FROM RIEMANN HYPOTHESIS TO RIEMANN THESIS

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Abstract:

This paper describes the Riemann (hypo)Thesis.

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1. THE RIEMANN ZETA FUNCTION

From studies of Gauss, Dirichlet, Euler the Riemann zeta function is defined as follows

$$\mathbf{Z(p)} = \prod_{n\text{-} \ primo} \left(1 - \frac{1}{n^p}\right) = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

valid for Re (s) > 1, and where the product is made of all primes p up to infinity.

Riemann extended the function of the complex plane through an analytic continuation using another property that satisfy zeta function through functional equation:

$$\mathbf{Z(p) = 2^p \pi^{p-1} \sin(\pi p/2)\, \acute{\Gamma}(1-p)\, Z(1-p)}$$

where Γ(p) is the Gamma function. This formula is an equality between functions valid over the entire complex plane. For p of negative real part, all the functions on the right of equality does not have poles and the even integers, the sine function has simple zeros; from this it follows that the zeta function has simple zeros (called trivial zeros) in the negative integers.

From functional equation we have:

$$\mathbf{Z(-2)=Z(-4)=Z(-6)=Z(-8)=....=0}$$

In addition, from Euler product we have:

Z(p) ≠0 for Re p > 1

The Euler product has as an immediate consequence of the Riemann zeta function has no zeros in the half-plane Re (p) > 1. Moreover, thanks functional equation, from this it follows that the only ones that the zeta function has zeros in the half-plane Re (p) < 0 are the trivial zeros. The remaining zeros can then be only in the strip 0 ≤ Re (p) ≤ 1.
Furthermore, the zeta function has no zeros in either straight Re (p) = 1 (and therefore, for the functional equation, even in Re(*p*) = 0).
In particular, all non-trivial zeros of the zeta function are in the strip 0 < Re (p) <1, which is therefore called the critical strip.
Another fundamental property of the zeta function is the position of symmetry with respect to the x axis (Im = 0 for the complex plane) of its values and as a result of its zeros.
If we choose a value z = a + b its value complex conjugate is z * = a-jb.
It is therefore important that equality:

Z(a+jb)=Z*(a-jb)

or even

Z(a-jb)=Z*(a+jb)

This also means that if a+jb≠a-jb≠1

Z(a+jb)=c+jd

then it is also

Z(a-jb)=c-jd

If we consider that a + jb is one of the zeros of the zeta function we have

Z(a+jb)=0

and then we also have

Z(a-jb)=0

and at the end

Z(a+jb)=Z(a-jb)=0

so that "automatically" we also have z = a-jb is a zero.

If the zeta function has a zero for some value of the variable z, there is also zero z* complex conjugate.
The zeta function has thus always two zeros to the complex variable z or equivalenttly has always double zeros.
The zeta function Z(p) has only one pole at p = 1 (Im = 0) and is holomorphic in the whole complex plane except at the pole C \ {1}.

2. HYPOTHESIS BECOMES THE THESIS OF RIEMANN

Rewrite the functional equation as follows:

Z(p) = K(p) Z(1-p)

The function K(p) consists of three functions and cannot never be equal to 0.
In fact, for any complex value of p = a + jb, or for the whole complex plane C, we have

2^p≠0

π^(p-1) ≠0

sin (πp/2)#0 only for πp / 2 = hπ and then only for p = 2h, and in fact we have seen that the zeta function vanishes for negative even integers.

Ѓ(1-p)≠0 by the definition of the gamma function .

Then there is always that

K(p)≠0 always has a certain value

then for Z (p) = 0 would be necessary that Z(1-p) = 0.
But we have seen before that if Z (p) = 0 then so is Z(p *) = 0.
The zeta function can not simultaneously have four zeros of which two complex conjugate pairs as to form the vertices of a rectangle (see Figure 1 below).
The zeros are NOT symmetrical with respect to Re p = 1/2, but they are only with respect to p (1/2, 0) and the line Re p = 1/2 and of course they always respect to the line Im (s) = 0. Consequently, for every non-trivial zero σ + it there is another in σ - en.

We have a contradiction and this implies that ONLY for Re (p) = 1/2 is actually that

Z(1/2+jb)=Z(1/2-jb)=0

This condition is the only acceptable and is a necessary and sufficient condition and proves unequivocally that all the infinite zeros of the zeta function can only stay on the line Re p = ½. because we always have K (p) ≠ 0.

!We can have 2 zeros but not 4 at the same time!

If K (p) “were” equal to zero then you would

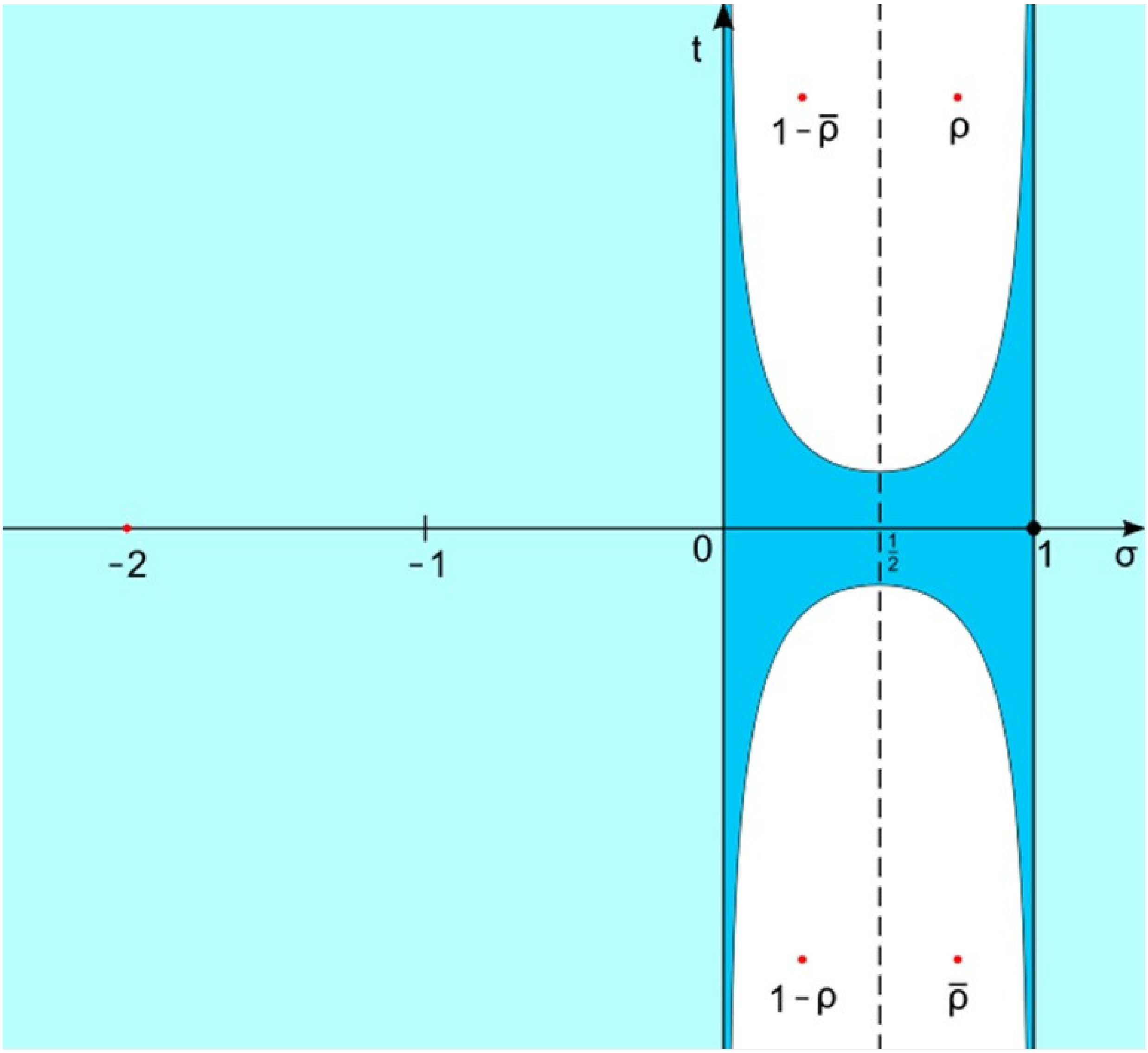
Z(p) = 0*Z(1-p)

Where

Z(1-p) can take on a finite value and would therefore Z(p) = 0 for any value of 0 <Re p <1 and any value Im p .
We would have that the zeros of Z(p) coincide with the zeros of K(p) → Z(p) = K p)=0 and this would not make sense.

But this is NEVER possible!

Figure 1



Example:

for p=(3/4+j10)

Z(3/4+j10)=1,4614…-j0,1141…
Z(3/4-j10)=1,4614…+j0,1141…

Z(1/4+j10)=1,6425…-j0,1117…
Z(1/4-j10)=1,6425…+j0,1117…

Z(a+jb)=c+jd

then it is also:

Z(a-jb)=c-jd

When we draw the graph of the zeta function Z (p) is drawn in the form of Z (p) - see picture below - it is impossible to draw a four-dimensional graph

Figure 2

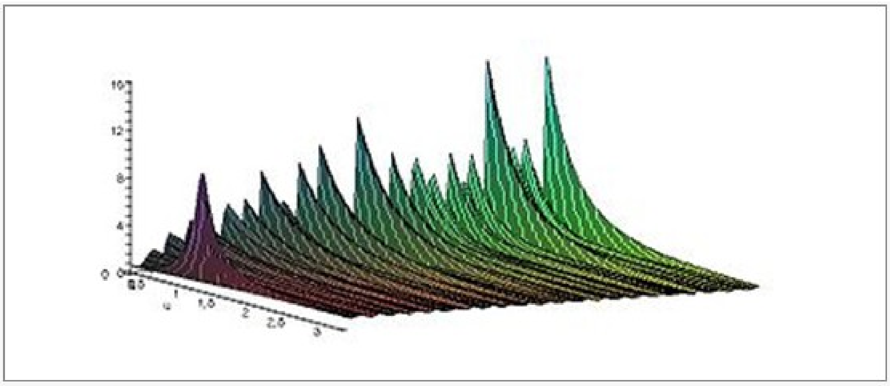


Grafico della funzione di Riemann $\zeta(a+ib)$ per $0 \leq a \leq 3$ e $0,1 \leq b \leq 200$.

However, we can also draw two Cartesian diagrams, one for the values of z = a + jb and another Cartesian diagram, screened and that is related to the 1st, with the values c + jd for the values of Z (a + jb) = c + jd.
In this way we can more easily understand that the values of the vertices of the rectangle are all completely different.
For example, the infinite zeros, all of which are on the critical line Re p = 1/2 are all projected to be the origin of the 2nd Cartesian graph.

3. PRIME NUMBER THEOREM

For any positive real number x, we define the function:

π (x): = number of primes less than or equal to x

The prime number theorem states that:

$$\pi(x) \approx \frac{x}{\ln(x)}$$

This notation meant to signify that the limit of the quotient of the two functions π (x) and x/ln (x) as x approaches infinity is 1; This does not mean, however, that the limit of the difference of the two functions as x approaches infinity is 0.

We can improve this approximation by considering the following relationship:

$$\pi(x) \approx \frac{x}{\ln(x) - 1}$$

Let's see how we improve with a few examples:

x	π(x)	π(x) − x / ln x	π(x) / (x / ln x)	x/(ln(x)-1)	π(x) /(x/ln(x)-1)
10	4	−0.3	0.921	7	0,571
10^{2}	25	3.3	1.151	27	0,926
10^{3}	168	23	1.161	169	0,994
10^{4}	1,229	143	1.132	1217	1,009
10^{5}	9,592	906	1.104	9512	1,008
10^{6}	78,498	6,116	1.084	78030	1,006
10^{7}	664,579	44,158	1.071	661458	1,005
10^{8}	5,761,455	332,774	1.061	5740303	1,004
10^{9}	50,847,534	2,592,592	1.054	50701542	1,003
10^{10}	455,052,511	20,758,029	1.048	454011971	1,002
10^{11}	4,118,054,813	169,923,159	1.043	4110416300	1,002
10^{12}	37,607,912,018	1,416,705,193	1.039	37550193649	1,001
10^{13}	346,065,536,839	11,992,858,452	1.034	345618860220	1,001
10^{14}	3,204,941,750,802	102,838,308,636	1.033	3201414635780	1,001
10^{15}	29,844,570,422,669	891,604,962,452	1.031	29816233849000	1,0009
10^{16}	279,238,341,033,925	7,804,289,844,393	1.029	279007258230819	1,0008
10^{17}	2,623,557,157,654,233	68,883,734,693,281	1.027	2621647966812031	1,0007
10^{18}	24,739,954,287,740,860	612,483,070,893,536	1.025	24723998785919976	1,0006
10^{19}	234,057,667,276,344,607	5,481,624,169,369,960	1.024	233922961602470390	1,0005
10^{20}	2,220,819,602,560,918,840	49,347,193,044,659,701	1.023	2219671974013732243	1,0005
10^{21}	21,127,269,486,018,731,928	446,579,871,578,168,707	1.022	21117412262909985552	1.0004
10^{22}	201,467,286,689,315,906,290	4,060,704,006,019,620,994	1.021	201381995844659893517	1.0004
10^{23}	1,925,320,391,606,803,968,923	37,083,513,766,578,631,309	1.020	1924577459166813514799	1,0003
10^{24}	18,435,599,767,349,200,867,866	339,996,354,713,708,049,069	1.019	18429088896563917716962	1,0003
10^{25}	176,846,309,399,143,769,411,680	3,128,516,637,843,038,351,228	1.018	176788931049963678496485	1,0003

As you can see in tab. 1 improvement is excellent.

Instead, using the approximation made by thei ntegral logarithmic function:

Li(x)= $\int_2^x \frac{1}{\ln t} dt$ **=li(x)-li(2)**

It has a further improvement, however, we also know that this function for n has a very large ratio

$$\frac{\pi(x)}{Li(x)}$$

that is oscillating between 0.9999 and 1.000000....1

That is, the approximation with Li (x) is sometimes higher than the true number of primes π(x) or sometimes instead is lower

In our case it has always for any x:

$$\pi(x) > \frac{x}{\ln(x) - 1}$$

and for x → ∞ we have π (x) = π(x) = $\frac{x}{\ln(x)-1}$

We can also define the lower and upper limits:

$$\frac{x}{\ln(x)-1+\varepsilon} < \pi(x) < \frac{x}{\ln(x)-1-\varepsilon}$$

where ἑ> 0

4. APPROXIMATION FOR n-th PRIME NUMBER p_n

As a result of previous to paragraph we obtain:

$p_n \approx n*(\ln(n)+1)$

best of the approximation classic

$p'_n \approx n*\ln(n)$

Here are some examples:

$p_{100} = 541$

$p_{100} \approx 561$

$p'_{100} \approx 461$

We see a number substantially larger than the 2 10^{17} th prime number:

$p_{2\ 10^{17}}$ = **8512677386048191063**

$p_{2\ 10^{17}}$ ≈ **8167418752291744387**

$p'_{2\ 10^{17}}$ ≈**7967418752291744387**

The approximation is much better.

Also in this case we have that the approximation is always less than the true value of p_n

$p_n > n*(\ln(n)+1)$

We can also define the lower and upper limits:

$n*(\ln(n)+1-\dot{\varepsilon}) < p_n < n*(\ln(n)+1+\dot{\varepsilon})$

where $\dot{\varepsilon}>0$

5. APPROXIMATION FOR the nth ZERO z_n OF ZETA FUNCTION

The approximate number of zeros of the Riemann zeta function under a certain value of x, given that they are all and only all on the line Re p = ½, is given by the following formula:

$$N(x) \approx \frac{x}{2\pi}\left(\ln\frac{x}{2\pi} - 1\right)$$

And then we get:

$$z_n \approx \frac{2\pi n}{\left(\ln\frac{n}{2\pi} - 1\right)}$$

of course there are also all zeros z * complex conjugate

Here are some examples:

N(100)=29

N(100)≈ 28,12

z_{100} = 236.52

z_{100} ≈ 355

N(74921)= 100000

N(74921)≈ 99998,78

z_{74921} = 57909,69

z_{74921} ≈ 56132,24

N(100000)=138069

N(100000)≈138067,68

z_{100000}=74920,82

z_{100000}≈72428

Note that there are many more zeros than primes and the number of zeros grow much faster than the primes.

For example, for x = 100000 and n = 100000 en we have the following values for the prime numbers using the two previous formulas

$\pi(100000)=9592$

$\pi(100000)\approx 9512$

$N(100000)=138069$

$N(100000)\approx 138067,68$

$p_{100000}=1299709$

$p_{100000}\approx 1251292$

$z_{100000}=74920,82$

$z_{100000}\approx 72428$

The relationship between the zeros and prime numbers is as follows:

$$\mathbf{N(x)} \approx \frac{x}{2\pi}\left(\ln\frac{x}{2\pi} - 1\right)$$

$$\boldsymbol{\pi}\mathbf{(x)} \approx \frac{x}{\ln(x) - 1}$$

$$\frac{N(x)}{\pi\,(x)} \approx \frac{1}{2\pi}\left(\ln\frac{x}{2\pi} - 1\right)\left(\ln(x) - 1\right) = \frac{\ln\left(\frac{x}{2\pi\,e}\right)\ln\left(\frac{x}{e}\right)}{2\pi}$$

With this formula, knowing the number of zeros N(x), below a certain limit x, it is possible to calculate the approximate number of the prime counting π(x) and viceversa.

While the relationship between the n-th prime number and n-th zero, is:

$$\mathbf{p_n} \approx \mathbf{n^*(ln(n)+1)}$$

$$\mathbf{z_n} \approx \frac{2\pi\,n}{\left(\ln\frac{n}{2\pi} - 1\right)}$$

$$\frac{p_n}{z_n} \approx \frac{(\ln(n)+1)\left(\ln\frac{n}{2\pi}-1\right)}{2\pi} = \frac{\ln\left(\frac{n}{2\pi e}\right)\ln(ne)}{2\pi}$$

With this formula knowing the n-th prime number p_n is possible to calculate roughly the nth zero z_n and viceversa.

Example:

$$\frac{N(100000)}{\pi\ (100000)} = \mathbf{14{,}39}$$

$$\frac{N(100000)}{\pi\ (100000)} \approx \mathbf{14{,}51}$$

$$\frac{p_n}{z_n} = \mathbf{17{,}34}$$

$$\frac{p_n}{z_n} \approx \mathbf{17{,}27}$$

As we can see from the ratios of the approximate values are very close to the true values

5.1. SPECIAL CASE WITH n=x

Consider the special case with n = x and let's us multiply the approximate formulas in this way:

n=x

$$\pi(x)\ p_x \approx x^2 \frac{\ln(x)+1}{\ln(x)-1}$$

$$N(x)\ z_x \approx x^2$$

Thus we can immediately calculate for example x-th zero z_x knowing the number of zeros N(x) below the threshold x and vice versa.

Oppue we can calculate the n-th prime p_x knowing the number of primes π(x) below the threshold x and vice versa.

The true values deviate very little from the real ones.

Moreover, for x→∞ we have more and more accurate approximations to the real values.

Example:

for x = 1000000

$\pi(1000000)=78498$

$\pi(1000000)\approx 78030,44$

$N(1000000)= 1747146$

$N(1000000)\approx 1747144,63$

$p_{1000000}=15485863$

$p_{1000000}\approx 14815510,55$

$z_{1000000}=600269,67$

$z_{1000000}\approx 572362,45$

$\pi(1000000)\ p_{1000000}=1215609273774$

$\pi(1000000)\ p_{1000000}\approx 1000000^2 \ \frac{\ln(1000000)+1}{\ln(1000000)-1} \quad 1156060891289$

$N(1000000)\ z_{1000000}=1048758752861$

$N(1000000)\ z_{1000000}\approx 1000000000000$

Approximate values are good very similar to the true values

If n → ∞ we can approximate the formula

$$\pi(x)\ p_x \approx x^2 \frac{\ln(x)+1}{\ln(x)-1} \quad \text{per } n\to\infty = x^2$$

and then

$$\pi(x)\ p_x \approx N(x)\ z_x = x^2 \qquad \text{per } n\to\infty$$

And we have a correlation of zeros and primes even easier!

Example:

for x = 1e12

π(1e12)=37.607.912.018

π(1e12)≈37.550.193.650

N(1e12)=

N(1e12)≈3.945.951.430.270

p_{1e12}= 29.996.224.275.833

$p_{1\ e12}$≈28.631.021.115.929

z_{1e12}= 267.653.395.647

z_{1e12}≈ 253.424.305.309

$\pi(x)\ p_x$= 1.128.095.363.437.723.227.660.994 ≈ 1e24

N(x) z_x ≈1.056.147.299.370.006.226.858.992 ≈ 1e24

So we have a better approximation to x^2=1e24

6. REFERENCES

- **Wikipedia**