PROOF OF THE RIEMANN HYPOTHESIS

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PRELIMINARY NOTES

Abstract

To prove the hypothesis, we relate the zeta function and its conjugate, under conditions that satisfy their functional and symmetrical relations. In geometrical words, we compare two curves, related to the same complex variable *s*, at their zero intersection point, to check if there is a constant behaviour of the real part of *s*. The result is a simple equation with a direct and unique solution of the problem.

Introduction

The purpose of this paper is to prove the *Riemann hypothesis*. We won't explain here the characteristics of the *zeta function* $\zeta(s)$, or the more generalised *L-functions*, for there is an extraordinary collection of documents about it, and for our proof, we need only the most elementary properties of $\zeta(s)$.

The *Riemann zeta function* is a function of complex variable *s*, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s \in \mathrm{C}, n \in \mathrm{N})$$

and in the whole complex plane by analytic continuation. The function has zeros at complex values of *s* (*nontrivial zeros*) and at the negative even integers (*trivial zeros*), and satisfies the functional equation

$$\Phi(s)\,\zeta(s) = \Phi(1-s)\,\zeta(1-s) \tag{1}$$

where $\Phi(z) = \pi^{-z/2}\,\Gamma(\frac{z}{2})$.

The ***Riemann Hypothesis*** states that

The nontrivial zeros of $\zeta(s)$ *have real part equal to* $\frac{1}{2}$.

Proof

From now on, we will concentrate on the behaviour of the variable *s* when $\zeta(s) = \underline{0}$ and we will use the notation $\underline{0}$ to underline that it is a nontrivial zero, a zero where $\Im(s) \neq 0$.

It is obvious that the functions Φ of the equation (1) never have zero values

$$\forall s \in \mathrm{C}, \quad \Phi(s) \neq 0 \ \wedge \ \Phi(1-s) \neq 0 \tag{2}$$

Then, by (1) and (2) we can state the well-known *functional property* of $\zeta(s)$

$$\zeta(s) = \underline{0} \quad \Leftrightarrow \quad \zeta(1-s) = \underline{0} \tag{3}$$

Another characteristic of $\zeta(s)$, as a function of complex variable, is that it satisfies the reflection principle

$$\forall s \in \mathrm{C} \quad \zeta(\bar{s}) = \overline{\zeta(s)}$$

Probably, we can follow here different ways to obtain the same results, but we prefer the selected one, for its easy geometrical visualisation.

We use the conjugation to define the *conjugate function* $\bar{\zeta}(s)$ of $\zeta(s)$

$$\bar{\zeta}(s) = \sum_{n=1}^{\infty} \frac{1}{n^{\bar{s}}} \qquad (s \in \mathrm{C}, n \in \mathrm{N})$$

Note that here, we introduce a different function and we expressly write $\bar{\zeta}(s)$ and not $\overline{\zeta(s)}$, to underline that it is another function. It is evident that the values of $\bar{\zeta}(s)$ are always the conjugates of $\zeta(s)$, but this is only a property between both functions, a consequence of their conjugate relation. It is also immediate that it satisfies the same functional equation but in conjugate form

$$\overline{\Phi}(s)\, \bar{\zeta}(s) = \overline{\Phi}(1-s)\, \bar{\zeta}(1-s)$$

that implies

$$\bar{\zeta}(s) = \underline{0} \quad \Leftrightarrow \quad \bar{\zeta}(1-s) = \underline{0}$$

The functions $\zeta(s)$ and $\bar{\zeta}(s)$ have symmetrical behaviours on the real axis, that is, their respective real parts are equal for the same value s, and their imaginary parts are opposed

$$\forall s \in \mathrm{C}, \quad \Re(\zeta(s)) = \Re(\bar{\zeta}(s)) \quad \wedge \quad \Im(\zeta(s)) = -\Im(\bar{\zeta}(s))$$

Note that this allows the Riemann Hypothesis to be stated in terms of $\bar{\zeta}(s)$. For our purposes, an important consequence of this *symmetrical property*, is their absolute coincidence at the zero output (coming from two opposed imaginary ways). Since $\zeta(s) = 0 + i0 = 0 - i0$

$$\zeta(s) = \underline{0} \quad \Leftrightarrow \quad \bar{\zeta}(s) = \underline{0} \tag{4}$$

At this zero point, we can construct an equated equation system that relates both functions and their respective symmetrical and functional properties, from (3) and (4)

$$\zeta(s) = \zeta(1-s) = \bar{\zeta}(s) = \bar{\zeta}(1-s) = \underline{0} \tag{5}$$

Now, we can solve the system (5) by taking from it two members. We can try it in several ways, but they *must* represent all the characteristics of both functions, for it is obvious that if we don't combine all their properties we will obtain indeterminate results (these other equations have results where $\Im(s) = 0$. Perhaps, the trivial zeros manifestation).

Since we also must consider $1-s$ as a function $\lambda(s) = 1-s$, the functions $\zeta(s)$, $\overline{\zeta}(s)$ and $\lambda(s)$ have to be involved in this equation. Then

$$\boxed{\zeta(s) = \overline{\zeta}(\lambda(s))} \tag{6}$$

that implies that s and $\lambda(s)$ are conjugate values

$$\zeta(s) = \overline{\zeta}(\lambda(s)) \quad \Rightarrow \quad \overline{s} = \lambda(s) \quad \equiv \quad \overline{s} = 1-s$$

and with $s = r + it$ we have

$$\overline{s} = 1-s \quad \equiv \quad r - it = 1-(r+it) \quad \Rightarrow \quad r - it = 1 - r - it \quad \Rightarrow \quad r = 1 - r$$

$$\Rightarrow \quad 2r = 1 \quad \Rightarrow \quad r = \frac{1}{2}$$

Note that we will obtain the same result, equating the other two descriptive members of (5), $\overline{\zeta}(s) = \zeta(\lambda(s))$.

Since $\zeta(s)$ is an injection on C, there are no other different real parts of s, with the same imaginary absolute value $|\Im(s)| \neq 0$, that satisfies the system (5), where $\zeta(s) = \underline{0}$

$$\forall \rho \neq \Re(s) \in \mathrm{R} : \quad \zeta(\rho \pm \Im(s)) \neq \underline{0} \quad \wedge \quad \overline{\zeta}(\rho \pm \Im(s)) \neq \underline{0}$$

Then, $\frac{1}{2}$ is the unique real value that satisfies the equation (6), that is, *the nontrivial zeros of* $\zeta(s)$ (and $\overline{\zeta}(s)$) *have real part equal to* $\frac{1}{2}$.

This result proves explicitly the *Riemann hypothesis*

$$\boxed{\zeta(s) = \underline{0} \quad \Rightarrow \quad \Re(s) = \frac{1}{2}}$$

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