 0.1. *(Artin-Mazur) For a C^1 dense set of diffeomorphisms $f : M \to M$ the associated zeta function $\zeta_f(z)$ has a non-zero radius of convergence, and therefore converges to an analytic function for $|z|$ sufficiently small.*

A fundamental problem in the study of zeta functions is to understand the analytic domain of such functions, or more generally, the meromorphic domain (i.e., where we allow isolated poles of finite order). In many interesting cases, these domains extend far beyond the initial domain of convergence of the series. The interest in these results goes beyond their natural aestetic beauty. They also lead (perhaps slightly surprisingly) to a deeper understanding of the dynamics of the underlying system.

An important class of C^1 diffeomorphisms for which the zeta function $\zeta_f(z)$ is better understood are *Axiom A diffeomorphisms.* We recall from the survey of Hasselblatt [**Hs**] that a diffeomorphism $f : M \to M$ is said to satisfy *Axiom A* if the non-wandering set Ω has the following properties:

1. The periodic points are dense in Ω,
2. The restriction $f : \Omega \to \Omega$ is hyperbolic (i.e. there exists a Df-invariant splitting $T_\Omega M = E^s \oplus E^u$ and $C > 0$, $0 < \lambda < 1$ such that $||Df^n|E^s||$, $||Df^{-n}|E^u|| \leq C\lambda^n$, for $n \geq 0$)

For Axiom A diffeomorphisms we have the following remarkable result.

THEOREM 0.2. *(Manning) For a C^1 Axiom A diffeomorphism $f : M \to M$ the associated zeta function $\zeta_f(z)$ has a meromorophic extension to the entire complex plane. Moreover, it takes the form of a rational function.*

To conclude the introduction, let us compare these results with those for a well known zeta function from algebraic topology. Using algebraic topology we can estimate the number of isolated periodic points $f^n x = x$. Let $L(f)$ denote the Lefschetz number of f defined in terms of the induced actions $f_{i,*} : H_i(M, \mathbb{Z}) \to H_i(M, \mathbb{Z})$ on homology by

$$L(f) = \sum_{i=0}^{\dim M} (-1)^i \mathrm{trace}(f_{i,*}).$$

Correspondingly, let $L(f^n)$ denote the Lefschetz number of the n-th iterate $f^n : M \to M$. The Lefschetz fixed point theorem gives that

$$L(f^n) = \sum_{f^n x = x} L_x(f^n)$$

where $L_x(f^n) = \pm 1$ is the Lefschetz index of the isolated periodic point $f^n x = x$ [**Br**] (i.e., $L(f^n)$ counts the number of periodic points, but with an appropriate sign reflecting how the map f^n behaves in a neighbourhood of its fixed point x).

DEFINITION. We define the *Lefschetz zeta function* by

$$\zeta_L(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\left(\sum_{i=0}^{\dim M} (-1)^i L(f^n)\right)\right)$$

From the definition of the Lefschetz numbers $L(f^n)$ we see that the zeta function $\zeta_L(z)$ is rational:

$$\begin{aligned}\zeta_L(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\left(\sum_{i=0}^{\dim M} (-1)^i \mathrm{trace}(f_{i,*}^n)\right)\right)\\ &= \frac{\prod_{i\mathrm{even}} \det(1 - zf_{i,*})}{\prod_{i\mathrm{odd}} \det(1 - zf_{i,*})}\end{aligned} \tag{0.2}$$

Of course, in the particular case that $L_x(f^n) = 1$ we immediately see that $L(f^n) = N(n)$ and so $\zeta_L(z) = \zeta_f(z)$, and conclude that $\zeta_f(z)$ is a rational map. More generally, the proof of Theorem 0.2 needs another approach.

1. Twisted orbits and zeta functions

We begin with a particularly simple, yet interesting, dynamical variant on the study of the Lefschetz zeta function. This also serves to illustrate an important theme in the study of dynamical zeta functions, namely that local data (e.g., periodic orbits) can have a bearing on the global properties of the transformation f. In particular, this involves the idea of characterizing closed orbits as either twisted or untwisted. The Lefschetz zeta function then reflects the balance between these two types of orbits.

More precisely, assume that $f : M \to M$ is an Anosov diffeomorphism (or more generally any diffeomorphism for which all periodic points $f^n x = x$ are isolated and there is a splitting $D_x f^n = E_x^u \oplus E_x^s$, where E^u and E^s correspond to the expanding and contracting eigenvalues).

DEFINITIONS. We say that the periodic orbit $\tau = \{fx, fx, \dots, f^{n-1}x\}$, where $f^n x = x$, is *twisted* if $D_x f^n : E^u \to E^u$ reverses the orientation. Correspondingly, we say that the periodic orbit τ is *untwisted* if $D_x f^n : E^u \to E^u$ preserves the orientation.

If the periodic point $f^n x = x$ is twisted we shall write $\Delta(\tau) = -1$, and if the point is untwisted then we write $\Delta(\tau) = 1$. For each orbit $\tau = \{x, fx, \dots, f^{n-1}x\}$, with $f^n x = x$, we denote the least period $|\tau| = n$. The following result simply gives the Euler product form of the Lefschetz zeta function, involving these notations (see [**Fr2**] for details).

LEMMA 1.1. *The Lefschetz zeta function can be written*

$$\zeta_L(z) = \prod_{\tau} \left(1 - \Delta(\tau) z^{|\tau|}\right)^{(-1)^{\dim E^u}} \tag{1.1}$$

(i.e. the exponenent is -1 *if dim*E^u *is odd and* 1 *if dim*E^u *is even).*

The Euler product presentation of the rational function $\zeta_L(z)$ has rather interesting implications. These are a consequence of the following abstract uniqueness result for general presentations of this form.

LEMMA 1.2. *Any formal power series $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ has a unique presentation:*

$$g(z) = \prod_{n=1}^{\infty} (1 - \Delta_n z^n)^{a_n} \tag{1.2}$$

where $a_n \geq 0$ and $\Delta = \pm 1$.

For many transformations $f : M \to M$ we can explicity compute the Lefschetz zeta function $\zeta_L(z)$ using the identity (0.2). For example, if $M = D^n$ is the n-disk then since $H_i(D^n, \mathbb{R}) = 0$, for $1 \leq i \leq n$, we see that we always have $\zeta_L(z) = \frac{1}{1-z}$.

Using the explicit form of the function $\zeta_L(z)$, and rewriting (1.1) in the terms of products of the form of (1.2), one can deduce the following result of Franks [**Fr2, Theorem A**].

THEOREM 1.1. *Let $f : M \to M$ be an Axiom A diffeomorphism.*

(i) *Assume there exists an odd $d \geq 1$ such that every periodic orbit τ of period $|\tau| = d$ is twisted, i.e., $\Delta(\tau) = -1$. Then if any such τ has least period p (which therefore divides d) then there are twisted periodic points of prime period $2^k d$, $k \geq 0$.*

(ii) *Assume there exists an odd $d \geq 1$ such that every periodic orbit τ of period $|\tau| = d$ is untwisted, i.e., $\Delta(\tau) = 1$. Then if any such τ has least period p (which therefore divides d) then there are periodic points of prime period $2^k d$, $k \geq 0$.*

More results of this nature can be found in [**Fr1**]. More generally, mixtures of dynamical and topological methods to prove the existance of periodic points, are important in recent results as new proofs of the Poincaré-Birkhoff theorem [**Fr2**], and the Klingenberg conjecture on closed geodesics on S^2 [**Fr4**]. However, such results lie beyond the scope of this survey, and we refer the reader to [**FM**].

The use of the rationality of the zeta function $\zeta_L(z)$ to deduce the existance of periodic points of different periods has been developed by a number of authors (cf. [**Ll**]). In the case of intervals, these can can be applied to give an interesting complement to the better known existance results of Sharkovski [**Sk**].

2. Axiom A diffeomorphisms

An important class of C^1 diffeomorphisms for which the zeta function $\zeta_f(z)$ is always rational are *Axiom A diffeomorphisms.* These form an open, if not dense, set in the space of C^1 diffeomorphisms of a compact manifolds (cf. the survey of Hasselblatt [**Hs**]). The following important result (previously given as Theorem 0.2 in the introduction) was proved by Manning [**Mn**] in his Ph.D. thesis, following earlier partial results by Guckenheimer [**Gu**].

THEOREM 2.1. *Let $f : M \to M$ be an Axiom A diffeomorphism then the zeta function $\zeta_f(z)$ is always a rational function.*

We shall outline the original proof of Manning in §3. For the remainder of this section we consider the immediate consequences of having at our disposal the rationality of the zeta function.

An interesting feature of the zeta function for Axiom A diffeomorphisms is the interpretation of its zeros and poles. As is well known, the periodic points lie in the non-wandering set $\Omega(f)$ for the diffeomophism, and by the Smale spectral decomposition theorem we can write this $\Omega(f) = \Lambda_1 \cup \ldots \cup \Lambda_n$, where the Λ_i are locally maximal closed disjoint f-invariant sets, and the restriction $f : \Lambda_i \to \Lambda_i$ is transitive. Those sets Λ_i which consist of more that a single closed orbit are called basic sets. Many dynamical properties, including rationality of the zeta function, are usually easier to establish first for the restriction to the basic sets.

Consider the restriction $f : \Lambda \to \Lambda$ of the Axiom A diffeomorphism to a basic set Λ. If $h(f|\Lambda) > 0$ denotes the topological entropy of $f : \Lambda \to \Lambda$ then the estimate

$$h(f|\Lambda) = \limsup_{n\to+\infty} \frac{1}{n} \log N_{f|\Lambda}(n)$$

implies that the zeta function $\zeta_f(z)$ converges to a non-zero analytic function for $|z| < e^{-h}$. Moreover, e^{-h} is a simple pole for the meromorphic extension of $\zeta_f(z)$.

If we assume that the rational meromorphic extension $\zeta_f(z)$ has zeros α_i (with multiplicity $n_i \geq 1$), for $i = 1, \ldots, k$, and poles β_i (with multiplicity $m_j \geq 1$), for $j = 1, \ldots, l$, then we can write

$$\zeta_f(z) = C \frac{\prod_{i=1}^k (\alpha_i - z)^{n_i}}{\prod_{j=1}^l (\beta_i - z)^{m_j}}, \tag{2.1}$$

for some constant $C \neq 0$. Taking the logarithic derivative of both the presentations (0.1) and (2.1) of the zeta function gives that

$$\begin{aligned}\sum_{n=0}^{\infty} z^n N_{f|\Lambda}(n+1) &= \sum_{i=1}^{k} \frac{n_i}{(\alpha_i - z)} - \sum_{j=1}^{l} \frac{m_i}{(\beta_i - z)} \\ &= \sum_{n=0}^{\infty} z^n \left(\sum_{i=1}^{k} \frac{n_i}{\alpha_i^{n+1}} - \sum_{j=1}^{l} \frac{m_i}{\beta_j^{n+1}} \right)\end{aligned}$$

and comparing coefficients gives explicit formulae for the number of periodic points of the form

$$N_{f|\Lambda}(n) = \sum_{i=1}^{k} \frac{n_i}{\alpha_i^n} - \sum_{j=1}^{l} \frac{m_j}{\beta_j^n}.$$

3. Symbolic dynamics and rationality

In this section we shall sketch the basic ideas in the proof of the rationality of the zeta function $\zeta_f(z)$ for Axiom A diffeomorphisms in Theorem 2.1.

3.1 Zeta functions and subshifts of finite type. As a starting point, we need to consider a simple symbolic model. Let

$$X_A = \left\{ \underline{x} = (x_n)_{n=-\infty}^{\infty} \in \prod_{k=-\infty}^{\infty} \{1, \ldots, k\} : A(x_n, x_{n+1}) = 1 \right\},$$

where A is a $k \times k$ irreducible matrix with entries either 0 or 1. The terms $1, \ldots, k$ are called symbols. Consider the subshift of finite type $\sigma : X_A \to X_A$ defined by $(\sigma \underline{x})_n = x_{n+1}$, for $n \in \mathbb{Z}$. The number of fixed points for σ is clearly given by the trace $N(1) = \mathrm{tr}(A)$.

Similarly the number of periodic points for σ of period n is given by $N(n) = \mathrm{tr}(A^n)$, where $A^n = \underbrace{A \cdots A}_{\times n}$ denotes n-fold matrix multiplication. In particular, we see

$$\zeta_\sigma(z) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\mathrm{tr}(A^n)\right) = \frac{1}{\det(I - zA)}.$$

In particular, we have that the zeta function $\zeta_\sigma(z)$ associated to a subshift of finite type is the reciprocal of a polynomial, as was originally observed by Bowen and Lanford [**BL**]. Moreover, the poles of $\zeta_\sigma(z)$ can be identified with the reciprocals $1/\lambda$ of the eigenvalues λ of the matrix A.

Aside. Whilest on the topic of subshifts of finite type, it is interesting to note that the zeta function is a useful invariant in the classification of subshifts of finite type. The most familiar type of classification is that of *conjugacy* between two subshifts of finite type $\sigma_1 : X_{A_1} \to X_{A_1}$ and $\sigma_2 : X_{A_2} \to X_{A_2}$, in which there is a homeomorphism $\psi : X_{A_1} \to X_{A_2}$ such that $\sigma_2 \circ \psi = \psi \circ \sigma_1$. Since there is a bijection between closed orbits, the zeta function is a complete invariant, i.e., $\zeta_{\sigma_1}(z) = \zeta_{\sigma_2}(z)$. In particular, since the zeros and poles of the zeta function coincide we see that this subsumes the familiar identity $h(\sigma_1) = h(\sigma_2)$.

Another weaker form of classification is that of *flow equivalence* for which an invariant is the special value of the zeta function $\zeta(1) = 1/\det(I - A)$, also known as the Parry-Sullivan invariant [**PaS**]. This was subsequently shown to be one of a pair of complete invariants for flow equivalence, by Franks [**Fr5**].

3.2 Markov partitions. Manning showed how the rationality of the zeta function associated to subshifts of finite type could be carried over to hyperbolic diffeomorphisms [**Mn**]. The key idea is to develop the familiar symbolic model of Axiom A diffeomorphisms, using Markov partitions, to accurately account for all periodic orbits.

From the definition of the zeta function we immediately see that the zeta function $\zeta_{f|\Omega}(z)$ is related to the zeta functions $\zeta_{f|\Lambda_i}(z)$ for the restrictions $f : \Lambda_i \to \Lambda_i$ to the basic sets by $\zeta_{f|\Omega}(z) = \prod_{i=1}^n \zeta_{f|\Lambda_i}(z)$. Thus it suffices to show that the Axiom A diffeomorphism restricted to a basic set $f : \Lambda \to \Lambda$ has a rational zeta function $\zeta_{f|\Lambda}(z)$. Assume that we have a Markov Partition $\mathcal{T} = \{T_1, \ldots, T_k\}$ for Λ, with a $k \times k$ transition matrix A defined by

$$A(i,j) = \begin{cases} 1 & \text{if } f(\mathrm{int}T_i) \cap \mathrm{int}T_j \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

(We refer the reader to the survey of Chernov for a detailed discussion of Markov partitions [**Ch**].) Using the transition matrix A we can naturally associate a subshift of finite type $\sigma : X_A \to X_A$ and a semi-conjugacy $\pi : X_A \to \Sigma$ defined by $\pi(\underline{x}) = \cap_{n\in\mathbb{Z}} f^{-n} T_{x_n}$ (i.e., $\pi \circ \sigma = f \circ \pi$). Unfortunately, there is not quite a one-to-one correspondence between the closed orbits of $f : \Lambda \to \Lambda$ and $\sigma : X_A \to X_A$, which would have allowed us to equate their zeta functions. This discrepancy is caused by the complication of closed orbits for f which are contained in the boundaries of elements of the Markov Partition $\mathcal{T}$.

To illustrate this difficulty, consider a periodic point $f^m x = x \in T_i \cap T_j$, say. We can associate two sequences $\underline{x} = (x_n)_{n=-\infty}^{\infty}$ and $\underline{x}' = (x'_n)_{n=-\infty}^{\infty}$ from X_A, with $x_0 = i$ and

$x_0 = j$, which both correspond to the point x (i.e., $f^n(x) \in T_{x_n}$ and $f^n(x) \in T_{x'_n}$, for $n \in \mathbb{Z}$). Consider the following two possibilities:

(a) $x_m = i$ and $x'_m = j$, in which case we have two distinct periodic sequences from X_A of period m, corresponding to the same periodic point x of period m; and

(b) $x_m = j$ and $x'_m = i$, in which case we have a single periodic sequence from X_A of period $2m$, corresponding to the periodic point x of period m.

One can interpret these two possibilities in terms of the behaviour of f^m in a neighbourhood of x. In the first case f^m maps a neighbourhood in T_i to a neighbourhood in T_i (and a neighbourhood in T_j to a neighbourhood in T_j). In the second case, such neighbourhoods are interchanged.

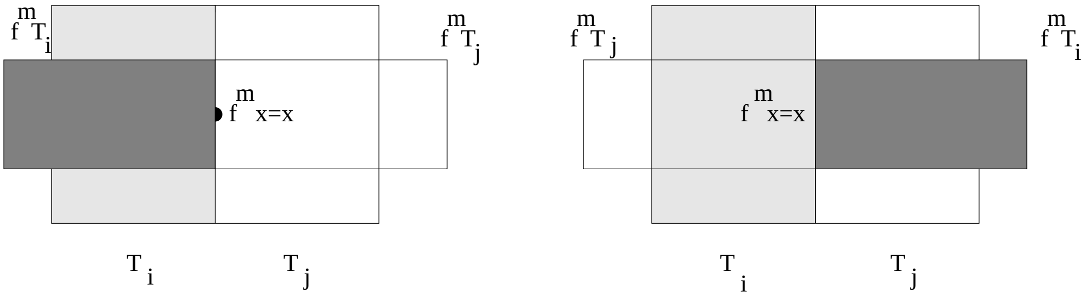

FIGURE 1. Cases (a) and Case (b)

A simple "fix" to account for such orbits would be to introduce a new subshift of finite type whose symbols are unordered pairs $\{i, j\}$ of distinct pairs, for which $T_i \cap T_j \neq \emptyset$. A new transition matrix $A_{[(2)]}$ can be defined by $A_{[(2)]}(\{i, j\}, \{i', j'\}) = 1$ if either $A(i, i') = A(j, j') = 1$ or $A(i, j') = A(j, i') = 1$. Under either of the scenarios (a) and (b) above we have for such the periodic points $f^m x = x$ then there is a corresponding periodic point for $\sigma : X_{A_{[(2)]}} \to X_{A_{[(2)]}}$ of period precisely m.

The proof of Theorem 2.1 involves a slightly more elaborate development of this basic idea, which we describe below.

3.3 Auxiliary subshifts and rationality. More generally, we have to consider the possibility that $f^m x = x \in T_{i_1} \cap \ldots \cap T_{i_N}$, where $1 \leq i_1 < \ldots < i_N \leq k$, where we assume that N is the maximal number of rectangles in which x lies. To each of the rectangles T_{i_j}, $j = 1, \ldots, N$ we can associate a sequence $\underline{x}^{(j)} = (x_n^{(j)}) \in X_A$ with $x_n^{(j)} = i_j$. The effect of f^m in a neigbourhood of the periodic x is to "permute" the rectangles $T_{i_1}, \ldots, T_{i_N}$ of the partition, giving rise to complicated families of periodic sequences in X_A. (More precisely, at the symbolic level we have that $\{x_0^{(1)}, \ldots, x_0^{(N)}\} = \{x_m^{(1)}, \ldots, x_m^{(N)}\}$.)

The remedy is to introduce *auxiliary transition matrices*, relating combinations of the original symbols, to keep track of these additional periodic sequences. For the record,

(a) for each $1 \leq l \leq N \leq k$ we associate a new family of symbols which are simply ordered l-tuples $\underline{i} = (\underline{i}_1, \ldots, \underline{i}_l)$ of (unordered) disjoint sub-collections of symbols from $\{1, \ldots, k\}$. If $\underline{i}_j$ has cardinality n_j, then we can write $\underline{n} = (n_1, \ldots, n_l)$ for the l-tuple of cardinalities and $|\underline{n}| = l$. Moreover, we require that the intersection over all sets T_i indexed by symbols appearing in any term $\underline{i}$ is non-empty, i.e., $\cap_{j=1}^{l} \cap_{i \in \underline{i}_j} T_i \neq \emptyset$;

(b) We associate a new transition matrix $A_{\underline{n}}$ whose rows and columns are indexed by l-tuples of disjoint sub-collections of rectangles (each with the cardinalities determined by the entries from $\underline{n}$). Consider two such l-tuples $\underline{i} = (\underline{i}_1, \ldots, \underline{i}_l)$ and $\underline{i}' = (\underline{i}'_1, \ldots, \underline{i}'_l)$. Assume that there are permutations ρ_j (on n_j-symbols), for each $1 \leq j \leq l$, such that if $\underline{i}_j = \{i_1^{(j)}, \ldots, i_{n_j}^{(j)}\}$ and $\underline{i}'_j = \{i'^{(j)}_1, \ldots, i'^{(j)}_{n_j}\}$ then $A(i_r^{(j)}, i'^{(j)}_{\rho_j(r)}) = 1$. We then we write $A_{\underline{n}}(\underline{i}, \underline{i}') = 1$. Otherwise we write $A_{\underline{n}}(\underline{i}, \underline{i}') = 0$.

These can be used to define new *auxiliary subshifts of finite type*. The role of these auxiliary subshifts of finite type is the following "adjustment lemma".

LEMMA 3.1. *For each $m \geq 1$, we have $N_f(m) = \sum_{\underline{n}} (-1)^{|\underline{n}|+1} tr(A_{|\underline{n}|}^m)$.* (3.1)

The proof of this lemma is essentially a straightforward "book-keeping" exercise. More precisely, consider any l-tuple $\underline{i} = (\underline{i}_1, \ldots, \underline{i}_l)$ such that there is a maximal (disjoint) set of symbols $\widehat{i}$ such that $(\cap_{i \in \widehat{i}} T_i) \cap (\cap_{j=1}^{l} \cap_{i \in \underline{i}_j} T_i) \neq \emptyset$. We can pair the l-tuple $\underline{i}$ with the the $(l+1)$-tuple $\underline{i}' = (\underline{i}_1, \ldots, \underline{i}_l, \widehat{\underline{i}})$. The contribution of these two terms to the sum on the Right Hand Side of (3.1) occurs with with opposite signs. This gives a cancellation, eventually leaving only the contribution of 1 coming from the largest singletons $\underline{i} = (\{i_1, \ldots, i_k\})$.

As a simple example, consider a fixed point $fx = x$ which lies on the boundaries of precisely two rectangles T_i and T_j. In this case, we would pair $\underline{i} = (\{i\})$ (a 1-tuple, consisting of the single symbol i), counted with weight 1, with $\underline{i}' = (\{i\}, \{j\})$ (a 2-tuple of singleton sets), counted with weight -1. Similarly, we would pair $\underline{i} = (\{j\})$ (a 1-tuple, consisting of a single symbol j), counted with weight 1, with $\underline{i}' = (\{j\}, \{i\})$ (a 2-tuple of singleton sets), counted with weight -1. The contributions from each of these two terms cancel, leaving the contribution from $(\{i, j\})$ counted with weight 1.

In particular, we can then write

$$\begin{aligned} \zeta_f(z) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} N_f(m)\right) &= \prod_{\underline{n}} \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} (-1)^{|\underline{n}|+1} \mathrm{tr}(A_{\underline{n}}^m)\right) \\ &= \frac{\prod_{|\underline{n}|=\mathrm{odd}} \det(I - zA_{\underline{n}})}{\prod_{|\underline{n}|=\mathrm{even}} \det(I - zA_{\underline{n}})} \end{aligned} \tag{3.2}$$

which is a rational function, as required. We can also identify $\zeta_\sigma(s) = 1/\det(I - zA_{(1)})$.

REMARK. There is a modified version of this proof using matrices which have entries ± 1 instead of 1 [**Sh**], [**B2**]. A different proof based on the use of the Lefschetz index was given by Fried [**Fd2**].

4. Zeta functions for interval maps

An interesting class of transformations for which the zeta function has been much studied are those of maps $T : I \to I$ of the interval $I = [0, 1]$, say. The class of intervals maps closest to the Axiom A systems we have described above are expanding Markov maps. More precisely, consider a collection of points $0 = z_0 < z_1 < \ldots < z_{k-1} < z_k = 1$ and the corresponding subintervals $I_j = [z_{j-1}, z_j]$, for $j = 1, \ldots, k$. Assume that on each subinterval I_j the restriction of the map $T : I_j \to \mathbb{R}$ is C^1.

The map is $T : I \to I$ is *Markov* if the image $T(I_j)$ of each subinterval I_j is a union of a collection of these subintervals. The map is $T : I \to I$ is *expanding* if $|T'(x)| \geq \alpha > 1$, for some $\alpha > 1$ and all points $x \in I$. For such Markov expanding maps there is a model by a one-sided subshift of finite type $\sigma : X_A^+ \to X_A^+$ on

$$X_A^+ = \left\{ \underline{x} \in \prod_{n=0}^{\infty} \{1, \ldots, k\} : A(x_n, x_{n+1}) = 1, n \geq 0 \right\},$$

as usual defined by $(\sigma \underline{x})_n = x_{n+1}$. The symbol space $\{1, \ldots, k\}$ indexes the subintervals I_i and the transition matrix A is defined by

$$A(i,j) = \begin{cases} 1 & \text{if } T(I_i) \supset I_j \\ 0 & \text{otherwise.} \end{cases}$$

There is a continuous semi-conjugacy $\pi : X_A \to I$ defined by $\pi(\underline{x}) = \cap_{n=0}^{\infty} T^{-n} I_{x_n}$, satisfying $\pi \circ \sigma = f \circ \pi$. In particular, the map π is a bijection, save for a minor correction due to the finitely many periodic points in the collection of boundary points $\{z_0, z_1, \ldots, z_{k-1}, z_k\}$, and the zeta function $\zeta_\sigma(z) = 1/\det(I - zA)$ is essentially the zeta function $\zeta_T(z)$ for the interval map (except for a correction for the finite number of periodic points in the boundary).

REMARK. If we allow f to have discontinuities at the points $\{z_0, z_1, \ldots, z_{k-1}, z_k\}$ then a convenient device is to "duplicate" these endpoints and to replace I by the disjoint union $I_1 \coprod \ldots \coprod I_k$.

An important generalization of this basic construction occurs when we remove the hypothesis that the interval map $f : I \to I$ is Markov. This leads to the more general notion of Markov extensions.

4.1 Markov extensions. The basic construction technical tool of the *Markov extension*, was introduced by Hofbauer, and developed by Hofbauer, Keller and others.

Assume only that $T : I \to I$ is a interval map which is expanding on each of the intervals $I_1, \ldots, I_k$. However, we do not need to assume that T is Markov.

A trivial property of intervals which is important is the following.

LEMMA 4.1. *Let J be a sub interval of one of the intervals $I_1, \ldots, I_k$. The image $T(J)$ will be a union of one of the following forms:*

(a) $T(J) = D_J \subset I_i$;
(b) $T(J) = \cup_{j=s}^{t} I_j$;
(c) $T(J) = D_J^- \cup \left(\cup_{j=s}^{t} I_j\right)$ *or* $T(J) = \left(\cup_{j=s}^{t} I_j\right) \cup D_J^+$
(d) $T(J) = D_J^- \cup \left(\cup_{j=s}^{t} I_j\right) \cup D_J^+$

where D_J, $D_J^- \subset I_{s-1}$ and $D_J^+ \subset I_{t+1}$ are each closed non-trivial and strict sub-intervals.

DEFINITION. We call $\mathcal{E}_0 = \{I_1, \dots, I_k\}$ the *zeroth generation of intervals*

We can apply Lemma 4.1 to each $J = I_i$ $(i = 1, \dots, n)$ to get new sub-intervals. For example, D_{I_i} (when $T(I_i)$ is as in case (a) or (c)) and $D_{I_i}^-, D_{I_i}^+$ (when $T(I_i)$ is as in case (d)). If $T(I_i)$ corresponds to case (b) then no new intervals are created.

DEFINITION. We call $\mathcal{E}_1 = \{D_{I_i}^-, D_{I_i}^+ \text{ or } D_{I_i} : i = 1, \dots, k\}$ the *first generation of intervals.*

We define collections of intervals $\mathcal{E}_n$, $n \geq 1$, (each a closed sub-interval of one of the original intervals $I_1, \dots, I_k$) inductively. Given a collection of intervals $\mathcal{E}_{n-1}$ (each a closed sub-interval of one of the original intervals $I_1, \dots, I_k$) then we can apply Lemma 4.1 to each interval $J \in \mathcal{E}_{n-1}$. We define the *nth generation of intervals* to be the collection of closed intervals $\mathcal{E}_n = \{D_J^-, D_J^+ \text{ or } D_J : J \in \mathcal{E}_{n-1}\}$. Finally, we write $\mathcal{E} = \cup_{n=0}^{\infty} \mathcal{E}_n$.

If $J \in \mathcal{E}_n$, say, then each of the sets J' in the appropriate union for $T(J)$ (in Lemma 4.1) is called a *successor* to J. We write $J \to J'$.

The essence of this construction is to compensate for the absence of the Markov property by introducing more and more intervals so that the image of any $J \in \mathcal{E}$ is again the union of other intervals from $\mathcal{E}$. The complications over and above those in the Markov case are that the larger family $\mathcal{E}$ will not in general be finite, and intervals in $\mathcal{E}$ will not in general have disjoint interiors.

The following simple example illustrates this construction.

EXAMPLE (β-TRANSFORMATION). Let $I = [0, 1]$ and choose $1 < \beta < 2$. We can define an interval map $T : I \to I$ by

$$Tx = \begin{cases} \beta x & \text{if } 0 \leq x \leq \frac{1}{\beta} \\ \beta x - 1 & \text{if } \frac{1}{\beta} < x \leq 1 \end{cases}$$

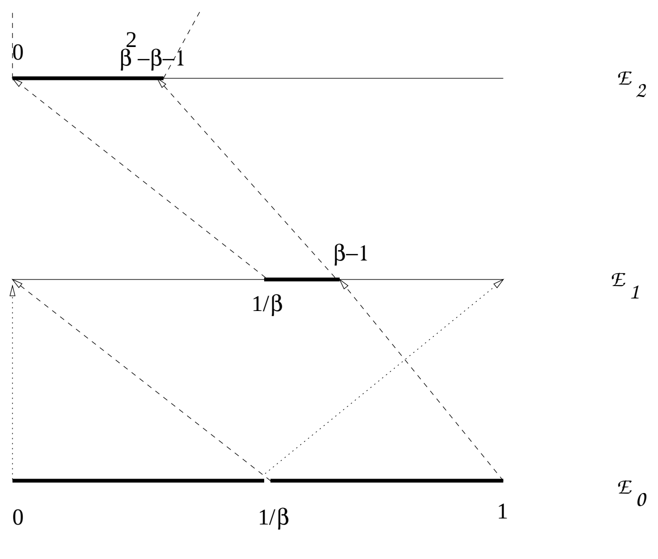

FIGURE 2. Generating new intervals for the Markov extension

In this case, we have that $\mathcal{E}_0 = \{[0, \frac{1}{\beta}], [\frac{1}{\beta}, 1]\}$. Since $\beta < 2$ we have that

$$T\left(\left[0, \frac{1}{\beta}\right]\right) = \left[0, \frac{1}{\beta}\right] \cup \left[\frac{1}{\beta}, 1\right] \text{ and } T\left(\left[\frac{1}{\beta}, 1\right]\right) = \left[0, \frac{1}{\beta}\right] \cup \left[\frac{1}{\beta}, \beta - 1\right].$$

Thus $\mathcal{E}_1 = \left\{\left[\frac{1}{\beta}, 1-\beta\right]\right\}$. If $\beta < (\sqrt{5}-1)/2$,say, then $T([\frac{1}{\beta}, \beta-1]) = [0, \beta^2-\beta-1] \subset [0, \frac{1}{\beta}]$ and so $\mathcal{E}_2 = \{[0, \beta^2 - \beta - 1]\}$.

For the purposes of exposition we want to make the following simplifying assumption. We shall assume that if $T^n(x_i) = T^m(x_j)$ then $x_i = x_j$ and $n = m$. This holds, for example, for the β-transformation when $\beta > 1$ is not an algebraic number.

Let $(J_n)_{n\in\mathbb{Z}^+}$ be a sequence of successor intervals (i.e., $J_0 \to J_1 \to J_2 \to \dots$) then we can associate a unique element $\pi\left((J_n)_{n\in\mathbb{Z}^+}\right) \in I$ by

$$\pi\left((J_n)_{n\in\mathbb{Z}^+}\right) = \cap_{n\in\mathbb{Z}^+} T^{-n} J_n$$

(in complete analogy with the Markov case).

There are very strong restrictions on the possible successors to a given interval in $\mathcal{E}_n$. Although $x \in I$ will not in general have a unique sequence $(J_n)_{n\in\mathbb{Z}^+}$ such that $\pi\left((J_n)_{n\in\mathbb{Z}^+}\right) = x$, the next lemma shows the remarkable fact that (generically) the "tail" of such a sequence is uniquely determined [**Ho**], [**HK**].

LEMMA 4.2. *Let $(J_n)_{n\in\mathbb{Z}^+}$ and $(J'_n)_{n\in\mathbb{Z}^+}$ be two sequences of successors such that $\pi\left((J_n)_{n\in\mathbb{Z}^+}\right) = \pi\left((J_n)_{n\in\mathbb{Z}^+}\right) = x \in I$. If $\forall m \geq 0$, $T^m(x) \notin \cup_{n\geq 0} T^n\{x_1, \dots, x_k\}$ then $\exists N \geq 0$ with $J_n = J'_n$ for $n \geq N$.*

If $T: I \to I$ is *not* Markov then $\mathcal{E} = \cup_{n=0}^{\infty}\mathcal{E}_n$ is infinite (by the simplifying assumption). We can define an *infinite* matrix A with entries 0 or 1, whose rows and columns are indexed by $\mathcal{E}$, in the following way

DEFINITION. For $J_1, J_2 \in \mathcal{E}$ we define

$$\mathcal{A}(J_1, J_2) = \begin{cases} 1 & \text{if } J_2 \to J_1 \\ 0 & \text{otherwise.} \end{cases}$$

The next obvious lemma gives us estimates on the number of periodic orbits for $T: I \to I$ in terms of $\mathcal{A}$.

LEMMA 4.3. *The number of periodic points $T^n x = x$ of period n is given by*

$$\#Fix(T^n) := \sum_{J\in\mathcal{E}} \mathcal{A}^n(J, J).$$

This is completely analogous to the Markov case, except that the summation is finite.

The idea is to study the number of periodic points $Card(Fix(T^n))$ via the matrix $\mathcal{A}$. Rather than try to analyze infinite matrices $\mathcal{A}$ it is simpler to fix some large $k \geq 1$ and consider the truncated (finite) matrix $\mathcal{A}_k$ defined by $\mathcal{A}_k(J, J') = \mathcal{A}(J, J')$ if $J, J' \in \cup_{n=0}^{k}\mathcal{E}_n$.

We can deduce that every string of successors corresponding to a fixed point for $T^n : I \to I$ must be of length n. Lemma 4.2 implies that, except for closed orbits passing through ends of intervals, we can always associated a unique periodic string. Since we can always choose a string starting in $\mathcal{E}_0$ then the unique string must contain intervals from $\mathcal{E}_0$. Finally, since the orbit has length at most $2n$ all the elements of the string must therefore be in $\cup_{i=1}^n \mathcal{E}_i$. Thus by Lemma 4.3 we see that

$$\#Fix(T^n) = \sum_{J\in\mathcal{E}} \mathcal{A}^n(J,J) = \sum_{J\in\mathcal{E}_k} \mathcal{A}_k^n(J,J) = \mathrm{tr}(\mathcal{A}_k^n)$$

whenever $k \geq n$.

Since $\mathcal{A}_k$ is a finite matrix, we have the standard matrix identity

$$Det(I - z\mathcal{A}_k) = exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m}\mathrm{tr}(\mathcal{A}_k^m)\right) \tag{4.1}$$

The Right Hand-Side carries information on the traces $\mathrm{tr}(\mathcal{A}_k^m)$. To study the Left Hand Side of (4.1), it helps to fix $1 \leq N \leq k$ and write $\mathcal{A}_k = \begin{pmatrix} A_N & B_N \\ C_N & D_N \end{pmatrix}$ where A_N is the square submatrix with indices $\cup_{i=0}^N \mathcal{E}_i$, and D_N is the square sub-matrix with indices $\cup_{i=N+1}^k \mathcal{E}_i$, etc. What makes the approximation by finite matrices useful is the following estimate, which has its origins in Lemma 4.1.

Lemma 4.4. *For any $k \geq N \geq 1$, the sub-matrix D_N has all of its eigenvalues with modulus at most $N^{1/N}$*

Since $D_N - zI$ is now known to be invertible whenever $|z| \geq N^{\frac{1}{N}}$ we can use the matrix identity

$$\begin{pmatrix} A_N - zI & B_N \\ C_N & D_N - zI \end{pmatrix} = \begin{pmatrix} (A_N - zI) - B_N(D_N - zI)^{-1}C & B(D_N - zI)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & D - zI \end{pmatrix}$$

and taking determinants gives that

$$\det(\mathcal{A}_N - zI) = \det\left((A_N - z) - B_N(D_N - z)^{-1}CB(D_N - z)^{-1}\right)\det\left(D_N - z\right).$$

Letting $k, N \to +\infty$, ultimately gives the following result.

Theorem 4.1. *Let T be a piecewise expanding interval map. The zeta function $\zeta_T(z)$ has a meromorphic extension to the disk $|z| < 1$.*

This result shows that the number of periodic points of period $N_T(n)$ is close to being determined by an explicit formula.

Corollary. *For any $\beta > 1$ there exist constants*
(i) $\lambda_1, \ldots, \lambda_M$, *with* $|\lambda_i| > \beta$ $(i = 1, \ldots, M)$, *and*
(ii) $C_1, \ldots, C_M \in \mathbb{C}$
such that

$$N_f(n) = \sum_{i=1}^{M} C_i \lambda_i^n + O(\beta^n).$$

The corollary follows from the theorem by taking λ_i, $i = 1, \ldots, M$, to be the eigenvalues of $\mathcal{A}_N$ of modulus at most $\beta > 1$, where N is chosen sufficiently large that $N^{1/n} < \beta$.

REMARK. There are echoes of this simple construction in the more recent work of Young on "Markov towers" which work, in particular, for a large class of invertible systems [**Yo**].

4.2 Unimodal maps and the Milnor-Thurston zeta function. In the case of unimodal maps of the interval, there is a very elegant approach due to Milnor and Thurston to studying a variant of the zeta function. We begin by recalling the class of interval maps to which the method applies.

DEFINITION. We call a continuous map $T : I \to I$ on the unit interval *unimodal* if $T(0) = 1 = T(1)$ and $\exists 0 < c < 1$ such that $T : [0, c] \to I$ is increasing and $T : [c, 1] \to I$ is decreasing. We call c the critical point.

The critical point plays an important role in understanding.

DEFINITION. For $x \in I$ we define the *itinerary* of this point to be the sequence $(x_n)_{n=0}^{\infty}$ where we write

(i) $x_n = L$ if $T^n(x) \in [0, c)$
(ii) $x_n = R$ if $T^n(x) \in (c, 1]$
(iii) $x_n = c$ if $T^n(x) = c$

Let $\Sigma = \{(x_n)_{n=0}^{\infty} : x \in I\}$. We can define a shift map $\sigma : \Sigma \to \Sigma$ by $\sigma(x_n)_{n=0}^{\infty} = (x_{n+1})_{n=0}^{\infty}$.

LEMMA 4.5. *If x has itinerary $(x_n)_{n=0}^{\infty}$ then $T(x)$ has itinerary $\sigma(x_n)_{n=0}^{\infty}$.*

There is a close connection between periodic points and periodic itineraries.

LEMMA 4.6. *If a point x is periodic under T (i.e. $\exists m \geq 1$ with $T^m(x) = x$) then its itinerary $(x_n)_{n=0}^{\infty}$ is periodic under σ (i.e. $\exists m \neq 1$ with $\sigma^m(x) = T^m(x)$).*

Conversely, assume that an itinerary $(x_n)_{n=0}^{\infty}$ is periodic under σ then these exists a periodic point w under T with this itinerary.

Rather than studying the Artin Mazur zeta function, it is more appropriate to consider a different zeta function which reflects the behaviour of the transformation T^m in a neighbourhood of the fixed points $T^m x = x$

DEFINITION. We denote the the number of periodic points $T^m x = x$, lying in neighbourhoods on which T^m is decreasing, by

$$N^-(T^m) = \#\{T^m x = x : T^m \text{ is decreasing in a neighbourhood of } x\}$$

and define the *Milnor-Thurston modified zeta function* by

$$\zeta^-(z) = \exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m} 2N^-(T^m)\right)$$

An important result on such zeta functions is the due to Milnor-Thurston [**MT**].

THEOREM 4,2 (MILNOR-THURSTON). *The zeta function takes the form* $\zeta^-(z) = 1/\det(1-D(z))$, *where* $D(z)$ *is an explicit square matrix ("kneading matrix") whose entries are explicit formal power series in* z.

We refer the reader to [**MT**] for a far more detailed account of this theory.

5. The Ruelle zeta functions

5.1 The analytic domain. There are natural generalizations of the Artin-Mazur zeta function which allow for the possibility of weighting a periodic orbit $\{x, fx, \ldots, f^{n-1}x\}$ by a (typically Hölder continuous) function $g : \Lambda \to \mathbb{R}$. Thus, in addition to the period n of the orbit we also have a useful weighting given by $g^n(x) := g(x) + g(fx) + \ldots + g(f^{n-1}x)$. The study of such zeta functions was developed by Ruelle.

DEFINITION. We define the *Ruelle zeta function* for an Axiom A diffeomorphism $f : \Lambda \to \Lambda$, on a basic set Λ, and a Hölder continuous function $g : \Lambda \to \mathbb{R}$ by

$$\zeta_f(z, g) = \exp\left(-\sum_{n=1}^{\infty} \frac{z}{n} \sum_{f^n x = x} e^{g^n(x)}\right) \tag{5.1}$$

This complex function converges to an analytic function on the disc $|z| < e^{-P(g)}$, where

$$P(g) = \limsup_{n \to +\infty} \frac{1}{n} \log\left(\sum_{f^n x = x} e^{g^n(x)}\right)$$

denotes the pressure of the real valued function $g : \Lambda \to \mathbb{R}$ (cf. [**Ch**]).

The main result on the domain of $\zeta_f(z, g)$ is the following (cf. [**Ru1**], [**Po1**], [**Ha**]).

PROPOSITION 5.1. *The zeta function* $\zeta_f(z, g)$ *has a non-zero meromorphic extension to a disk* $|z| \leq \theta e^{-P(g)}$, *for some* $0 < \theta < 1$, *with a simple pole at* $z = e^{-P(g)}$. *If* $f : \Lambda \to \Lambda$ *is topologically mixing then there are no other poles of modulus* $e^{-P(g)}$.

The value of $0 < \theta < 1$ depends on the Hölder exponent of g and the hyperbolicity of $f : \Lambda \to \Lambda$.

In the special case $g = 0$ the Ruelle zeta function $\zeta_T(z, g)$ simply reduces to the Artin-Mazur zeta function, and the pole occurs at $e^{-P(0)} = e^{-h}$. Of course, for questions of analyticity it is natural to consider the more general case of *complex valued* Hölder continuous functions $g : \Lambda \to \mathbb{C}$. As before we can write $g^n(x) := g(x) + g(fx) + \ldots + g(f^{n-1}x)$ and define the generalized Ruelle zeta function again by (5.1).

We can write $g = Re(g) + iIm(g)$ in terms of its real and imaginary parts and then the zeta function $\zeta_f(z, g)$ converges to an analytic function on the disc $|z| < e^{P(Re(g))}$. The corresponding result to Proposition 5.1 on the domain of $\zeta_f(z, g)$ is the following.

PROPOSITION 5.2. *The zeta function* $\zeta_f(z, g)$ *has a non-zero meromorphic extension to a disk* $|z| \leq \theta e^{-P(Re(g))}$, *for some* $0 < \theta < 1$.

The proof of these results uses symbolic dynamics. As described in §3, we can associate to the Axiom A map $f : \Lambda \to \Lambda$, restricted to a basic set, a subshift of finite type

$\sigma : X_A \to X_A$ and a Hölder semi-conjugacy $\pi : X_A \to \Lambda$ [**Ch**]. We can associate to the function g a (Hölder) continuous function $\overline{g} = g \circ \pi : X_A \to \mathbb{R}$. The definition of the zeta function $\zeta_\sigma(z, \overline{g})$ for σ and $\overline{g}$ is exactly as in (5.1).

The construction of Manning described in §3 naturally generalizes to give

$$\zeta_f(z, g) = \prod_{\underline{n}} \zeta_{\sigma_{\underline{n}}}(z, \overline{g}_{\underline{n}})^{(-1)^{|\underline{n}|}} \tag{5.2}$$

where $\overline{g}_{\underline{n}} : X_{A_{\underline{n}}} \to \mathbb{C}$ and $\sigma_{\underline{n}} : X_{A_{\underline{n}}} \to X_{A_{\underline{n}}}$ are interpreted in an obvious sense, and $\zeta_\sigma(z, \overline{g}) = \zeta_{\sigma_{(1)}}(z, \overline{g}_{(1)})$. In particular, Propositions 5.1 and 5.2 follow from the corresponding results for subshifts of finite type, whose proofs we shall now briefly outline. A key ingredient is the Ruelle transfer operator. This is a bounded linear operator which can usefully be viewed as a natural generalization of the transition matrix A. However, we need to replace the *two sided* subshift of finite $\sigma : X_A \to X_A$ by the corresponding *one sided* subshift of finite $\sigma : X_A^+ \to X_A^+$ on the space

$$X_A^+ = \{\underline{x} = (x_n)_{n=0}^\infty : A(x_n, x_{n+1}) = 1, n \geq 0\},$$

defined, as before, by $(\sigma\underline{x})_n = x_{n+1}$. The essential difference is that the one sided subshift is no longer a homeomorphism, and points typically have more than one pre-image.

We can construct a suitable Banach space as follows. Given an appropriate value $0 < \theta < 1$, we define for $h : X_A^+ \to \mathbb{C}$ a semi-norm

$$|h|_\theta = \sup \left\{ \frac{|h(\underline{x}) - h(\underline{y})|}{\theta^n} : x_i = y_i \text{ for } i = 0, \ldots, n-1 \right\}$$

Let $\mathcal{F}_\theta = \{h : X_A^+ \to \mathbb{C} : |h|_\theta < +\infty\}$, then this is a Banach space with norm $||h|| = |h|_\theta + |h|_\infty$ (where $|h|_\infty$ denotes the usual supremum norm).

Given $g_0 \in \mathcal{F}_\theta$, we define the *(Ruelle) transfer operator*

$$\begin{cases} \mathcal{L}_{g_0} : \mathcal{F}_\theta \to \mathcal{F}_\theta \\ \mathcal{L}_{g_0} w(\underline{x}) = \sum_{\sigma\underline{y} = \underline{x}} e^{g_0(\underline{y})} w(\underline{y}) \end{cases}$$

(i.e., we average the function over pre-images by the one sided shift $\sigma : X_A^+ \to X_A^+$ with a weighting given by e^g)

We call that the essential spectrum of a bounded linear operator consists of that part of the spectrum that remains once isolated eigenvalues (with eigenprojections of finite dimensional kernel and range [**PP1**]) The main spectral properties of the operators $\mathcal{L}_{g_0}$ are the following.

PROPOSITION 5.3.
(i) *The spectrum of* $\mathcal{L}_{g_0} : \mathcal{F}_\theta \to \mathcal{F}_\theta$ *is contained in the disk* $|z| \leq e^{P(Re(g_0))}$.
(ii) *The essential spectrum of the operator is contained in the disk* $|z| \leq \theta e^{P(Re(g_0))}$.
(iii) *If* g_0 *is real valued then* $\mathcal{L}_{g_0}$ *has* $e^{P(g_0)}$ *as a simple isolated eigenvalue. Moreover, the associated eigenfunction* h *satisfying* $\mathcal{L}_{g_0} h = e^{P(g_0)} h$ *is strictly positive*

The last part of this proposition is usually called the *Ruelle operator Theorem* [**Ru3**].

We can define a zeta function $\zeta_\sigma(z, g_0)$ by analogy with (5.1). A direct link between $\zeta_\sigma(z, g_0)$ and the Ruelle transfer operators is the following.

LEMMA 5.1. *Let $\underline{x}_i \in X_A^+$ with $\underline{x}_i = i$, for $1 \leq i \leq k$. We have the identity*

$$\sum_{\sigma^n \underline{x}=\underline{x}} e^{g_0^n(\underline{x})} = \sum_{i=1}^{k} \mathcal{L}_{g_0}^n \chi_{[i]}(\underline{x}_i) + O(\theta^n). \tag{5.3}$$

In particular, the zeta function $\zeta_\sigma(z, g_0)$ has a singular value $z = z_i$ in its meromorphic extension to a disk $|z| \leq e^{-P(Re(g_0))}/\theta$ if $\lambda_i = 1/z_i$ is an eigenvalue for $\mathcal{L}_{g_0}$.

The last part of the Lemma 5.1 follows easily from the straightforward estimate (5.3) and the definition of $\zeta_\sigma(z, g_0)$ (cf. (5.1)).

In order to apply this result to $\zeta_f(z, g)$ we need only replace $\bar{g} : X_A \to \mathbb{R}$ by a suitable function $g_0 : X_A^+ \to \mathbb{R}$ which takes the same values around corresponding periodic orbits (i.e., $g_0^n(\underline{x}) = \bar{g}^n(\underline{x})$ whenever $\sigma^n \underline{x} = \underline{x}$, viewed as a point in X_A or X_A^+, as appropriate) and then $\zeta_\sigma(z, \bar{g}) = \zeta_\sigma(z, g_0)$. The existence of such a function $g_0 : X_A^+ \to \mathbb{R}$ is shown in [**Si**] and [**B3**].

We can now invoke (5.2) to translate the results on zeta functions for the symbolic model (and auxiliary shifts) to the original Axiom A diffeomorphism to deduce both Propositions 5.1 and 5.2 and the following result on the location of poles.

THEOREM 5.1. *There exists $0 < \theta < 1$ such that $\zeta_f(z, g)$ has a meromorphic extension to the disk $|z| \leq e^{-P(Re(g))}/\theta$. Moreover, the zeta function $\zeta_f(z, g)$ has a singular value $z = z_i$ in its meromorphic extension precisely when $\lambda_i = 1/z_i$ is an eigenvalue for $\mathcal{L}_g$.*

Let us consider a couple of applications of the results and ideas from this subsection

APPLICATION I: DISTRIBUTION OF PERIODIC ORBITS. In a similar way to that in which the zeros and poles of the Artin-Mazur influenced the asymptotic behaviour of the number $N(n)$ of periodic points for f of period n, so the poles of $\zeta_f(g, z)$ give information on the distribution of orbits. Assume that $g : \Lambda \to \mathbb{R}$ is a Hölder continuous function. Taking a logarithmic derivative of $t \mapsto \zeta_g(z, tg)$ at $t = 0$ gives

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} \sum_{f^n x = x} g^n(x) &= \left. \frac{\partial \zeta_f(tg, z)}{\partial t} \right|_{t=0} \zeta_f(0, z)^{-1} \\ &= \frac{\int g d\mu}{e^{P(g)} - z} + \sum_{\substack{\lambda_i \neq e^{P(g)} \\ |\lambda_i| > \theta' e^{P(g)}}} \frac{C_i}{\lambda_i - z} + \chi(z),\end{aligned}$$

where we fix $\theta' > \theta$ and we use that $\frac{de^{P(tg)}}{dt} = \int g d\mu$, i.e., the integral with respect to the measure of maximal entropy [**Ru3**]. Using the expansions $\frac{1}{\lambda_i - z} = \sum_{n=0}^{\infty} \frac{z^n}{\lambda_i^{n+1}}$ and comparing coefficients we see that

$$\frac{1}{N_f(n)} \sum_{f^n x = x} \frac{g^n(x)}{n} = \int g d\mu + \sum_{\substack{\lambda_i \neq e^{P(g)} \\ |\lambda_i| > \theta' e^{P(g)}}} C_i \left(\lambda_i e^{-P(g)} \right)^n + O(\theta'^n).$$

Application II: The Dimension of Julia sets. A rather nice application of the results on zeta functions gives results on the dimension of Julia sets.

Consider the quadratic map $T_c : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by $T_c(z) = z^2 + c$, for $c \in \mathbb{C}$. We define the Julia set J_c to be the closure of the repelling periodic points, i.e.,

$$J_c = \overline{\{z \in \mathbb{C} : T_c^n(z) = z, \text{ for some } n \geq 1, \text{ and } |(T^n)'(z)| > 1\}}.$$

The Julia set is always a closed invariant set [**CG, St**].

Let us briefly recall the definition of Hausdorff dimension. Given a subset $E \subset \mathbb{R}^n$ we can consider a cover $\mathcal{U} = \{U_i\}_i$ for E by open sets. For $\delta > 0$ we can define $H_\epsilon^\delta(E) = \inf_{\mathcal{U}} \{\sum_i \operatorname{diam}(U_i)^\delta\}$ where the infimum is taken over all open covers $\mathcal{U} = \{U_i\}$ such that $\operatorname{diam}(U_i) \leq \epsilon$. We define $H^\delta(E) = \lim_{\epsilon \to 0} H_\epsilon^\delta(E)$ and, finally, $\dim_H(E) = \inf\{\delta : H^\delta(E) = 0\}$.

The simplest quadratic map is $T_0 : z \to z^2$ for which the Julia set is simply the unit circle. i.e., $J_0 := \{z \in \mathbb{C} : |z| = 1\}$. For $|c|$ sufficiently small, the Julia set J_c for $T_c(z) = z^2 + c$ is still a Jordan circle, but it has $\dim_H(J_c) > 1$. Ruelle showed that for c sufficiently small the Hausdorff dimension of the Julia set J_c is a solution $d = \dim_H(J_c)$ to the identity $\zeta_{T_c}(d, \log |T_c'|) = 0$ [**Ru7**]. In particular, it follows by the implicit function theorem that $c \mapsto \dim_H J_c$ is analytic in a neighbourhood of $c = 0$. Ruelle also calculated the second order approximation

$$\dim_H J_c \sim 1 + \frac{|c|^2}{4 \log 2}, \quad \text{as } |c| \to 0.$$

Remark. Of course, in order to use the previous analysis on zeta functions we need the existance of a *Markov Partition* for J, which is a finite collection of closed sets $R_i \subset J$, $i = 1, \ldots, k$ such that

1. $\cup_i R_i = J$
2. $\operatorname{int}(R_i) \cap \operatorname{int}(R_i) = \emptyset$ for $i \neq j$
3. $T(R_i)$ is a union of sets from $\{R_i : i = 1, \ldots, k\}$

Markov partitions always exist for expanding maps $T_c : J_c \to J_c$ (e.g., when $|c|$ is small) [**Ru7**].

5.3 One dimensional maps. In section 4.1 we described the use of Markov extensions to extend the analytic domain of the associated Artin Mazur zeta function $\zeta_T(z)$. This method can be extended to weighted zeta functions. Consider a continuous function $g : I \to \mathbb{R}$. We say that g has bounded variation if

$$|g|_{BV} := \sup \left\{ \sum_{i=0}^{n-1} |g(x_{i+1}) - g(x_i)| \, : \, 0 = x_0 < x_1 < \ldots < x_n = 1 \right\}$$

We can again use the weighting given by $g^n(x) := g(x) + g(Tx) + \ldots + g(T^{n-1}x)$ and define the weighted zeta function by

$$\zeta_f(g, z) = \exp \left(-\sum_{n=1}^{\infty} \frac{z}{n} \sum_{T^n x = x} e^{g^n(x)} \right) \tag{5.4}$$

The $\zeta_T(z,g)$ function converges to an analytic function on the disc $|z| < e^{-P(g)}$, where (as in §5.1) we write

$$P(g) = \limsup_{n \to +\infty} \frac{1}{n} \log \left(\sum_{f^n x = x} e^{g^n(x)} \right).$$

Let us denote

$$\theta = \limsup_{n \to +\infty} \left(\sup_{x \in I} \frac{1}{|(T^n)'(x)|} \right).$$

then the analogue of Theorem 5.1 ia the following.

THEOREM 5.2. *Let $T : I \to I$ be an expanding interval map and let $g : I \to \mathbb{R}$ be a function of bounded variation. Then $\zeta_T(z,g)$ has a meromorphic extension to $|z| < e^{-P(g)}/\theta$.*

This theorem was originally proved by Hofbauer and Keller in the case that g is constant on each of a finite number of intervals [**HK**]. The extension to the case that $g : I \to \mathbb{R}$ is of bounded variation was made by Baladi and Keller [**BK**]. The method of proof is basically analogous to that in the case of Hölder functions on subshifts of finite type. One considers the operator $\mathcal{L}_g : B_{BV} \to B_{BV}$ on the Banach space B_{BV} of functions of bounded variation, with norm $||h||_{BV} = |h|_{BV} + |h|_\infty$, defined as before by

$$\mathcal{L}_g h(x) = \sum_{Ty = x} e^{g(y)} h(y).$$

As usual, the poles z_i for the zeta function $\zeta_f(z,g)$ which satisfy $e^{-P(g)} < |z_i| < e^{-P(g)}/\theta$ correspond to eigenvalues $\lambda_i = 1/z_i$ for $\mathcal{L}_g$.

A particularly interesting weight function is the choice $g(x) = 1/|T'(x)|$. In part, this is since the function is closely related to the existence an absolutely continuous invariant measure $d\mu = \rho dx$, where the density ρ is a fixed point $\mathcal{L}_{1/|T'|}\rho = \rho$. In particular, as one might expect, the zeta function $\zeta_T(z, 1/|T'|)$ is analytic on the unit disk, with a simple pole at $z = 1$. For this weight it is possible to further weaken expanding hypotheses. Consider a unimodal interval map $T : I \to I$ with critical point c. We say that an interval map $T : I \to I$ satisfies the Collet-Eckmann condition if

$$\lambda := \limsup_{n \to +\infty} \frac{1}{n} \log |(T^n)'(Tc)| > 0. \tag{5.5}$$

We also need to make two other growth assumptions:

$$\lambda_{\text{per}} := \inf_{n \geq 1} \inf_{T^n x = x} \frac{1}{n} \log |(f^n)'(x)| > 0 \text{ and } \lambda_{\text{I}} := - \inf_{n \geq 1} \inf_{A \in \alpha^n} \frac{1}{n} \log l(A) > 0, \tag{5.6}$$

where $\alpha^n = \alpha \vee T^{-1}\alpha \vee \cdots \vee T^{-(n-1)}\alpha$ is the refinement of the partition $\alpha = \{I_1, \ldots, I_k\}$ and l is Lebesgue measure.

Keller and Nowicki [**KN**] established the following general result.

THEOREM 5.3. *Let $T : I \to I$ be a unimodal map which satisfies the mild conditions (5.5) and (5.6). Then $\zeta_T(z, 1/|T'|)$ has a meromorphic extension to a disk $|z| < R$, where $R = \max\{e^{\lambda/2}, e^{\lambda_{\mathrm{per}}/2}, e^{\lambda_{\mathrm{I}}}\} > 1$.*

A very different approach to studying the zeta function $\zeta_T(z, 1/|T'|)$ for interval maps $T : I \to I$ was used by Ruelle in [**Ru6**]. The basic idea there is to introduce a second complex variable, and then to invoke a theorem of Bochner on the analytic domain of complex functions in two variables. More precisely, let us denote $g = \log |T'|$ we define a zeta function of two variables by

$$\zeta(z, s) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} e^{s(g(x)+g(Tx)+\ldots+g(T^{n-1}x))}\right).$$

We want to make the assumption that

$$N := \limsup_{n\to+\infty} \left(\#\{x \in I : T^n x = x\}\right)^{1/n} < +\infty. \tag{5.7}$$

The simplest form of Ruelle's theorem takes the following form [**Ru6**].

THEOREM 5.4. *Let $T : I \to I$ be a interval map which satisfies (5.7). The zeta function $\zeta_T(z, 1/|T'|)$ has a non-zero meromorphic extension to the region $|z| < \theta$.*

We can legitimately assume that $g \in B_{BV}$, then the result of Baladi-Keller tells us that $1/\zeta(z, s)$ is analytic on the domain $\{(z, s) \in \mathbb{C}^2 : Re(s) > 0 \text{ and } |z|\theta^{Re(s)} < 1\}$. It follows by a simple consideration of the radius of convergence of the series in its definition that $\zeta(z, s)$ is also analytic provided $Re(s) < 0$ and $|z|\theta^{|Re(s)|} < 1$.

The bridge between these two domains, which is the core of Ruelle's proof, follows from the following general result of Bochner on the domain of bi-analytic functions comes by writing $z = e^w$.

LEMMA 5.2. *(Bochner's Theorem) Assume that a complex function $f(s, w)$ in two variables is analytic for $(Re(s), Re(w)) \in U \subset \mathbb{R}^2$, for a connected open domain U. Then $f(s, w)$ is analytic for $(Re(s), Re(w)) \in co(U)$, where $co(U)$ is the convex hull of U.*

REMARK. Another direction in which the result of Baladi and Keller can be generalized is to consider interval maps whose hyperbolicity fails at an indifferent fixed point. For example, assume that $T : I \to I$ satisfies $T(0) = 0$ with $T'(0) = 1$ and $|T'(x)| > 1$ for $0 < x \leq 1$. This has been studied in [**Rg**].

5.4 Sharp determinants. Finally, we shall conclude this section by mentioning recent work of Ruelle, Baladi and others on Fredholm determinants [**BR**]. This subject is similar in spirit to that of zeta functions, although one of the features of this study is that it deals with transfer operators which do not necessarily arise from a single transformation. We refer to [**Ba**] for a survey which contains extensive references.

Let $\mathcal{M}$ be an operator acting on the Banach space B_{BV} of functions of bounded variation (on the real line, but compactly supported) of the form

$$\mathcal{M}w(x) = \sum_i g_i(x) w(\psi_i x), \tag{5.8}$$

where g_i is continuous, of bounded variation and with compact support, and and ψ_i is a homeomorphism of an interval containing the support of g_i onto an interval in $\mathbb{R}$. The space of such operators is a Banach space whose norm $||\mathcal{M}||$ is the infimum of the $\sum_i |g_i|_{BV}$ for all the representations (5.8) of $\mathcal{M}$.

Definition. The *sharp trace* of $\mathcal{M}$ is defined by

$$\mathrm{Tr}^\sharp \mathcal{M} = \sum_\omega \int d(g_\omega(x)) \tfrac{1}{2}\,\mathrm{sgn}(\psi_\omega(x) - x), \text{ where } \mathrm{sgn}(\xi) = \begin{cases} +1 & \text{if } \xi > 0 \\ 0 & \text{if } \xi = 0, \\ -1 & \text{if } \xi < 0 \end{cases}$$

The *sharp determinant* is defined formally by

$$\Delta(z) = \det{}^\sharp(1 - z\mathcal{M}) = \exp - \sum_{m=1}^\infty \frac{z^m}{m} \mathrm{Tr}^\sharp \mathcal{M}^m.$$

There is a natural dual operator to $\mathcal{M}$ which we denote by $\widehat{\mathcal{M}}$. Let R and $\widehat{R}$ denote the norms of $\mathcal{M}$ and $\widehat{\mathcal{M}}$, respectively, as operators acting on the space of bounded functions on $\mathbb{R}$ with the uniform norm. The following result was proved by Baladi and Ruelle [**BR**].

Theorem 5.5. *The spectral radius of $\mathcal{M}$ is at least $\max(R, \widehat{R})$ and the essential spectral radius of $\mathcal{M}$ is at most $\widehat{R}$. Furthermore, $\Delta(z) = \det^\sharp(1 - z\mathcal{M})$ is holomorphic for $|z| < R^{-1}$, and λ with $\widehat{R} < |\lambda| \leq R$ is an eigenvalue of $\mathcal{M}$ if and only if λ^{-1} is a zero of $\Delta(z)$ with the same multiplicity.*

6. Zeta functions for Axiom A flows

A more delicate problem than studying the domain of zeta functions for Axiom A diffeomorphisms is that of understanding the behaviour of the corresponding zeta function for Axiom A flows. Let M be a C^∞ compact manifold. We call a closed invariant set $\Lambda \subset M$ for a C^1 flow $\phi_t : M \to M$ *hyperbolic* if:

1. the tangent bundle $T_\Lambda M$ has a continuous splitting $T_\Lambda M = E^0 \oplus E^u \oplus E^s$ into $D\phi_t$-invariant subbundles E^0 is the one-dimensional bundle tangent to the flow; and there exist $C, \lambda > 0$ such that
$$\begin{cases} ||D\phi_t|E^s|| \leq Ce^{-\lambda t} \text{ for } t \geq 0 \\ ||D\phi_{-t}|E^u|| \leq Ce^{-\lambda t} \text{ for } t \geq 0; \end{cases}$$
2. $\phi_t : \Lambda \to \Lambda$ is transitive;
3. the periodic orbits are dense in Λ;
4. there exists an open neighbourhood $U \supset \Lambda$ such that $\cap_{t\in\mathbb{R}}\phi_t U = \Lambda$.

We call $\phi_t : \Lambda \to \Lambda$ a hyperbolic flow. An Axiom A flow is one for which the non-wandering set is a finite union of hyperbolic sets and periodic orbits [**Ch**]. A particular case are transitive Anosov flows, where $\Lambda = M$.

Smale proposed a definition for the zeta function of an Axiom A flow in his famous survey [**Sm**], motivated by the definition of the classical Selberg zeta function corresponding to the geodesic flow on manifolds of constant negative curvature. Subsequently, Ruelle proposed a modified definition which has proved better suited to analysis [**Ru1**]. It is helpful to motivate the definition by recalling first the more familiar definition of the Riemann zeta function from analytic number theory.

6.1 The Riemann zeta function in prime number theory. Consider the asymptotic behaviour of the number of prime numbers less than x, i.e.,

$$\pi(x) = \#\{p_n \le x : p_n \text{ is a prime }\}.$$

In 1896, Hadamard and de la Vallée Poussin independently showed the asymptotic estimate $\pi(x) \sim \frac{x}{\log x}$, as $x \to +\infty$. Subsequent improvements showed that $\exists A > 0$

$$\pi(x) = \mathrm{li}(x)\left(1 + O\left(e^{-A(\log x)^{1/2}}\right)\right), \tag{6.1}$$

where $\mathrm{li}(x) = \int_2^x \frac{1}{\log u} du \quad (\sim x/\log x)$. The original analysis of $\pi(x)$ came from the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_n \left(1 - p^{-s}\right)^{-1},$$

which converges to an analytic non-zero function for $Re(s) = 1$. For future reference, let us recall a few important properties and conjectures on the domain of $\zeta(s)$.

1. Hadamard and de la Vallée Poussin showed that $\zeta(s)$ has an analytic non-zero extension to a neighbourhood of $Re(s) > 1$, *except* for a simple pole at $s = 1$ (which was sufficient to establish the principal term in (6.1));
2. The error term estimate in (6.1) comes from knowing that $\zeta(s)$ has no poles in a region
$$\left\{ s = \sigma + it : \sigma \ge 1 - \frac{C}{\log |t|} \text{ for } |t| \ge 2 \right\}$$
3. The *Riemann Conjecture* (or *Riemann Hypothesis*) states that $\zeta(s)$ has all of its zeros on the line $Re(s) = 1/2$. This would imply that for any $\epsilon > 0$ we can estimate $\pi(x) = \mathrm{li}(x) + O(x^{1/2+\epsilon})$.

6.2 The zeta function for Axiom A flows. Consider an Axiom A flow $\phi_t : \Lambda \to \Lambda$, restricted to a basic set Λ with topological entropy $h = h(\phi) > 0$. For closed orbits of the flow $\phi_t : \Lambda \to \Lambda$ we want to study the analogous dynamical zeta function. We shall be interested in prime closed orbits τ and write $\lambda(\tau)$ for their least period (i.e., $\phi_{\lambda(\tau)}(x_\tau) = x_\tau$ whenever $x_\tau \in \tau$).

DEFINITION. We define the *dynamical zeta function* for $\phi_t : M \to M$ by

$$\zeta_\phi(s) = \prod_\tau \left(1 - e^{-sh\lambda(\tau)}\right)^{-1}, \quad s \in \mathbb{C}. \tag{6.2}$$

Here the product is over all prime orbits. This converges to a non-zero analytic function for $Re(s) > 1$ cf. [**PP1**].

Comparing (6.1) and (6.2) we see that in some formal sense we are replacing the primes numbers p by the values $e^{-h\lambda(\tau)}$ associated to closed orbits τ.

Selberg's definition (corresponding to geodesic flows for surfaces with $\kappa = -1$) was of a zeta function $Z(s) = \prod_{n=0}^{\infty}\prod_{\gamma}\left(1 - e^{-(s+n)l(\gamma)}\right)$, where γ denotes a closed geodesic of length $l(\gamma)$. Smale proposed the natural extension to Anosov flows of

$$Z(s) = \prod_{n=0}^{\infty}\prod_{\tau}\left(1 - e^{-(s+n)h\lambda(\tau)}\right), \quad s \in \mathbb{C}.$$

Clearly, there is the simple relationship $\zeta_\phi(s) = Z(s+1)/Z(s)$. However, the definition we have given (and due to Ruelle) has proved the most useful.

We recall that $\phi_t : \Lambda \to \Lambda$ is *topologically weak mixing* if there is a constant $a > 0$ and a non-trivial function $F : \Lambda \to \mathbb{C}$ such that $F(\phi_t x) = e^{ia}F(x)$. The first main result we have on the domain of $\zeta_\phi(s)$ is the following [**PP1**].

THEOREM 6.1. *There exists $\epsilon > 0$ such that $\zeta_\phi(s)$ has a non-zero meromorphic extension to the larger half-plane $Re(s) > 1 - \epsilon$, and the value $s = 1$ is a simple pole.*

(i) *If $\phi_t : \Lambda \to \Lambda$ is topologically weak mixing then there are no other poles on the line $Re(s) = 1$.*

(ii) *If $\phi_t : \Lambda \to \Lambda$ is not topologically weak mixing then $\zeta_\phi(s)$ has poles at $s = 1 + iak$, $k \in \mathbb{Z}$.*

In the not weak mixing case then $\zeta_\phi(s)$ actually has a meromorphic extension to $\mathbb{C}$ with periodicity $\zeta_\phi(s) = \zeta_\phi(s + iak)$, for $k \in \mathbb{Z}$. By contrast, there exist examples of weak mixing Axiom A flows for which the zeta function $\zeta_\phi(s)$ does not have a meromorphic extension to the entire complex plane $\mathbb{C}$.

6.3 The use of symbolic dynamics. The method of proof of Theorem 6.1 is to use symbolic dynamics, and then to reduce the problem to that for the Ruelle zeta functions in §3. Consider an Axiom A flow $\phi_t : \Lambda \to \Lambda$ restricted to a hyperbolic set Λ. Given $x \in \Lambda$ we can define *stable and unstable manifolds*

$$W^s(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \to 0, \text{ as } t \to +\infty\},$$

$$W^u(x) = \{y \in M : d(\phi_{-t}(x), \phi_{-t}(y)) \to 0, \text{ as } t \to +\infty\}$$

These form leaves of transverse foliations which we denote by $\mathcal{W}^u$, $\mathcal{W}^s$ which are tangent to E^s and E^u, respectively. For practical purposes we like to study, for $\epsilon > 0$, pieces of leaves called *local stable and unstable manifolds*:

$$W^s_\epsilon(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \le \epsilon, \forall t \ge 0\}$$
$$W^u_\epsilon(v) = \{w \in M : d(\phi_{-t}(x), \phi_{-t}(y)) \le \epsilon, \forall t \ge 0\}$$

which are transverse embedded discs.

We want to consider a finite family of sections $\mathcal{R} = \{R_1, \ldots, R_k\}$ such that each R_i is transverse to the flow (and, for technical reasons, that each section is the closure of its interior). If $x \in R_i$ then we write $W^s(x, R_i) = W^s_\epsilon(x) \cap R_i$. Given points $x_i \in R_i$ we want to denote and $W^u(x_i, R_i) = W^u_\epsilon(x_i) \cap R_i$. We can arrange that the sections R_i have the convenient form

$$R_i = \cup_{x \in W^u(x_i, R_i)} W^s(x, R_i).$$

We also ask that the sections are *Markov* which (essentially) means that

1. If we push the stable pieces $W^s(x, R_i)$ forward under the flow (for the same time $T > 0$, say, since stable manifolds map to stable manifolds) they lie wholly within a piece $W^s(\phi_T x, R_j)$, say, of the next section; and
2. If we push the unstable pieces $W^u(x_k, R_j)$ backward under the flow (usually for varying times) then they lie wholly within a piece $W^s(x', R_i)$, say, of the last section.

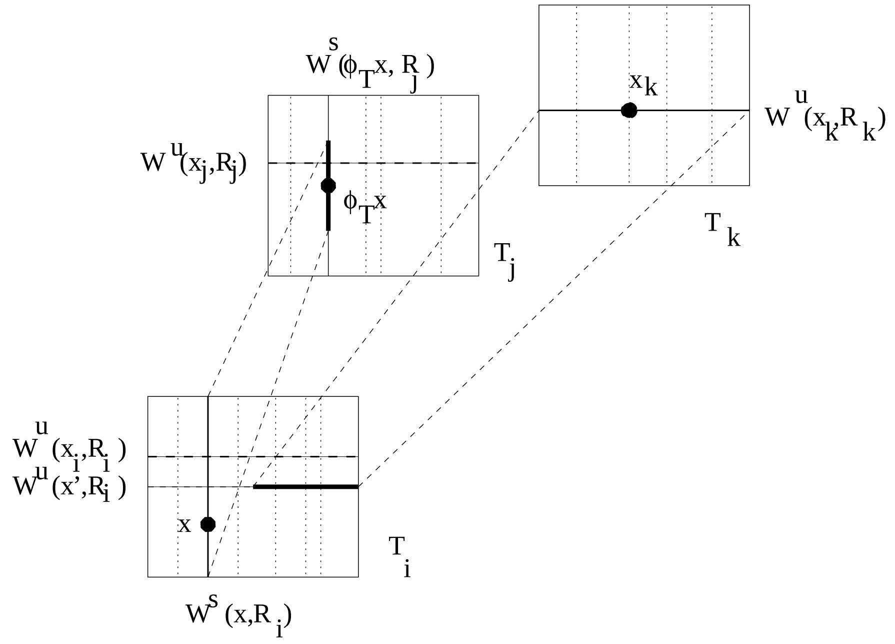

FIGURE 3 . Markov sections for Axiom A flows

One can always find a Markov family of sections for Anosov or Axiom A flows [**Ra, Bo3**]. Using Markov sections we can associate a subshift of finite type $\sigma : X_A \to X_A$, which corresponds to the Poincaré map between sections. If $r : X_A \to \mathbb{R}$ corresponds to the transition time between sections then it is a strictly positive Hölder function. We define the space $X_A^r = \{(\underline{x}, u) \in X_A \times \mathbb{R} : 0 \le u \le r(\underline{x})\}$, where we identify $(\underline{x}, r(\underline{x})) \sim (\sigma\underline{x}, 0))$, and a suspended flow $\sigma^r : X_A^r \to X_A^r$ defined locally by $\sigma_t^r(\underline{x}, u) = (\underline{x}, u+t)$. This provides a symbolic model for the original flow $\phi_t : \Lambda \to \Lambda$.

For suspended flows there is a natural correspondence between closed orbits $\{\underline{x}, \sigma\underline{x}, \dots, \sigma^{n-1}\underline{x}\}$ for $\sigma : X_A \to X_A$ and closed orbits $\overline{\tau}$ for the suspended flow $\sigma^r : X_A^r \to X_A^r$ of least period $\lambda(\overline{\tau}) = r^n(\underline{x}) := r(\underline{x}) + r(\sigma\underline{x}) + \cdots + r(\sigma^{p-1}\underline{x})$. We can define a zeta function $\zeta_{\sigma^r}(s)$ to the suspended flow $\sigma^r : X_A^r \to X_A^r$ exactly as in (6.2).

The first step is to compare $\zeta_{\sigma^r}(s)$ and $\zeta_\phi(s)$. Manning's construction of auxiliary subshifts of finite type to correct for closed orbits lying in the boundary of Markov partitions for diffeomorphisms carries over in a natural way to flows. In particular, we have shifts $\sigma_{\underline{i}} : X_{A_{\underline{i}}} \to X_{A_{\underline{i}}}$ and positive functions $r_{\underline{i}} : X_{A_{\underline{i}}} \to \mathbb{R}$.

$$\zeta_\phi(s) = \frac{\prod_{|\underline{n}|=\mathrm{odd}} \zeta_{\sigma_{\underline{i}}^{r_{\underline{n}}}}(s)}{\prod_{|\underline{n}|=\mathrm{even}} \zeta_{\sigma_{\underline{i}}^{r_{\underline{n}}}}(s)} \tag{6.3}$$

and $\zeta_{\sigma^r}(s) = \zeta_{\sigma_{(1)}^{r_{(1)}}}(s)$.

The second step is to establish the required extension for $\zeta_{\sigma^r}(s)$ and, more generally, $\zeta_{\sigma_{\underline{i}}^{r_{\underline{i}}}}(s)$. corresponding to that for $\zeta_\phi(s)$ in Theorem 6.1. The following simple expansion relates the zeta functions for $\sigma^r : X_A^r \to X_A^r$ and $\sigma : X_A \to X_A$

$$\begin{aligned}
\zeta_{\sigma^r}(s) &= \prod_{\overline{\tau}} \left(1 - e^{-sh\lambda(\overline{\tau})}\right)^{-1} = \exp\left(\sum_{\tau}\sum_{m=1}^{\infty} \frac{(e^{-sh\lambda(\overline{\tau})})^m}{m}\right) \\
&= \exp\left(\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \sum_{\substack{\text{prime orbits} \\ \{\underline{x},\dots,\sigma^{p-1}\underline{x}\}}} \frac{e^{-shmr^p(x)}}{m}\right) \\
&= \exp\left(\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \sum_{\substack{\text{periodic points} \\ \sigma^{pm}\underline{x}=\underline{x}}} \frac{e^{-shmr^p(\underline{x})}}{pm}\right) \\
&= \exp\left(\sum_{n=1}^{\infty}\sum_{\sigma^n\underline{x}=\underline{x}} \frac{e^{-shr^n(\underline{x})}}{n}\right) = \zeta_\sigma(-shr, 1)
\end{aligned} \tag{6.4}$$

Thus the results in §5.1 on the domain of weighted zeta functions and subshifts of finite type, and the identity (6.3), gives the following corresponding result (Theorem 6.1) for the zeta function $\zeta_\phi(s)$ for flows.

Furthermore, the poles s for $\zeta_\phi(s)$ in the region in Theorem 6.1 are related to the condition that $\mathcal{L}_{-shr}$ has 1 as an isolated eigenvalue.

7. Zeta functions for analytic Anosov flows

In the previous section we considered the use of symbolic dynamics to study the zeta function $\zeta_\phi(s)$. However use of subshifts of finite and suspended flows essentially involves sacrificing the differentiable structure of the flow whilest retaining only the Hölder structure.

We shall now consider the case where $\phi : M \to M$ is a real analytic (C^ω) Anosov flow. Initially, we shall assume a stronger technical hypothesis. The strong stable manifolds $W^{ss}(x) = \{y \in M : \lim_{t\to+\infty} d(\phi_t(x), \phi_t(y)) = 0\}$, $x \in M$, are C^ω immersed submanifolds, but we shall also assume that the associated foliation $\mathcal{W}^{ss} = \{W^{ss}(x)\}$ is C^ω (i.e., the associated holonomy map is C^ω).

Given a family $\mathcal{R} = \{R_1, \dots, R_k\}$ of Markov sections for flow, we can assume without loss of generality that each of the sections is foliated by pieces of strong stable manifold i.e. sections are *aligned* in the sense that for each R_i there exists $x_i \in \mathrm{int}(R_i)$ with $R_i = \cup_{x\in W^u(x_i,R_i)} W^s(x, R_i)$ (if this is not the case then it is easy arranged by an appropriate perturbation in the sections in the flow direction).

Using Markov sections we can get a well-defined expanding interval map $f : I \to I$ by the following simple device

1. identify all points on a section R_i which lie on the same stable piece. This collapses R_i to an interval I_i;
2. The Poincaré map between sections defined by ϕ induces a map $f : I \to I$ (where $I = \cup_{i=1}^k I_i$ represents the disjoint union of the intervals I_i)
3. The transition time for a point to move from one section to the next under the flow is constant on stable pieces. This induces a map $r : I \to \mathbb{R}^+$

Because of the assumption that $\mathcal{W}^{ss}$ is an analytic foliation the sets $I_i = R_i/\sim$ can be considered as a C^ω d-dimensional manifolds (where $d = \dim(E^u)$). Similarly, the piecewise C^ω Poincaré map induces a C^ω expanding map $f : I \to I$. Finally, the piecewise C^ω return time function induces a positive C^ω function $r : X \to \mathbb{R}$.

Since I_i is real analytic this means we can choose an open neighbourhood (in the complexification) $V_i \supset I_i$ and denote the disjoint union by $V = \cup_{i=1}^k V_i$. Moreover, we can assume V is chosen so that the function r and the local inverses to f have (complex) analytic extensions to V.

Consider the Banach space $\mathcal{A}_0$ of bounded analytic functions $h : U \to \mathbb{C}$ with the supremum norm $|h|_\infty = \sup_{z \in V} |h(z)|$. Given $g \in \mathcal{A}$, we define a transfer operators by complete analogy with the definition for subshifts of finite type in §3, i.e.,

$$\begin{cases} \mathcal{L}_g : \mathcal{A}_0 \to \mathcal{A}_0 \\ \mathcal{L}_g w(x) = \sum_{fy=x} e^{g(y)} h(y). \end{cases}$$

Ruelle observed that this operator is *nuclear* (or trace class, in the sense of Grothendieck) and concluded that its operator determinant satisfies

$$\det(I - z\mathcal{L}_g) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{f^n x = x} \frac{e^{g^n(x)}}{\det(I - (D_x f)^{-1})} \right).$$

This is very close to the expansion (6.4) for the zeta function (except for the denominator inside the final summation). To obtain the exact expression in (6.4) one needs to study the natural generalizations $L_{g,j} : \mathcal{A}_j \to \mathcal{A}_j$, $1 \le j \le n$ of the transfer operator to the space. The associated determinants then recover the zeta function via the identity

$$\zeta_{\sigma^r}(s) = \prod_{j=0}^{n} \det(I - z\mathcal{L}_{g,j})^{(-1)^{j+1}}.$$

The determinants are entire functions. More precisely, one can write $\det(I - z\mathcal{L}_{g,j}) = 1 + \sum_{n=1}^{\infty} a_n z^{n^{1+1/d}}$, where $|a_n| = O(\theta^n)$, for some $0 < \theta < 1$. The Bowen-Manning lemma again relates $\zeta_\phi(s)$ to $\zeta_{\sigma^r}(s)$. This gives the following result of Ruelle [**Ru2**].

THEOREM 7.1. *For C^ω Anosov flows $\phi_t : M \to M$ for which the foliations $\mathcal{W}^{ss}$ are real analytic the zeta function $\zeta_\phi(s)$ is meromorphic on $\mathbb{C}$.*

The very stringent assumption that the foliations $\mathcal{W}^{ss}$ are real analytic holds only for very special examples (e.g. geodesic flows on manifolds with constant negative sectional curvatures). However, there have been subsequent developments which have allowed this assumption to be relaxed. In particular, work of Rugh [**Rg**] and Fried [**Fd3**] shows that the conclusion of Theorem 7.1 is the same for any C^ω Anosov flow without any additional assumption on $\mathcal{W}^{ss}$.

For the suggestive case of expanding C^k maps, Tangerman showed that although the associated operator is not nuclear the essential part of its spectrum shrinks to zero as k increases, and that this leads to corresponding improved estimates on the domain of the zeta function [**Ta**] (cf. also [**Ru4**]) Kitaev has results pertaining to Axiom A transformations in the C^∞ case [**Kt**].

8. L-functions and special values

For an Anosov flow $\phi_t : M \to M$ we want to associate a generalization of the zeta function $\zeta_\phi(s)$ which carries additional information about the algebraic topology of the manifolds M, and the location therein of individual orbits.

Given a closed orbit τ we can denote by $[\tau] \in \pi_1(M)$ the homotopy class in which it lies. We can consider the unstable bundle $E^u(\tau)$ restricted to the orbit τ and if this bundle is orientable we write $\Delta(\tau) = 1$, and if the bundle is not orientable then we can write $\Delta(\tau) = -1$.

Let $R_\chi : \pi(M) \to U(n)$ be any unitary representation of fundamental group. We can define the associated L-function by

$$L(s, \chi) = \prod_\tau \det\left(I - \Delta(\tau) e^{-sh\lambda(\tau)} R_\chi[\tau]\right)^{-1}, s \in \mathbb{C},$$

which converges for $Re(s) > 1$. The definition depends only on the character $\chi = \mathrm{tr}(R_\chi)$, rather than on the particular choice of representation. In the special case that $R_\chi = 1$ is the trivial representation we see that $L(s, \chi)$ simply reduces to the zeta function $\zeta_\phi(s)$ (as defined in (6.2)).

The results and analysis we described in the previous section carry over to L-functions for C^ω Anosov flows. The generalization of Theorem 7.1 is the following.

THEOREM 8.1. *For C^ω Anosov flows $\phi_t : M \to M$ for which the foliations $\mathcal{W}^{ss}$ are real analytic the L-function $L(s, R_\chi)$ is the reciprocal of an entire function.*

As before, the assumption that the foliations $\mathcal{W}^{ss}$ are real analytic can be relaxed to just being a C^ω Anosov flow [**Fd3**].

In the particular case of L-functions, there is some interest in understanding the value of $L(s, \chi)^{-1}$ at $s = 0$. For zeta functions, the value $\zeta(s)^{-1}$ is typically zero. However, for the case of general L-functions if typically takes a value which has a simple, purely topological, interpretation which is independent both of the geometry of the manifold and the particular flow $\phi_t : M \to M$.

Consider a triangulization of M and a differential complex $\mathcal{C} = \{d_i : C_i \to C_{i-1}\}$. For a given unitary representation $R_\chi : M \to U(n)$ we say that $\mathcal{C}$ is cyclic if $H^*(M, R_\chi) = 0$. Alternatively, we can choose a chain contraction $\delta = \{\delta_i : C_i \to C_{i+1}\}$ such that $\delta_{i-1} d_i + d_{i+1}\delta_i = I$. By a suitable choice of basis we can write $d_{\mathrm{odd}} = \oplus_i d_{2i+1}$ and $\delta_{\mathrm{odd}} = \oplus_i \delta_{2i+1}$ as square matrices. Finally, we can define the *Reidmeister torsion* by $\tau = \pm\det(d_{\mathrm{odd}} + \delta_{\mathrm{odd}})$ which, except for an ambiguity in the sign, is independent of all choices.

There are a number of interesting results by Fried relating the special value $L(0, R_\chi)$ to the torsion. The following illustrates the theme [**Fd4**].

THEOREM 8.2. *Let $\phi_t : M \to M$ be a C^ω geodesic flow and let $R_\chi : M \to U(n)$ be an acyclic unitary representation then $L(0, R_\chi)^{-1} = \tau$.*

9. Counting closed orbits for flows

An interesting application of dynamical zeta functions is to analyzing the growth and distribution of closed orbits for flows. Given a closed orbit τ of least period by $\lambda(\gamma)$. We write $\pi(T) = \#\{\tau : \lambda(\tau) \le T\}$, for $T > 0$. The analogue of the prime number theorem of Hadamard and de la Vallée Poussin is the following.

THEOREM 9.1. *Let $\phi_t : \Lambda \to \Lambda$ be a weak mixing Axiom A flow restricted to a basic set Λ then*

$$\pi(T) \sim \frac{e^{hT}}{hT}, \text{ as } T \to +\infty, \tag{9.1}$$

where $h = h(\phi)$ is the topological entropy.

Here we use the notation that $f(T) \sim g(T)$ if $\frac{f(T)}{g(T)} \to 1$ as $T \to +\infty$.

This theorem was proved in 1969 by Margulis (without the use of zeta functions) for geodesic flows. An alternative proof for Axiom A flows was made by Parry and Pollicott in [**PP2**]. Prior to this Sinai had shown in 1966 that $\lim_{T\to+\infty} \frac{1}{T}\log \pi(T) = h$. For the special case of geodesic flows on surfaces of constant curvature $\kappa = -1$, Huber showed in 1959, using the Selberg trace formula, that $\pi(T) = \mathrm{li}(e^T) + O\left(e^{cT}\right)$, where $c < 1$ is related to the first non-zero eigenvalue of the Laplacian [**Hu**].

By way of illustration we shall first briefly recall what happens in the case of prime numbers and the Riemann zeta function.

9.1 Comparison with counting prime numbers. The connection between $\zeta(s)$ and the number $\pi(x)$ of primes p_n less than x can be expressed by the following expressions

$$\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty t^{-s} d\Pi_0(t), \text{ where } \Pi_0(t) = \sum_{\substack{p_n^k \le t \\ k\ge 1}} \log p_n,$$

$$\text{and } \sum_{\substack{p_n^k \le x \\ k\ge 1}} k \log p_n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(\frac{\zeta'(s)}{\zeta(s)}\right) ds, \tag{9.2}$$

for any $c > 1$ [**El, pp. 48-49**]. Knowledge of the domain of $\zeta(s)$ allows us to move the curve of integration further to the left in the complex integral in (9.1) getting asymptotic formulae (plus error terms) for $\pi(x)$.

The basic principle is that the further we move the curve of integration to the left, the better the error term, and how far the curve can be moved to the left is determined by the zero-free region for $\zeta_\phi(s)$.

The proof in [**PP2**] follows the lines of the classical proof of the Prime Number Theorem and involves analyzing the logarithmic derivative of the zeta function

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty}\sum_{\tau}\frac{\lambda(\tau)}{n}e^{-sn\lambda(\tau)} = \int_1^{\infty} t^{-s}d\Pi(t), \text{ where } \Pi(t) = \sum_{\substack{e^{n\lambda(\tau)}\leq t\\ n\geq 1}} \log\lambda(\tau)$$

By Theorem 6.1 (i) we see that the above expression converges to an analytic function for $Re(s) > 1$ and has a meromorphic extension of the form

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{-1}{s-1} + \psi(s), \tag{9.3}$$

where $\psi(s)$ is analytic in a neighbourhood of $Re(s) > 1$. An application of a standard tauberian theorem (i.e., the Ikehara-Wiener tauberian theorem [**El**]) to (9.3) implies that $\Pi(t) \sim t$, from which (9.2) follows without too much difficulty.

Application : Distribution of closed orbits. Given a hyperbolic flow $\phi_t : \Lambda \to \Lambda$, a classical result of Bowen shows that the closed orbits τ are distributed according to the measure of maximal entropy μ. Consider a Hölder continuous function $g : \Lambda \to \mathbb{R}$, then we can weight a given closed orbit τ by $\lambda_g(\tau) = \int_0^{\lambda(\tau)} g(\phi_t x_\tau)dt$ (for $x \in \tau$). The following result was originally proved by Bowen [**Bo4**], with a subsequent proof by Parry [**Pa**] using zeta functions.

Theorem 9.2. *Given a hyperbolic flow $\phi : \Lambda \to \Lambda$ we have*

$$\sum_{\lambda(\tau)\leq T}\lambda_g(\tau)/\sum_{\lambda(\tau)\leq T}\lambda(\tau) \to \int g d\mu$$

This should be compared with the discrete case in Application I, in §5 (from which the non-weak-mixing case follows). The proof of Parry (for the weak-mixing case) requires the following generalization of the original zeta function $\zeta_\phi(s)$:

$$\zeta_\phi(s,g) = \prod_{\tau}\left(1 - e^{\lambda_g(\tau)-s\lambda(\tau)}\right)^{-1}$$

whose properties are the same as those of the symbolic Ruelle zeta function

$$\zeta_\sigma(1, G - sr) := \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\sigma^n \underline{x}=\underline{x}} e^{G^n(\underline{x})-sr^n(\underline{x})}\right),$$

where $G(x) = \int_0^{r(\underline{x})} g\circ\pi(\underline{x}, t)dt$ (cf. §5). In particular,

$$\frac{\partial}{\partial t}\log\zeta_\sigma(tg - sr, 1)|_{t=0} = \sum_{\tau}\sum_{n=0}^{\infty}\lambda_g(\tau)e^{-sn\lambda(\tau)} = \frac{\int g d\mu}{s-1} + \psi_1(s)$$

where $\psi_1(s)$ is analytic in a neighbourhood of $Re(s) \geq 1$. As in the case of the outline of the proof of Theorem 9.1 we see that $\sum_{\lambda(\tau)\leq T}\lambda_g(\tau) \sim (\int g d\mu)e^{hT}$, which gives the conclusion of Theorem 9.2.

9.2 Error terms for geodesic flows. In this subsection and the next we want to describe sharper estimates than (9.1) in special cases. For this problem we can no longer rely on tauberian theorems, and more information on $\zeta_\phi(s)$ is required.

Let V be a C^∞ n-dimensional compact manifold with negative sectional curvatures. Let $< \cdot, \cdot >_{TV}$ denote the Riemannian metric on the tangent bundle. Let $M = SV = \{(x,v) \in TV : ||v||_{TV} = 1\}$ denote the unit tangent bundle. (Clearly, TV is $2n$ dimensional and M is $2n-1$ dimensional). Given any $(x,v) \in M$ we let $\gamma_{(x,v)} : \mathbb{R} \to M$ be the unit speed geodesic with $\gamma_{(x,v)}(0) = x$ and $\dot\gamma_{(x,v)}(0) = v$. We define the *geodesic flow* $\phi_t : M \to M$ by $\phi_t(x,v) = (\gamma_{(x,v)}(t), \dot\gamma_{(x,v)}(t))$ [**Kn**].

There is an obvious one-one correspondence between closed orbits for the geodesic flow and (directed) closed geodesics γ for the geodesic flow. In particular, we can interpret $\pi(T) = \#\{\gamma : l(\gamma) \le T\}$, for $T > 0$. In this case, a more precise formula for $\pi(T)$ can be given. It turns out that it is more convenient to replace the principle asymptotic term by $\mathrm{li}(e^{hT}) \sim \frac{e^{hT}}{hT}$, as $T \to +\infty$.

NOTATION. We write $f(T) = g(T) + O(w(T))$ if there exists $C > 0$ such that $|f(T) - g(T)| \le C|w(T)|$ for sufficiently large T.

The following result shows that for geodesic flows we can improve Theorem 9.1 to exponential error terms (cf. [**D2**] [**PS1**]).

THEOREM 9.3. *Let $\phi_t : M \to M$ be the geodesic flow on for a compact surface (i.e. $dim(M) = 2$) with negative curvature. There exists $\epsilon > 0$ such that*

$$\pi(T) = li(e^{hT})\left(1 + O\left(e^{-\epsilon T}\right)\right), \ \ as\ T \to +\infty \tag{9.4}$$

In fact, the estimate (9.4) generalizes to counting closed geodesics on compact manifolds of arbitrary dimension providing that the sectional curvature is pinched $-4 \le \kappa \le -1$. We shall explain the actual technical conditions we require in §11. The following result on $\zeta_\phi(s)$ the main ingredient in the proof of Theorem 9.2.

PROPOSITION 9.1. *For a geodesic flow there exists $c < h$ such that $\zeta_\phi(s)$ is analytic in the half-plane $Re(s) \ge c_0$, except for a simple pole at $s = h$. Moreover, there exists $0 < \alpha < 1$ such that $\zeta'_\phi(\sigma + it)/\zeta_\phi(\sigma + it) = O(|t|^\alpha)$, as $|t| \to +\infty$.*

This can be viewed as an analogue of the classical Riemann Hypothesis (iii) for the Riemann zeta function for prime numbers. It is well-known for the case of constant negative curvature.

If we define $\psi(T) := \sum_{e^{h\lambda(\tau)} \le T} h\lambda(\tau)$ and $\psi_1(T) := \int_0^T \psi(y)dy$ then we have the following

$$\psi_1(T) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \frac{T^{s+1}}{s(s+1)}\left(-\frac{\zeta'_\phi(s)}{\zeta_\phi(s)}\right) ds,$$

for $d > 1$ (by integrating term by term), which is an analogue of the identity (9.1) for primes.

We want to move the line of integration to some line $Re(s) = c < h$. From Cauchy's theorem and the bounds in Proposition 9.1 we can estimate

$$\psi_1(T) = \frac{T^2}{2} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'_\phi(s)}{\zeta_\phi(s)}\right)\frac{T^{s+1}}{s(s+1)}ds. \tag{9.4}$$

In particular, Proposition 9.1 can be used to bound the integral in such a way that $\psi_1(T) = \frac{T^2}{2} + O(T^{c+1})$. Using the trivial identity

$$\frac{\psi_1(T+\epsilon) - \psi_1(T)}{\epsilon} = \frac{1}{\epsilon}\int_T^{T+\epsilon} \psi(t)dt$$

it is then a simple deduction that $\psi(T) = T + O(T^{(c+1)/2})$. From this it is a straightforward derivation (using integration by parts) that $\pi(T) = \mathrm{li}(e^{hT})\left(1 + O\left(e^{-\epsilon T}\right)\right)$, with $\epsilon = 1 - \frac{c}{h}$, as required.

Remark on Related counting problems. There is a companion result to Theorem 9.3 which it is interesting to consider. Let V be a compact surface with negative curvature and let $\tilde{V}$ be its universal cover (with the lifted metric d). The fundamental group $\pi_1(V)$ can be realized as a group of isometries Γ acting on $\tilde{V}$ with $\tilde{V}/\Gamma = V$. Given a point $x \in \tilde{V}$, we can introduce the orbital counting function $N(T)$ defined by

$$N(T) = \#\{g \in \Gamma : d(x, gx) \leq T\}.$$

(Equivalently, this is the number of geodesic arcs from a point $x_0 (= x\Gamma)$ on M to itself, of length at most T.)

Theorem 9.4. [**PS1**] *There exists $\epsilon > 0$ and $C > 0$ such that*

$$N(T) = Ce^{hT}\left(1 + O(e^{-\epsilon T})\right).$$

This result was proved by Huber for surfaces of constant curvature $\kappa = -1$. In 1969, Margulis showed the principle asymptotic $N(T) \sim Ce^{hT}$ [**Ma**]. Related results are due to Lalley and Babillot-Ledrappier [**BL**].

9.3 Error terms for general Anosov flows. Even in the context of general Anosov flows there are error terms in the asymptotic (9.1), allbethey some what weaker than in the case of geodesic flows in Theorem 9.3. The following result is elaborated in [**PS1**], using ideas formulated for decay of correlations for Anosov flows by Dolgopyat [**D1, D2**].

Theorem 9.5. *Let $\phi_t : M \to M$ be a weak-mixing transitive Anosov flow. There exists $\delta > 0$ such that*

$$\pi(T) = \frac{e^{hT}}{hT}\left(1 + O\left(\frac{1}{T^\delta}\right)\right). \tag{9.5}$$

Notice that this asymptotic estimate is stronger than in Theorem 9.1, but weaker than in Theorem 9.3. Observe also that since we have $\mathrm{li}(e^{hT}) = \frac{e^{hT}}{hT}\left(1 + O\left(\frac{1}{T}\right)\right)$ there is nothing gained by using $\mathrm{li}(e^{hT})$ for the principal term in (9.5). (The estimate for (9.5) holds for any hyperbolic flow for which the box dimension of the hyperbolic set Λ is strictly greater than one.)

Let us briefly describe the proof of Theorem 9.5. We begin with the following technical result on the zeta function (which is correspondingly weaker than Proposition 9.1).

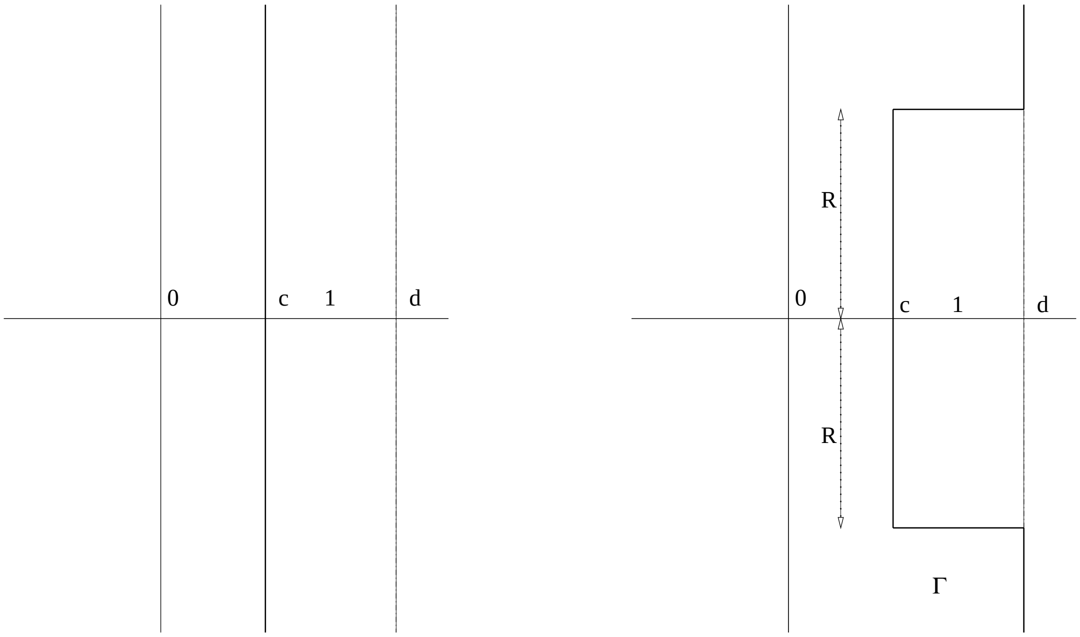

FIGURE 4. Curves of integration for the proofs of Theorems 9.4 and 9.5

PROPOSITION 9.2. *For a weak-mixing Anosov flow $\phi_t : M \to M$ there exist* $0 < \rho, \beta < 1$ *such that $\zeta_\phi(s)$ is analytic in the region*

$$\left\{ \sigma + it \,:\, 1 - \frac{C}{|t|^\rho} < \sigma < 1 \text{ and } |t| > 1 \right\}.$$

Moreover, we have the bound $\zeta'_\phi(s)/\zeta_\phi(s) = O(|t|^\beta)$.

In order to obtain the optimal error term for $\pi(T)$ we want move the line of integration $Re(s) = d$ to the left to the curve

$$\Gamma = [c + i\infty, c + iR] \cup [c + iR, d + iR] \cup [d + iR, c - iR] \cup [c - iR, c - iR] \cup [c - iR, c - i\infty]$$

(where we choose $R = R(T) = (\log T)^{[1/\rho]}$). The identity (9.4) then gives

$$\psi_1(T) = \frac{T^2}{2} + \frac{1}{2\pi i} \int_\Gamma \left(-\frac{\zeta'_\phi(s)}{\zeta_\phi(s)} \right) \frac{T^{s+1}}{s(s+1)} ds.$$

In particular, Proposition 9.2 can be used to show that

$$\psi_1(T) = \frac{T^2}{2} + O\left(\frac{T^2}{(\log T)^{(1-\beta)[1/\rho]}} \right).$$

From this we can now easily deduce that $\psi(T) = T + O\left(\frac{T}{(\log T)^{(1-\beta)[1/\rho]}}\right)$ and finally, after a little manipulation, $\pi(T) = \mathrm{li}(e^{hT})(1 + O(1/T^\delta))$ with $\delta = (1 - \beta)[1/\rho]$.

9.4 Relationship with decay of correlations. A closely related problem is that of decay of correlations. We complete this section by recalling the link.

Let $\phi_t : \Lambda \to \Lambda$ be a weak-mixing Axiom A flow let μ be the measure of maximal entropy. The flow $\phi_t : (\Lambda, \mu) \to (\Lambda, \mu)$ is strong mixing i.e.

$$\rho_{F,G}(t) := \int F \circ \phi_t G d\mu - \int F d\mu \int G d\mu \to 0, \text{ for all } F, G \in L^2(X, \mu).$$

(i.e. The "correlation of the flow tends to zero" [**B1**].)

Dolgopyat proved the following deep result on exponential decay of correlations in the case of geodesic flows on compact negatively curved surfaces.

THEOREM 9.6. *Let $\phi_t : M \to M$ be a geodesic flow on a negatively curved surface. There exists $\epsilon > 0$ such that for any Hölder $F, G : M \to \mathbb{R}$ one can choose $C > 0$ with $|\rho_{F,G}(t)| \leq Ce^{-\epsilon|t|}$* [**D2**].

The similarity of the error terms in Theorem 9.6 and 9.3 results from some similarity in their proofs. The complex function used in the study of $\rho_{F,G}(t)$ is its Fourier transform $\hat{\rho}_{F,G}(s) = \int_{-\infty}^{\infty} e^{ist} \rho_{F,G}(t) dt$. Moreover, since its analysis also depends on the Ruelle transfer operator there is a direct relationship between the poles of $\hat{\rho}_{F,G}(s)$ and $\zeta_\phi(s)$ (described in [**Po2**]). By the same token, for transitive Anosov flows and Gibbs measures Dolgopyat showed that $\exists \gamma > 0$ such that $\rho_{F,G}(t) = O(1/|t|^\gamma)$[**D3**], which corresponds to the error term in Theorem 9.5.

10. L-functions and homology

Consider a geodesic flow $\phi_t : M \to M$ on the unit tangent bundle M of a negatively curved manifold. To understand the growth of closed orbits in a given homology class, it is useful to consider the analogy with an important problem in number theory.

10.1 Sums of squares in number theory. Consider the asymptotic behaviour of the number of integers less than x which can be written as a square or as the sum of two squares, i.e.,

$$B(x) = \#\{1 \leq n \leq x : n = u_1^2 + u_2^2, u_1, u_2 \in \mathbb{Z}\}.$$

Landau [**La**] showed in 1908 that $B(x) \sim Kx/(\log x)^{1/2}$, for some $K > 0$, as $x \to +\infty$, and the same result appears in Ramanujan's famous letter to Hardy in 1913. A fuller asymptotic expansion involves studying the complex function

$$f(s) = \frac{1}{1 - 2^{-s}} \prod_q \frac{1}{1 - q^{-s}} \prod_r \frac{1}{1 - r^{-2s}}$$

where q runs through all primes equal to 1 (mod 4) and r runs through all primes equal to 3 (mod 4). Observe that this differs from the Riemann zeta function only in the factor of 2 in the last exponent, but this difference results in there being a singularity of the form $(s-1)^{-1/2}$. The asymptotic expansion for $B(x)$ has the following simple form. For every $N \geq 1$

$$B(x) = \frac{Kx}{(\log x)^{1/2}} \left(1 + \sum_{n=1}^{N} \frac{\alpha_n}{(\log T)^n} + O\left(\frac{1}{(\log x)^N}\right)\right)$$

[**Hd, pp.61-63**].

10.2 Counting geodesic flows in homology classes. We now return to the geometric analogue of the number theory result described above. For a compact n-dimensional manifold V with negative sectional curvatures, we want to count the number of closed geodesics γ lying in a given homology class. However, despite the analogy, the nature of the singularity in the complex function associated to our problem is somewhat more complicated and the detailed arguments are consequently more elaborate.

Let $\alpha \in H_1(V, \mathbb{Z})$ be a fixed element in the first homology. Let $[\gamma]$ be the homology class associated to a closed geodesic V and let $\pi(T)$ be the number of closed geodesics in the homology class α of length at most T, i.e.,

$$\pi(T, \alpha) = \#\{\gamma : l(\gamma) \leq T, \quad [\gamma] = \alpha\}.$$

The following formula is due to Anantharaman [**An**] and Pollicott-Sharp [**PS3**].

THEOREM 10.2. *Let $b = dim(H_1(M, \mathbb{R}))$ be the first Betti number for V. There exist $C_0, C_1, C_2, \ldots$ (with $C_0 > 0$) such that*

$$\pi(T, \alpha) = \frac{e^{hT}}{T^b} \left(\sum_{n=0}^{N} \frac{C_n}{T^{n/2}} + o\left(\frac{1}{T^{N/2}} \right) \right)$$

for any $N > 0$.

For surfaces of constant curvature $\kappa = -1$ this was proved by Phillips and Sarnak in 1987 (when $b = 2g$, where g is the genus of the surface). Katsuda-Sunada [**KS1, KS2**], Lalley [**Ly**] and Pollicott [**Po3**] each subsequently showed that the basic asymptotic $\pi(T, \alpha) \sim \frac{e^{hT}}{T^{b/2}}$ still held for variable curvature. Kotani has more detailed estimates on the constants C_n, and Babillot and Peigne have generalizations to non-compact surfaces.

The proof of Theorem 9.4 is very similar in nature to that of Theorem 10.2. We again need to study a suitable complex function, which in this case is

$$\eta(s) = \sum_{\gamma} \sum_{\substack{n=1 \\ n[\gamma]=\alpha}}^{\infty} (-hl(\gamma))^{g+1} n^g e^{-shnl(\gamma)},$$

where we let $g = [b/2]$. The function $\eta(s)$ converges to an analytic function which converges to an analytic function on $Re(s) > 1$. The complex function $\eta(s)$ has a somewhat more complicated singularity at $s = 1$ then the simple poles we saw for $\zeta(s)$ and $\zeta'(s)/\zeta(s)$.

PROPOSITION 10.1. *$\eta(s)$ has an analytic extension to a neighbourhood of $Re(s) \geq 1$, except at $s = 1$, where:*

1. *if b is even then there exist analytic functions $f_1(s)$ and $f_2(s)$ such that*

$$\eta(s) = f_1(s) + \frac{c_0}{s-1} + \frac{c_1}{(s-1)^{1/2}} + f_2(s) \log(s-1) + \sum_{n=0}^{\infty} a_n (s-1)^{n+1/2};$$

2. *if b is odd then there exist analytic functions $f_1(s)$ and $f_2(s)$ such that*

$$\eta(s) = f_1(s) + \frac{c_1}{(s-1)^{1/2}} + f_2(s) \log(s-1) + \sum_{n=0}^{\infty} a_n (s-1)^{n+1/2}$$

(for some $a_0, a_1, a_2, \dots$).

In this context it is convenient to define a counting function

$$\psi(T,\alpha) = \sum_{n=1}^{\infty} \sum_{\substack{N(\gamma)^n \le T \\ n[\gamma]=\alpha}}{}' \, (h^* l(\gamma))^{g+1} \, n^g$$

where the prime denotes that for terms where we have an equality $N(\gamma)^n = T$ the summand is multiplied by $1/2$. In particular, for $d > 1$ we can write

$$\psi(T,\alpha) = \frac{1}{2\pi i} \int_{d+i\infty}^{d-i\infty} \frac{\eta(s)}{s} T^s ds$$

Again, the asymptotic formulae depend on moving the curve of integration to the left. *However, because of the essential singularity at $s = 1$ this point must stay on the left of the path of integration* giving a more complicated curve of integration. We refer to [**PS3**] for details.

10.3 Error terms for Anosov flows. Again, as in the case of estimates for $\pi(T)$, we still have a mild asymptotic error term. We need the following technical assumption (which automatically holds for the case of geodesic flows). We say that the flow ϕ is *homologically full* if every element of $H_1(M,\mathbb{Z})$ contains at least one periodic orbit γ.

Given $\alpha \in H_1(M,\mathbb{Z})$ we count the number of orbits lying in γ, of least period at least T,

$$\pi(T,\alpha) = \#\{\gamma : l(\gamma) \le T, [\gamma] = \alpha\}.$$

The next result gives a partial generalization of Theorem 10.2 to Anosov flows [**PS3**].

THEOREM 10.3. *Let M be a compact manifold with first Betti number $b > 0$. Let $\phi_t : M \to M$ be a homologically full, weak mixing, transitive Anosov flow. There exists $C_0 > 0$, $0 < h^* \le h$ and $\delta > 0$ such that*

$$\pi(T,\alpha) = \frac{C_0 e^{h^* T}}{T^{b/2+1}} \left(1 + O\left(\frac{1}{T^\delta}\right)\right)$$

for any $N > 0$.

11. Pole free regions for the zeta function

We want to explain the ideas behind the the proof of the results Proposition 10.1 and 10.2 on the domain of the zeta function. Indeed, these results have been seen to be central to the proofs of Theorems 9.3 and 10.2.

11.1 Transfer operators and geodesic flows. Geodesic flows on the unit tangent bundle to a negatively surface V are canonical examples of Anosov flows $\phi_t : M \to M$ on the compact three dimensional unit tangent bundle with the following additional properties:

(a) The two foliations of M by stable and unstable manifolds are C^1;
(b) The flow is contact (i.e. the stable and unstable manifolds foliations are not integrable).

(cf. [**Ch**]). In particular, the proof shows that estimate (9.3) hold for any Anosov flows with these extra properties.

Using Markov sections we can get a well-defined expanding interval map $f : I \to I$ and an associated function $r : I \to \mathbb{R}$ by the steps (1)-(3) in §7. The only difference in the present setting is that the foliation is merely C^1 (rather than C^ω) and the resulting map $f : I \to I$ and function $r : I \to \mathbb{R}$ are also only C^1. These important properties for $f : I \to I$ are collected together in the following lemma.

LEMMA 11.1. *For a geodesic flow on a negatively curved surface:*
1. $f : I \to I$ *is* C^1 *(since the foliation* $\mathcal{W}^s$ *is* C^1*); and*
2. *there exists* $0 < \gamma < 1$ *such that* $|f'(x)| \geq \frac{1}{\gamma}$ *for all* $x \in I$ *(since the* ϕ_t *expands distances on unstable manifolds).*

For geodesic flows we can consider the transfer operators acting on C^1 functions:

$$\begin{cases} \mathcal{L}_{-shr} : C^1(I) \to C^1(I) \\ \mathcal{L}_{-shr} w(x) = \sum_{fy=x} e^{-shr(y)} w(y). \end{cases}$$

The connection between transfer operators and the zeta function comes from the following result of Ruelle [**Ru4**] (by comparing the first term with the expansion for the zeta function in (6.4))

LEMMA 11.2. *For each* $1 \leq i \leq k$ *fix choices* $x_i \in I_i$. *There exists* $\epsilon > 0$, $0 < \theta < 1$ *and* $C > 0$ *such that*

$$\left| \sum_{f^n x = x} e^{-sh(r(s)+r(fx)+\ldots+r(f^{n-1}x))} - \sum_{i=1}^{k} \mathcal{L}^n_{-shr} \chi_{I_i}(x_i) \right| \leq C \left(\theta e^{P(-Re(s)hr)} \right)^n$$

for $Re(s) > 1 - \epsilon$

We need to convert property (b) of the geodesic flow (and the foliations $\mathcal{W}^s$ and $\mathcal{W}^u$) into a property for the interval map $f : I \to I$. Given $n \geq 1$ we want to consider for each $x \in I$ a pair of f^n-preimages y_1 and y_2 with $x = f^n(y_1) = f^n(y_2)$. In particular, if we fix a value x_0 with f^n-preimages y_1^0 and y_2^0 then for a nearby value x (and corresponding pre-images y_1 and y_2) consider the quantity

$$\Delta(x, x^0) = (r^n(y_1) - r^n(y_2)) - \left(r^n(y_1^0) - r^n(y_2^0)\right),$$

for x sufficiently close to x^0. With such choices the maps $I_i \ni x \mapsto \Delta(x, x^0)$ are C^1. Moreover, one deduces from (b) that these maps are locally strictly monotonic and there exist constants $B_1, B_2 > 0$ with

$$B_1|x - x^0| \leq |\Delta(x, x^0)| \leq B_2|x - x^0|. \tag{11.1}$$

The bound (11.1) is important in proving the following estimate [**D2**].

LEMMA 11.3. *There exist constants* $\epsilon > 0$, $C > 0$, $A > 0$ *and* $0 < \rho < 1$ *such that whenever* $s = \sigma + it$ *(with* $|\sigma| \leq \epsilon$, $t \neq 0$*) and* $n = k[A \log|t|] + l$ *(*$k \geq 0$, $0 \leq l \leq [A \log|t|] - 1$*) then*

$$||L^n_{-(\sigma+it)r}||_{C^1} \leq C \left(\rho e^{P(-\sigma r)} \right)^{k[A \log t]} . ||L^l_{-\sigma r}||_{C^1}$$

Combining the estimates in Lemmas 11.2 and 11.3 leads to the proof of Proposition 9.1.

11.2 Transfer operators and Anosov flows. For more general Anosov flows we can consider the transfer operators associated to Hölder continuous functions on subshifts of finite type.

$$\begin{cases} \mathcal{L}_{-sh\bar{r}} : C^\alpha(X_A^+) \to C^\alpha(X_A^+) \\ \mathcal{L}_{-sh\bar{r}} w(\underline{x}) = \sum_{\sigma \underline{y} = \underline{x}} e^{-sh\bar{r}(\underline{y})} w(\underline{y}). \end{cases}$$

The main result on the norm of the operator in this context is the following weaker version of Lemma 11.3.

LEMMA 11.4. *There exist constants $\tau > 0$, $C_0 > 0$, and $C > 0$, such that whenever $s = \sigma + it$ and $|t| \geq 1$,*

$$||\mathcal{L}_{-(\sigma+it)hr}^{2Nn}|| \leq C_0 |t| ||\mathcal{L}_{-\sigma r}^{2Nn}||_{C^\alpha} \left(1 - \frac{1}{|t|^\tau}\right)^{n-1}, \quad \forall n \geq 1,$$

where $\sigma \geq 1 - \frac{1}{|t|^\rho}$ and $N = [C \log |t|]$.

In this case the connection between the transfer operator and the zeta function corresponding to Lemma 11.3 is the following.

LEMMA 11.5. *For each cylinder $[i] = \{\underline{x} \in X_A^+ : x_0 = i\}$ fix a choice $\underline{x}_i \in [i]$. There exists a constant $K > 0$ such that*

$$\left| \sum_{\sigma^n \underline{x} = \underline{x}} e^{-sh(\bar{r}(\underline{x}) + \bar{r}(\sigma \underline{x}) + \ldots + r(\sigma^{n-1} \underline{x}))} - \sum_{i=1}^k \mathcal{L}_{-sh\bar{r}}^n \chi_{[i]}(\underline{x}_i) \right| \leq K |t|^2 ||L_{-\sigma h \bar{r}}^n|| \left(1 - \frac{1}{|t|^\tau}\right)^{\frac{n}{2N}},$$

where $N = [A \log |t|]$ for some $A > 0$.

Comparing Lemmas 11.4 and 11.5 (and equation (6.4)) leads to a proof of Proposition 9.2.

REMARK ON POINCARÉ SERIES. To complete this section, we shall briefly describe a closely related complex function, used in proving results such as Theorem 9.4. The associated complex function in the proof of Theorem 9.4 is the Poincaré series

$$\eta(s) = \sum_{g \in \Gamma} e^{-shd(x, gx)}, s \in \mathbb{C}$$

(which converges to an analytic function for $Re(s) > 1$).

Using a completely different approach, rooted in geometric group theory, we can associate a (non-transitive) subshift of finite type $\sigma : X_B^+ \to X_B^+$ and a Hölder continuous function $R : X_B^+ \to \mathbb{R}$ [**PS1, PS2**].

LEMMA 11.6. *We can write*

$$\eta(s) = 1 + \sum_{n=1}^\infty \mathcal{L}_{-sR} \chi_{[*]}(\underline{x}_0),$$

where $[*]$ *is a the one cylinder associated to a distinguished state* $*$ *and* $\underline{x}_0 \in X_B^+$ *is a distinguished sequence.*

Using a similar sort analysis for $\eta(s)$ as we used for $\zeta'(s)/\zeta(s)$ one can derive Theorem 9.4 [**PS2**].

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