A proof of the Riemann Hypothesis

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July 21, 2011

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Abstract

In this paper, we give a proof of the Riemann Hypothesis.
Also, we discuss in details on a weak part of our argument.

1 Introduction

Let $\zeta(s)$ denote the Riemann ζ-function.

In this paper, we give a proof of the following theorem.

Theorem 1. *All the nontrivial zeros of the Riemann ζ-function have the real part equal to* $1/2$.

The key lemma to a proof of Theorem 1 is the following integral inequality:

Lemma 1. *Let*

$$U(x+iu) := \frac{\zeta(1-x-iu)}{|\zeta(1-x-iu)|}$$

and

$$\eta(x+iu) := \pi^{-(x+iu)/2}\Gamma[(x+iu)/2].$$

There is a constant A_β, depending on β, such that

$$\begin{aligned} &\left|\int_{-\infty}^{\infty}(1-\beta^{(1-x)/2+iu/2})\frac{\zeta'(x-iu)U(x+iu)}{|\zeta(1-x-iu)|}|\eta(x+iu)|^2du\right| \\ &\leq A_\beta\sum_{q\geq 1}\frac{\log q}{q^{1+2(x-1/2)}}, \end{aligned} \tag{1}$$

valid for all $x \in (1, 3/2)$ and all $\beta > 0$.

A proof of this integral inequality is given in Section 3.

The following lemma is used in the proof of Lemma 1.

Lemma 2. *For any $m, n > 0$ and any x_1, x_2 with $\Re x_1, \Re x_2 > 0$, we have*

$$\begin{aligned} &\int_{-\infty}^{\infty} m^{-x_1+it}n^{-x_2+it}\Gamma(x_1+it)\Gamma(x_2+it)dt \\ &= 2\pi m^{-2x_1}n^{-x_2-x_1}\alpha(m,n;x_1,x_2), \end{aligned} \tag{2}$$

where

$$\alpha(m,n;X,Y) := \int_0^\infty e^{-v(nm)^{-1}}e^{-v^{-1}}v^{X-Y-1}dv.$$

We prove this lemma in Section 2.

In Section 4, we complete a proof of the Riemann Hypothesis using what have been shown.

In Section 5, we make a note on our argument, which seems to be unavoidable for the validation of this paper.

Since Lemma 2 is founded on the fact that the Γ-function $\Gamma(x+it)$ is a rapidly decreasing function of t for each fixed $x > 0$, we end this section by giving a proof of this fact, for the sake of completeness.

Lemma 3. *For each fixed $x > 0$, $\Gamma(x+it)$ is a rapidly decreasing function of the real variable t.*

Proof. The integral representation of the Γ-function is

$$\Gamma(x+iy) = \int_0^\infty e^{-t}t^{x+iy-1}dt.$$

In this equation, we make the change of variable $t = e^{-2\pi u}$, and obtain

$$\Gamma(x+iy) = \int_{-\infty}^{\infty} 2\pi e^{-e^{-2\pi u}} e^{-2\pi x u} e^{-2\pi i y u} du.$$

If we denote the Fourier transform of a function f by

$$\mathcal{F}[f](y) := \int_{-\infty}^{\infty} f(t) e^{-2\pi i y u} du,$$

then the function $\Gamma(x+iy)$ is regarded as the Fourier transform of the function

$$\gamma(t) := 2\pi e^{-e^{-2\pi t}} e^{-2\pi x t} = \gamma_1(t)\gamma_2(t),$$

where x is a fixed positive real number and

$$\gamma_1(t) := 2\pi e^{-e^{-2\pi t}}, \quad \gamma_2(t) := e^{-2\pi x t}.$$

Since the Fourier transform of a rapidly decreasing function is also rapidly decreasing, we prove below that $\gamma(t)$ is rapidly decreasing. Then the lemma follows immediately.

First, we note that

$$\gamma_2^{(k)}(t) = (-1)^k (2\pi x)^k e^{-2\pi x t},$$

and so $\gamma_2^{(k)}(t)$ decreases exponentially as $t \to +\infty$ for each $k \geq 0$, and it increases like $e^{2\pi x|t|}$ as $t \to -\infty$ for each $k \geq 0$.

Second, we have

$$\gamma_1^{(1)}(t) = 2\pi(-1)(-2\pi)e^{-2\pi t}e^{-e^{-2\pi t}},$$

and so $\gamma_1^{(1)}(t)$ decreases like $e^{-2\pi t}$ as $t \to +\infty$ and does so like $e^{2\pi|t|}e^{-e^{2\pi|t|}}$ as $t \to -\infty$.

By induction on k, we find out that $\gamma_1^{(k)}(t)$ has the form

$$\gamma_1^{(k)}(t) = \sum_{i=1}^{n_k} a_{i,k} e^{-2\pi b_{i,k} t} \gamma_1(t),$$

for some positive integers n_k and $b_{i,k}$, and for some real numbers $a_{i,k}$.

Hence, we see that $\gamma_1^{(k)}(t)$ decreases exponentially as $t \to +\infty$ for each $k \geq 0$, and does so in the order of magnitude at most $e^{2\pi|t|}e^{-e^{2\pi|t|}}$ as $t \to -\infty$ for each $k \geq 0$.

Now by Leibniz's formula, we have

$$\gamma^{(k)}(t)=\frac{d^k}{dt^k}[\gamma_1(t)\gamma_2(t)]=\sum_{i=1}^{k}\frac{k!}{i!(k-i)!}\gamma_1^{(i)}(t)\gamma_2^{(k-i)}(t).$$

Since each term of this sum decays faster than $|t|^{-m}$ for any positive integer m, we see that $\gamma(t)$ is a rapidly decreasing function.

As noted above, this establishes the lemma. $\square$

2 Proof of Lemma 2

In this section, we give a proof of Lemma 2.

We begin by the following expression:

$$\int_{-\infty}^{\infty} m^{-x_1+it}n^{-x_2+it}\Gamma(x_1+it)\Gamma(x_2+it)dt. \tag{3}$$

We apply to (3) the multiplication formula [1]

$$\int_{-\infty}^{\infty}\mathcal{F}[f](t)g(t)dt=\int_{-\infty}^{\infty}f(t)\mathcal{F}[g](t)dt,$$

which is valid when f, g are rapidly decreasing functions; this argument is validated by Lemma 3.

In doing so, we recognize on (3)

$$\Gamma(x+it)=\mathcal{F}[2\pi e^{-e^{-2\pi u}}e^{-2\pi xu}](t),$$

and

$$m^{it}=e^{2\pi(\log m/2\pi)it},\quad n^{it}=e^{2\pi(\log n/2\pi)it}.$$

We recall that

$$\mathcal{F}[f(x+h)](\xi)=\mathcal{F}[f](\xi)e^{2\pi ih\xi}$$
$$\mathcal{F}[f(x)e^{-2\pi ixh}](\xi)=\mathcal{F}[f](\xi+h).$$

With these results, we can rewrite (3) as follows:

$$\begin{aligned}
&m^{-x_1}n^{-x_2}\int_{-\infty}^{\infty}m^{it}n^{it}\Gamma(x_1+it)\Gamma(x_2+it)dt\\
&=m^{-x_1}n^{-x_2}\int_{-\infty}^{\infty}e^{2\pi(\log m/2\pi)it}\mathcal{F}[2\pi e^{-e^{-2\pi u}}e^{-2\pi x_1u}](t)\\
&\times e^{2\pi(\log n/2\pi)it}\mathcal{F}[2\pi e^{-e^{-2\pi u}}e^{-2\pi x_2u}](t)dt\\
&=m^{-x_1}n^{-x_2}\int_{-\infty}^{\infty}e^{2\pi(\log m/2\pi)it}\mathcal{F}[2\pi e^{-e^{-2\pi u}}e^{-2\pi x_1u}](t)\\
&\times e^{2\pi(\log n/2\pi)it}\mathcal{F}^{-1}[2\pi e^{-e^{2\pi u}}e^{2\pi x_2u}](t)dt\\
&=4\pi^2m^{-x_1}n^{-x_2}\int_{-\infty}^{\infty}e^{-e^{-2\pi(u+\log m/2\pi)}}\\
&\times e^{-2\pi x_1(u+\log m/2\pi)}e^{-e^{2\pi(u-\log n/2\pi)}}e^{2\pi x_2(u-\log n/2\pi)}du\\
&=4\pi^2m^{-x_1}n^{-x_2}\int_{-\infty}^{\infty}e^{-m^{-1}e^{-2\pi u}}e^{-2\pi ux_1}\\
&\times m^{-x_1}e^{-e^{2\pi u}n^{-1}}e^{2\pi ux_2}n^{-x_2}du.
\end{aligned}\tag{4}$$

(We used the multiplication formula in the third equality.)

Making the change of variable $v = e^{-2\pi u}$, we have

$$\begin{aligned} &4\pi^2 m^{-2x_1} n^{-x_2} \int_{-\infty}^{\infty} e^{-m^{-1}e^{-2\pi u}} e^{-2\pi u x_1} \\ &\times e^{-e^{2\pi u} n^{-1}} e^{2\pi u x_2} n^{-x_2} du \\ &= 2\pi m^{-2x_1} n^{-2x_2} \int_0^{\infty} e^{-vm^{-1}} e^{-v^{-1}n^{-1}} v^{x_1 - x_2 - 1} dv. \end{aligned} \tag{5}$$

In addition, making the change of variable $vn = w$ in the last expression of (5), we get

$$\begin{aligned} &4\pi^2 m^{-2x_1} n^{-x_2} \int_{-\infty}^{\infty} e^{-m^{-1}e^{-2\pi u}} e^{-2\pi u x_1} \\ &\times e^{-e^{2\pi u} n^{-1}} e^{2\pi u x_2} n^{-x_2} du \\ &= 2\pi m^{-2x_1} n^{-x_2 - x_1} \int_0^{\infty} e^{-w(nm)^{-1}} e^{-w^{-1}} w^{x_1 - x_2 - 1} dw. \end{aligned} \tag{6}$$

Combining (4), (5), and (6), we have

$$\begin{aligned} &m^{-x_1} n^{-x_2} \int_{-\infty}^{\infty} m^{it} n^{it} \Gamma(x_1 + it)\Gamma(x_2 + it) dt \\ &= 2\pi m^{-2x_1} n^{-x_2 - x_1} \alpha(m, n; x_1, x_2). \end{aligned} \tag{7}$$

This completes the proof the lemma.

3 Proof of Lemma 1

In this section, we prove Lemma 1.

Choose x to be $0 < \Re x < 1$.

In Lemma 2, we put $x_1 = (1-x)/2$ and $x_2 = x/2$; also, making the change of variable $t = u/2$, we have

$$\begin{aligned} &2^{-1} \int_{-\infty}^{\infty} m^{-(1-x)/2 + iu/2} n^{-x/2 + iu/2} \Gamma[(1-x)/2 + iu/2]\Gamma(x/2 + iu/2) du \\ &= 2\pi m^{-(1-x)} n^{-1/2} \alpha(m, n; (1-x)/2, x/2). \end{aligned} \tag{8}$$

Next, replacing in (8) m and n by $\pi^{-1} r$ and $\pi^{-1} q^2$, respectively, and then multiplying both sides by $2\pi^{-1} r^{1-x} = 2\pi^{-2(1-x)/2} \pi^{-2x/2} r^{2(1-x)/2}$, we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} r^{(1-x)/2 + iu/2} q^{-x+iu} \eta(1 - x + iu)\eta(x + iu) du \\ &= 4\pi^{1/2 + 2(1-x)} q^{-1} \alpha(\pi^{-1} r, \pi^{-1} q^2; (1-x)/2, x/2). \end{aligned} \tag{9}$$

Substituting $r = 1$ and then $r = \beta$ ($\beta > 0$) in (9), and taking the difference of the left and right members of the obtained formulas, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} (1 - \beta^{(1-x)/2 + iu/2}) q^{-x+iu} \eta(1 - x + iu)\eta(x + iu) du \\ &= 4\pi^{1/2 + 2(1-x)} B(x; q) q^{-1}, \end{aligned} \tag{10}$$

where

$$B(x;q) := \alpha(\pi^{-1}, \pi^{-1}q^2; (1-x)/2, x/2) - \alpha(\pi^{-1}\beta, \pi^{-1}q^2; (1-x)/2, x/2).$$

By the factor $1-\beta^{(1-x)/2+iu/2}$, the left integral of (10) has the analytic continuation to the region $1 \leq \Re x < 3$ with its integral representation being kept valid there.

In addition, it is easy to see that the integral for the factor α on the right also converges for each x in the right half-plane $\Re x > 0$.

Thus we see that the equation (10) extends to the region $0 < \Re x < 3$.

From here on, we choose x to be $1 < \Re x < 3/2$.

Multiplying both sides of (10) by $\Lambda(q)$ (the arithmetical Mangoldt Λ-function) and summing over all $q \geq 1$, we get

$$\begin{aligned}&\int_{-\infty}^{\infty}(1-\beta^{(1-x)/2+iu/2})\frac{-\zeta'(x-iu)}{\zeta(x-iu)}\eta(1-x+iu)\eta(x+iu)du \\ &= 4\pi^{1/2+2(1-x)}\sum_{q\geq 1}\frac{\Lambda(q)B(x;q)}{q}.\end{aligned} \tag{11}$$

At this point, we recall that

$$\zeta(x+iu)\eta(x+iu) = \zeta(1-x-iu)\eta(1-x-iu),$$

or

$$\frac{1}{\zeta(1-x-iu)}\eta(x+iu) = \frac{1}{\zeta(x+iu)}\eta(1-x-iu). \tag{12}$$

The complex conjugate of the left member of (12) is that of the right member of the same equation. Hence, we see that

$$\frac{1}{\zeta(1-x-iu)}\eta(x+iu)\frac{1}{\zeta(x-iu)}\eta(1-x+iu) = \frac{1}{|\zeta(1-x-iu)|^2}|\eta(x+iu)|^2.$$

Using this in (11) and noting that

$$1 = \frac{\zeta(1-x-iu)}{\zeta(1-x-iu)},$$

we have

$$\begin{aligned}&\int_{-\infty}^{\infty}(1-\beta^{(1-x)/2+iu/2})\frac{-\zeta'(x-iu)U(x+iu)}{|\zeta(1-x-iu)|}|\eta(x+iu)|^2du \\ &= 4\pi^{1/2+2(1-x)}\sum_{q\geq 1}\frac{\Lambda(q)B(x;q)}{q},\end{aligned} \tag{13}$$

where

$$U(x+iu) := \frac{\zeta(1-x-iu)}{|\zeta(1-x-iu)|}.$$

Now we give an estimate for the factor $B(x;q)$.

We begin with the definition of $B(x;q)$

$$\begin{aligned}B(x;q) &= \alpha(\pi^{-1}, \pi^{-1}q^2; (1-x)/2, x/2) - \alpha(\pi^{-1}\beta, \pi^{-1}q^2; (1-x)/2, x/2) \\ &= \int_0^{\infty}[e^{-v\pi^2q^{-2}} - e^{-v\pi^2\beta^{-1}q^{-2}}]e^{-v^{-1}}v^{-1/2-x}dv.\end{aligned}$$

Making the change of variable $u = vq^{-2}$, we have

$$B(x;q) = q^{1-2x}\int_0^{\infty}[e^{-u\pi^2} - e^{-u\pi^2\beta^{-1}}]e^{-u^{-1}q^{-2}}u^{-1/2-x}du. \tag{14}$$

If we use the bound

$$|e^{-u\pi^2} - e^{-u\pi^2\beta^{-1}}|u^{-1} \begin{cases} \leq c & : u \in (0,1) \\ \leq |e^{-u\pi^2} - e^{-u\pi^2\beta^{-1}}| & : u \in [1,\infty), \end{cases}$$

in the integral of (14), then we find out that

$$\left| \int_0^\infty [e^{-u\pi^2} - e^{-u\pi^2\beta^{-1}}] e^{-u^{-1}q^{-2}} u^{-1/2-x} du \right|$$
$$\leq \int_0^1 cu^{1/2-x} du + \int_1^\infty |e^{-u\pi^2} - e^{-u\pi^2\beta^{-1}}| u^{1/2-x} du, \quad x \in (1/2, 3/2).$$

Since both integrals on the right are finite values whenever $x \in (1/2, 3/2)$, and they do not depend on q, we have

$$B(x;q) = O(q^{1-2x}), \quad x \in (1/2, 3/2), \tag{15}$$

where the absolute constant depends only on β.

By (15), the equation (13) becomes

$$\int_{-\infty}^\infty (1 - \beta^{(1-x)/2+iu/2}) \frac{-\zeta'(x-iu)U(x+iu)}{|\zeta(1-x-iu)|} |\eta(x+iu)|^2 du$$
$$= O\left(\sum_{q\geq 1} \frac{\log q}{q^{1+2(x-1/2)}} \right), \quad x \in (1, 3/2).$$

This completes the proof of Lemma 1.

4 Proof of Theorem 1

In this section, we complete a proof of Theorem 1.

There are mainly two more things to accomplish.

They are expressed in the following lemmas:

Lemma 4. *Let $1/2+\delta-iT$ be a nontrivial zero of the Riemann ζ-function and $\epsilon > 0$ arbitrary. Then the following integral*

$$\int_{T-\epsilon}^{T+\epsilon} \frac{\zeta'(x-iu)}{|\zeta(1-x+iu)|} du$$

diverges to ∞ in magnitude as $x \to 1/2+\delta$.

Lemma 5. *For any fixed real numbers a and τ, there exists a constant B_β depending only on $\beta > 0$ such that*

$$\left| \int_{-\infty}^\infty (1 - \beta^{(1-x)/2+iu/2}) \zeta(x - 1/2 + a - iu + i\tau) \eta(x+iu) \eta(1-x+iu) du \right|$$
$$\leq B_\beta \sum_{q=1}^\infty \frac{1}{q^{3/2-a+2(x-1/2)}}. \tag{16}$$

So let us explain how we use these lemmas.

With the well-known identity

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho_n} \left[\frac{1}{s-\rho_n} + \frac{1}{\rho_n} \right] - \frac{1}{s-1} + R(s), \tag{17}$$

where the sum runs over all the nontrivial zeros of the Riemann ζ-function, and $R(s)$ increases in the order of magnitude at most $\log|t|$ as $|t| = |\Im(s)| \to \infty$, and it is analytic in the right half-plane $\Re x > 0$, we could rewrite the integral of (11) as

$$\int_{-\infty}^{\infty}(1-\beta^{(1-x)/2+iu/2})\frac{-\zeta'(x-iu)}{\zeta(x-iu)}\eta(1-x+iu)\eta(x+iu)du$$
$$= \int_{-\infty}^{\infty}(1-\beta^{(1-x)/2+iu/2})[L_1(x+iu)+L_2(x+iu)]\eta(1-x+iu)\eta(x+iu)du + E, \tag{18}$$

where $L_1(x+iu)$ is a part of the sum on the right of (17) all of whose terms have poles on the closed left half-plane $\Re s \leq 1/2+\delta$ and $L_2(x+iu)$ is the leftover all of whose terms have poles on the open right half-plane $\Re s > 1/2+\delta$; the term E stands for the integral which contains the part $R(s)$ of (17).

First, if we recall the precise expression for the right member of the inequality (1), then it is easy to see that the analytic continuation of the sum

$$\int_{-\infty}^{\infty}(1-\beta^{(1-x)/2+iu/2})L_1(x+iu)\eta(x+iu)\eta(1-x+iu)du$$
$$+\int_{-\infty}^{\infty}(1-\beta^{(1-x)/2+iu/2})L_2(x+iu)\eta(x+iu)\eta(1-x+iu)du$$
$$=: \mathcal{L}_1(x)+\mathcal{L}_2(x)$$

to the half-plane $\Re x > 1/2$ exists.

At this point, we note that by existence of the analytic continuation of $\mathcal{L}_1(x)+\mathcal{L}_2(x)$ to the open half-plane $\Re x > 1/2$ and by Lemma 4, the only two cases where such an existence is possible are

$$|\mathcal{L}_1(x)| \to \infty \qquad \text{and} \quad |\mathcal{L}_2(x)| \to \infty, \quad \text{as } x \to 1/2+\delta,$$

and

$$|\mathcal{L}_1(x)| < \infty \qquad \text{and} \quad |\mathcal{L}_2(x)| < \infty, \quad \text{as } x \to 1/2+\delta.$$

Each of these cases can be treated by the same method below, and so we do not mention a specific choice here.

Since it is hard to get an explicit expression for the analytic continuation of $\mathcal{L}_2(x)$ for $\Re x \leq 1$, we work with the analytic continuation of $\mathcal{L}_1(x)$ whose integral representation is valid for $\Re x > 1/2+\delta$.

Now, if we recall Lemma 4, then all the zeros ρ_n on the vertical line $\Re s = 1/2-\delta$ (the minus sign attached to δ is by the factor $|\zeta(1-x-iu)|^{-1}$) are the *only* zeros which contribute to the ∞-divergence of the term $\mathcal{L}_1(x)$. (Note that $|\zeta(1-x-iu)| = |\zeta(1-x+iu)|$ whenever x is real.) Hence, if we subtract *only one* such diverging term, denoted by $\mathcal{Q}(x)$, from $\mathcal{L}_1(x)$ by means of Lemma 5, then the difference

$$\mathcal{L}_1(x)+\mathcal{L}_2(x)-\mathcal{Q}(x)$$

now diverges to ∞ in magnitude as $x \to 1/2+\delta$.

(In particular, we may choose in Lemma 5, $a=-\delta$ and $\tau = T$.)

A contradiction then follows (since by the right side of Lemma 5, any such term $\mathcal{Q}(x)$ does *not* diverge to ∞ in magnitude as $x \to 1/2+\delta$), which completes the proof of Theorem 1.

In the rest of this section, we prove Lemmas 4 and 5.

4.1 Proof of Lemma 4

In this subsection, we prove Lemma 4.

Using the power series expansions about the point $s = 1/2 + \delta - iT$ and $s = 1/2 - \delta + iT$

$$\begin{aligned}
\zeta'(x - iu) &= \sum_{k \geq n} kc_k[x - iu - (1/2 + \delta - iT)]^{k-1} \\
\zeta(1 - x + iu) &= \sum_{k \geq n} d_k[1 - x + iu - (1/2 - \delta + iT)]^k \\
&= \sum_{k \geq n} (-1)^k d_k[x - 1/2 - iu - \delta + iT)]^k \\
&= \sum_{k \geq n} (-1)^k d_k[x - iu - (1/2 + \delta - iT)]^k,
\end{aligned}$$

respectively, we have for $\epsilon > 0$ sufficiently small and x real numbers,

$$\begin{aligned}
&\int_{T-\epsilon}^{T+\epsilon} \frac{\zeta'(x - iu)}{|\zeta(1 - x + iu)|} du \\
&= \int_{T-\epsilon}^{T+\epsilon} \frac{nc_n[x - iu - (1/2 + \delta - iT)]^{n-1} + o^n}{|d_n[x - iu - (1/2 + \delta - iT)]^n + o^{n+1}|} du \\
&= \int_{T-\epsilon}^{T+\epsilon} \frac{H(x + iu) + o}{|g_0^{-1}||d_n[x - iu - (1/2 + \delta - iT)] + o^2|} du \\
&= \int_{T-\epsilon}^{T+\epsilon} \frac{H(x + iu) + o}{|g_0^{-1}||d_n||x - iu - (1/2 + \delta - iT)|(1 + o)} du \\
&= \int_{T-\epsilon}^{T+\epsilon} \frac{H(x + iu) + o}{|g_0^{-1}||d_n||x - iu - (1/2 + \delta - iT)|} du + O(1);
\end{aligned}$$

here g_0 is defined by

$$|g_0| = |nc_n|,$$

and

$$H(x + iu) = \frac{[x - iu - (1/2 + \delta - iT)]^{n-1}}{|x - iu - (1/2 + \delta - iT)|^{n-1}};$$

in addition, we used the following estimate

$$\frac{1}{1 - r} = 1 + O(r), \quad r < 1$$

in the fourth equality.

Since $|H| = 1$ by definition, and it is continuous, we have either $|\Re H| > 1/4$ or $|\Im H| > 1/4$ for $x + iu$ sufficiently close to the point $1/2 + \delta + iT$.

Finally, since the integral

$$\int_{T-\epsilon}^{T+\epsilon} \frac{1}{|x - iu - (1/2 + \delta - iT)|} du$$

diverges to ∞ as $x \to 1/2 + \delta$, we obtain the lemma.

To see the divergence of the last integral,

$$\int_{T-\epsilon}^{T+\epsilon} \frac{1}{[(x - 1/2 - \delta)^2 + (u - T)^2]^{1/2}} du = \int_{-\epsilon/(x-1/2-\delta)}^{\epsilon/(x-1/2-\delta)} \frac{1}{[1 + v^2]^{1/2}} dv,$$

and so as $x \to 1/2 + \delta$, we have

$$\int_{T-\epsilon}^{T+\epsilon} \frac{1}{[(x - 1/2 - \delta)^2 + (u - T)^2]^{1/2}} du \to \infty. \tag{19}$$

Note 1. By choosing β appropriately, we can make the integrand of (1) nonzero and real at the point $1/2+\delta+iT$. This is required for the ∞-divergence of $\mathcal{L}_1(x)$ caused by a pole, (not by a rare case of divergence by the sum of finite values; and we can cancel out this singularity and get a contradiction as mentioned above) as $x \to 1/2+\delta$, since its imaginary part, whether diverging or not, cancels out with its complex conjugate.
Note 2. The product of H and U also has the same property that $|HU| = 1$. Hence, Lemma 4 holds for our original integrand in (1).

4.2 Proof of Lemma 5

In this section, we give a proof of Lemma 5.

Multiplying both sides of (10) by $q^{-1/2+a-i\tau}$, we have

$$\begin{aligned}&\int_{-\infty}^{\infty}(1-\beta^{(1-x)/2+iu/2})q^{-x-1/2+a+iu-i\tau}\eta(1-x+iu)\eta(x+iu)du\\&=4\pi^{1/2+2(1-x)}B(x;q)q^{-3/2+a-i\tau}.\end{aligned} \tag{20}$$

Summing through all $q \geq 1$, we obtain the lemma.

5 Discussion for our method

In this final section, we make some notes on our argument.

The author was extremely confused by the fact that while Lemma 4 holds, the diverging integral of the same lemma can be actually canceled out by integral of the form appearing in Lemma 5, which, as the right member of the same equation implies, does *not* diverge as $x \to 1/2+\delta$.

In particular, one may argue that our whole argument of this paper is wrong by applying the argument just described above to the zeros on the critical line.

But the proof of Lemma 4 seems to be correct.

We present here one reason why our argument can actually work.

The key point in resolving this difficulty lies in the fact that all the integrals we have dealt with in this paper converges only conditionally, if it can converge, as $x \to 1/2+\delta$.

On the other hand, we note that *if the integrand is shown to have the form*

$$\int_{T-\epsilon}^{T+\epsilon}\frac{1}{|\zeta(x-iu)|}g(x+iu)du, \quad \textit{as } x \to 1/2+\delta \quad (g \neq 0),$$

then the integral does diverge to ∞ *in magnitude as* $x \to 1/2+\delta$. But again, we *can* cancel this diverging integral with ones appearing in Lemma 5, which, on the contrary, converges.

What makes this phenomenon, that is, the same integral expression can be both diverging to ∞ in magnitude or converging, possible is that *the integral expression in question has the property which conditionally-converging series have*; one may recall that a conditionally-converging series can be rearranged so that we can make it converge to any finite number, or even we can make it diverge to ∞ in magnitude.

If we see our confusing situation from this point of view, we can realize that in fact, whether it diverges or not, subtracting the same integral term (with its integrand having a singularity) from one integral term does make the difference 0; that is, in the terminology of rearrangements of conditionally-converging series $\sum a_n$, whether the partial sums of the rearranged series $\sum a'_n(x)$ diverge or not as $x \to 1/2+\delta$, the partial sums of the difference

$$\lim_{N\to\infty}\left[\sum_{i=1}^{N}(a'_n(x)-a'_n(x))\right], \quad \sum_{i=1}^{\infty} a'_n(x)-\sum_{i=1}^{\infty} a'_n(x)$$

is zero for $x > 1/2+\delta$, and so it can not diverge to ∞ in magnitude as $x \to 1/2+\delta$.

We end this discussion by advising the reader to find an error in this paper, and/or to check the argument of this section.

References

[1] E. M. Stein and R. Shakarchi, *Fourier Analysis* (Princeton Lectures in Analysis, Princeton University Press, Princeton, 2003).

[2] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function* (Oxford, 1951).