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RESEARCH ARTICLES

Number Theory as the Ultimate Physical Theory*

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Abstract—At the Planck scale doubt is cast on the usual notion of space-time and one cannot think about elementary particles. Thus, the fundamental entities of which we consider our Universe to be composed cannot be particles, fields or strings. In this paper the numbers are considered as the fundamental entities. We discuss the construction of the corresponding physical theory. A hypothesis on the quantum fluctuations of the number field is advanced for discussion. If these fluctuations actually take place then instead of the usual quantum mechanics over the complex number field a new quantum mechanics over an arbitrary field must be developed. Moreover, it is tempting to speculate that a principle of invariance of the fundamental physical laws under a change of the number field does hold. The fluctuations of the number field could appear on the Planck length, in particular in the gravitational collapse or near the cosmological singularity. These fluctuations can lead to the appearance of domains with non-Archimedean p-adic or finite geometry. We present a short review of the p-adic mathematics necessary, in this context.

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The whole thing is a number
Pythagoras

1. INTRODUCTION

Contemporary superstring theory is a beautiful theory which is capable of explaining many things about the elementary particles (for a review, see [1]). String theory has accumulated many profound ideas such as the dual approach, the Kaluza-Klein approach, and supersymmetry, and to a considerable extent has realized the dreams of theorists about a unified theory and, in particular, it has avoided the appearance of divergences. In contrast to the usual field theory it does not deal with point-like objects, as does the local field theory, but with extended objects such as strings.

Nevertheless there are questions which are difficult to understand in string theory. In particular, gravitational collapse and the cosmological singularity are such problems. The present string theories do not tell us what to do with these problems. It seems that to understand these problems one needs to understand better what is space-time.

In string theory, as in other modern physical theories, the usual conception, going back to Euclid, Riemann and Einstein about space-time as a manifold with a metric, has been unquestioningly accepted. However, the superstring theory deals with very small distances of Planck order and it is very questionable that the standard structure of space-time, based on the notion of real numbers and the usual geometry, could indeed be applicable here. It seems also that a proper understanding of the problems of gravitational collapse and the cosmological singularity cannot be obtained in terms of the standard concept of space-time (for a discussion, see Ref. [2]).

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Thus, one has to abandon the usual concept of space-time as well as the usual concept of quantum field theory or quantum string and must construct a new theory from which the usual quantum theory and the usual concept of space-time follow only in some macroscopic limit. Such a theory was proposed in a previous paper [3] in which a hypothesis on the possible non-Archimedean structure of space-time on the sub-Planckian scale was considered and the corresponding dual approach was initiated. A possible role of the non-Archimedean p-adic space in physics was noted by Vladimirov and the author in Ref. [4].

In this paper this approach will be developed further. In particular it will be discussed in what sense numbers could be the fundamental entities of which we consider our Universe to be composed and the construction of the corresponding physical theory will be outlined. A hypothesis on the possible quantum fluctuations of the number field is also discussed. A short review of the necessary p-adic mathematics in this context is presented.

2. QUANTUM GRAVITY, STRINGS AND NON-ARCHIMEDEAN GEOMETRY

Certainly, on distances of the Planck order, the effects of quantum gravity have to play a crucial role. It is known that gravitational measurement cannot be localized in a sub-Planckian domain (see the discussion of these questions in Refs. [5] and [6]). More exactly it was shown in quantum gravity that in contrast to quantum electrodynamics [7] principal limitations on field measurements arise which follow from the equivalence principle. In particular one has the inequality $\Delta\Gamma \geq l_{\rm Pl}^2/l^3$ where $\Delta\Gamma$ denotes the uncertainty in determination of a connection coefficient, l is the length scale, $l_{\rm Pl} = \left(\hbar G c^{-3}\right)^{-\frac{1}{2}}$ is the Planck length and G is Newton's constant.

Then there is an absolute limitation on length measurements $l \geq l_{\rm Pl}$, so the Planck length is the smallest possible distance that can in principle be measured. The reason is the following. If we want to locate an elementary particle or string we need to have energy greater than the Planck mass $m_{\rm Pl} = \left(\hbar c G^{-1}\right)^{\frac{1}{2}}$. But in such a case the corresponding gravitational field will have a horizon at $r = 2Gm_{\rm Pl}c^{-2} = 2l_{\rm Pl}$, shielding whatever happens inside the Schwarzschild radius. So no information of geometry in a sub-Planckian region is available. We see that in some sense a smallest quantum of space and time exists and it is of the order of the Planck scale. Really this means that we need a new theory instead of the usual quantum gravity or string theory in this high energy region.

Nevertheless, people sometimes hope that a satisfactory theory of quantum gravity can be found by means of a technical trick, i.e., one can hope to join the concept of Riemannian geometry with the quantum theory at a formal level. From our point of view the main problem with quantum gravity is not technical, but is a more fundamental one - we have to understand what quantum space-time on the Planck scale itself means. Note that a string is a curve in the usual space and therefore the string theory cannot help us here.

Indeed, what do we mean by usual space? In fact this mathematical notion expresses our point of view about the motion of solid objects. This question was cleaned up by Poincare [8] and Weyl [9] . To good precision we can consider points, straight lines, planes, etc., to satisfy the axioms of Euclidean geometry (at least in small macroscopic distances). Let us stress that to actually realize these geometrical figures we have to deal with macroscopic bodies. So we can ask ourselves whether it is possible to construct a segment of a straight line, a plane, etc., from electrons and other elementary particles taken in small numbers in such a way that one can move them according to the axioms of Euclidean geometry. The answer is of course no. Therefore the usual meaning of space is inapplicable to them. The notion of space occurs only in the case when we deal with a large number of particles so this is a macroscopic notion.

Our intuitive understanding of properties of space is expressed in the axioms of elementary geometry. Let us consider elementary geometry. Because the geometrical postulates are intuitively self-evident it proved very difficult to compile a complete list of geometrical axioms. This goal was reached at the turn of the 19th century and found its expression in Hilbert's famous "Grundlagen der Geometrie" [10]. Hilbert arranged the axioms in five groups: the axioms of incidence, of order ("betweenness"), of congruence, of parallelity and the Archimedean axiom.

According to the Archimedean axiom any given large segment on a straight line can be surpassed by successive addition of small segments along the same line. Really, this is a physical axiom which concerns the process of measurement. Two different scales are compared in this axiom. If one imagines that the small segment has a size less than the Planck length and the large one has a macroscopic or even cosmological size, then the application of this axiom becomes questionable. As we just discussed, the Planck length is the smallest possible distance that can in principle be measured. So the suggestion emerges to abandon the Archimedean axiom.

How can one construct a physical theory corresponding to a non-Archimedean geometry? In Ref. [3], the following approach was proposed. Ordinary geometry is constructed under the field of real numbers. Here the field means not a quantum field but a mathematical notion, i.e., a set together with operations of addition and multiplication (see the next section for a more detailed description). If we want an essential departure from ordinary geometry and at the same time to retain the correspondence principle we ought to construct the theory on some field which has common features with the real field and at the same time on which it is possible to construct a non-Archimedean geometry.

It has been shown that this approach leads in fact to a unique theory. Indeed any field contains the field of rational numbers or the finite Galois field as a subfield (see the next section). So we have only the two simplest possibilities. The first is to consider the theory on the field of rational numbers and the second one on a finite field. Then, in order to develop mathematical analysis or a physical theory we need a norm. On the field of rational numbers there are only two norms, the usual absolute value and a non-Archimedean p-adic norm (here p is a prime number) . On the finite Galois field, there does not exist a non-trivial norm.

What could the new theory be? Here it is appropriate to recall an old physical question: what are the fundamental entities of which we consider our Universe to be composed? Usually one considers particles (quarks, preons, ...), quantum fields or strings as such entities (cf. an interesting discussion by Weinberg and by Chew of this question in Ref. [11]). But at the Planck scale doubt is cast on the usual notion of space-time and one cannot think about elementary particles. Therefore the fundamental entities cannot be particles, quantum fields or strings. We shall consider numbers as the fundamental entities. Such an answer at first sight may seem rather extraordinary for our time, but it does not look so unexpected if one recalls the physical origin of the underlying mathematical notion of number [8, 9].

What could the mathematical language for a new theory be? The dynamical tool of the usual quantum field theory is in fact adequate only for the macroscopic space-time continuum because it essentially uses real numbers. Fortunately, in this situation it is possible to develop a dynamical theory along the line of dual models. The analytical theory of the S-matrix was forced out by quantum field theory and the dual theory was transformed into string theory. Note here that in spite of the fact that the bootstrap idea has not been effectively realized (except for the two-dimensional case [12]) it looks in general rather attractive. From a mathematical point of view, S-matrix theory, as well as the dual theories, deals with sets of functions with special properties such as analyticity and crossing symmetry [13]. As is known [14], the dual approach is equivalent to the string model over the field of real numbers. But over other fields the situation may be more complicated. It was shown [3] that there is a natural analogue of the dual theory for the case when the p-adic numbers instead of the real numbers are considered. Such a theory can be considered as a predual theory. In principle it is possible to construct a p-adic string theory, but the equivalence of the theory and the p-adic dual theory is not clear; we will discuss this question m the next sections. It has been proposed [3] to consider the p-adic dual theory as a description of the Planck region. This theory was obtained by means of p-adic interpolation from the usual dual theory. Let us stress that the p-adic interpolation is a very restrictive and non-trivial procedure and it is remarkable that the dual theory admits a p-adic interpolation.

When $p \to \infty$ one expects the correspondence with the standard theory to emerge so $1/p$ formally plays the role of a fundamental length. It is possible to consider p-adic differential equations, in particular for gravitational equations. One can speculate that the classical p-adic gravity may be a good approximation to the usual quantum gravity in the sub-Planckian region. It would be interesting to investigate a relation between the p-adic approach and the foamlike picture [15] or the discrete space-time approach [16].

The usual string theory uses point-like fields for the description of the world sheet. Recently Green [17] has proposed to avoid the use of point-like fields by introducing world sheets for world sheets. In order to avoid point-like fields also in the underlying string theory it should be natural to describe the "second" world sheet by another two-dimensional sheet, and so on ad infinitum. It seems that to eliminate point-like structure from the theory we need to avoid the use of the standard geometry of space-time which is based on real numbers.

Recently, de Vega and Sanchez [18] have shown that one has to introduce a cut-off in the string theory when the strings are considered in an accelerated Rindler frame. Perhaps there is a relation between this cut-off and a prime number p in the p-adic string because formally p could be considered as a cut-off. For interesting discussions of quantum gravity and the string theory see also Refs. [6], [19]–[21]. It has been proposed to reformulate the usual string theory in terms of more abstract principles such as universal moduli space [22] or loop space [23]. These formulations use the standard picture of space-time which is based on the real numbers, but they may also be useful for consideration of the p-adic approach.

Now let us discuss the theory on the finite Galois field, which is the second possibility mentioned above. If the number of points is sufficiently large there can be no experimental objections to the world being finite. Note also that this is not a lattice theory. At first look, the theory on the finite field is just opposite to the theory on the continuous field of the p-adic numbers. Riemann states that "for a discrete manifold the principle of measurement is already contained in the concept of this manifold, but that for a continuous one it must come from elsewhere" [24]. But in fact there is a deep relation between the corresponding dual theories [3]. First of all it is remarkable that there is a natural analogue of the Veneziano amplitude in the finite world. It is the Jacobi sum which is well known in number theory. Then if one considers the p-adic Veneziano amplitude for rational points one can use the Gross-Koblitz formula [25] and get the Jacobi sum. Therefore there is a close connection between the p-adic string and the finite string.

In the next section we will give a review of the necessary p-adic mathematics and after that we return to a consideration of the physical problems.

3. p-ADIC MATHEMATICS

3.1. Fields and norms

Let us recall [26] the basic algebraic notions which are needed for the construction of a physical theory. A field K is a set together with two operations, addition and multiplication, such that K is a commutative group under addition, $K\backslash\{0\}$ is a commutative group under multiplication and the distributive law holds. The famous examples of a field are the field of real numbers, the field of complex numbers and the field $\mathbb{Q}$ of rational numbers. They are the infinite fields. The simplest example of a finite field is the integer modulo a prime p, i.e., the field $\mathbb{F}_p$ of numbers $0, 1, 2, ..., p-1$ modulo p for some prime number p, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, where $\mathbb{Z}$ are integers. This is a finite Galois field. The field $\mathbb{F}_2$ (or $\mathbb{Z}_2$) is well known in supersymmetry theory.

If K is a field, it either contains the field of rational numbers $\mathbb{Q}$ as a subfield (in this case it is said to have characteristic 0) or the field $\mathbb{F}_p$ for some prime number p (in this case K is said to have characteristic p). So, there are two simplest fields: $\mathbb{Q}$ and $\mathbb{F}_p$.

Some fields have an additional property of being "ordered" according to a relationship of "greater than" or "less than". For example the field of rational numbers and the field of real numbers are ordered. This order is customarily expressed in terms of the relation $a < b$ (a is less than b). The relation $a < b$ means that the difference $b - a$ is a positive number. Consequently, every property of the relation $a < b$ can be derived from properties of the class of positive numbers. A field K is said to be ordered if it contains a set P of "positive" elements with the additive, multiplicative and trichotomic properties. The real numbers can be described as a complete ordered field. But there is no possible definition of a "positive complex number" that would make the field of complex numbers an ordered field. Moreover, the field of p-adic numbers and the field of characteristic p are not ordered. Nevertheless it is sometimes useful to introduce a partial order for the field of p-adic numbers (see [27]).

Note that the notion of the arrow of time is closely connected with the order of the field of real numbers. Usually, one considers time as the real axis. But in such a case it is difficult to understand the appearance of the arrow of time (see an interesting discussion of this problem in [28], and [29]), because the field of real numbers has a natural order. It seems that only if we try to represent time as a non-ordered field do we get a possibility to understand the appearance of the arrow of the time.

To develop mathematical analysis we need a norm. A norm on the field K is a map, denoted $|\cdot|$, from K to the non-negative real numbers such that

$$1)\ |x+y| < |x| + |y| \text{ (triangle inequality)}\,,$$

$$2)\ |xy| = |x||y|,$$

$$3)\ |x| = 0 \ \text{ if and only if } \ x = 0\,.$$

A basic example of a norm on the field $\mathbb{Q}$ of rational numbers is the usual absolute value. It is known that every non-trivial norm on $\mathbb{Q}$ is equivalent to the usual absolute value or p-adic norm for some prime p.

3.2. p-Adic numbers [26, 30]

The p-adic norm is as follows. Let p be any prime number, $p = 2, 3, 5, ..., 137, \ldots$. Any non-zero rational number $x = M/N$ where M and N are integers can be represented in the form $x = p^f m/n$ where m and n are integers which are not divisible by p and f is an integer. Such a representation is possible because each integer greater than 1 can be written as a product of primes. Then the p-adic norm is $|x|_p = |p^f m/n|_p = p^{-f}$. Examples:

$$|4|_2 = \frac{1}{4}, \quad |\frac{1}{4}|_2 = 4, \quad |4|_3 = 1.$$

We have $|m|_p \leq 1$ for any m an integer.

The p-adic norm is a norm on $\mathbb{Q}$. This norm satisfies not only the triangle inequality but also a stronger inequality

$$|x + y|_p \leq \max(|x|_p, |y|_p).$$

A norm is called non-Archimedean if this inequality holds. Thus, the p-adic norm is a non-Archimedean norm on $\mathbb{Q}$.

The completion of $\mathbb{Q}$ with the usual absolute value is the field of real numbers. The completion of $\mathbb{Q}$ with the p-adic norm is the p-adic number field $\mathbb{Q}_p$. Any p-adic number $x \in \mathbb{Q}_p$ can be represented as the series

$$x = p^f(a_0 + a_1 p + a_2 p^2 + \ldots)$$

Here a_i are integers, $0 \leq a_i \leq p - 1$. This series is convergent in the p-adic norm because $|a_n p^n|_p = p^{-n}$. This series can be compared to a decimal which represents a real number: $1/10 \sim p$. The p-adic numbers can be defined by such series. Addition and multiplication are defined in an obvious way.

For example, we have

$$-1 = p - 1 + (p - 1)p + (p - 1)p^2 + \ldots$$

and for $p = 2$,

$$-1 = 1 + 2 + 2^2 + 2^3 + \ldots .$$

This convergent series can be obtained from the equality

$$-1 = \frac{p - 1}{1 - p}$$

as a formal expansion with respect to p. It is possible that p-adic analysis will be useful for the investigation of strong coupling theories.

We denote $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. This is the set of all numbers in $\mathbb{Q}_p$ whose p-adic expansion involves no negative powers of p. An element of $\mathbb{Z}_p$ is called a p-adic integer. The usual integers are dense in $\mathbb{Z}_p$. If $x, y \in \mathbb{Q}_p$, we write $x \equiv y (\operatorname{mod} p^n)$ if $|x - y|_p \leq p^{-n}$, i.e., if the first non-zero digit in the p-adic expansion of $x - y$ occurs no sooner than the p^n place. The p-adic number field $\mathbb{Q}_p$ contains p solutions $a_0, a_1, \ldots, a_{p-1}$ to the equation $x^p - x = 0$, where $a_n \equiv n (\operatorname{mod} p)$. These p numbers are called the Teichmüller representatives of $0, 1, 2, \ldots, p - 1$ and are sometimes used as a set of p-adic digits instead of $0, 1, \ldots, p - 1$.

Our intuition about distance is based on the Archimedean metric. Some properties of the non-Archimedean metric seem very strange at first. Here are some examples.

1) In non-Archimedean geometry all triangles are isosceles. If $|x|_p < |y|_p$, then $|y|_p = |y - x|_p$.

2) Any point in a disc is its centre. Define a disc of radius r centered at a: $D_a(r) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\}$. Then if $b \in D_a(r)$, we have $D_a(r) = D_b(r)$.

3) If we consider two balls then either a ball is contained inside the other one or they do not intersect. Such balls remind us of the elementary particles.

3.3. *p-Adic analysis [30]*

We can repeat the familiar definition of differentiable functions and say that a function $f : \mathbb{Q}_p \to \mathbb{Q}_p$ is differentiable at $x \in \mathbb{Q}_p$ if

$$\left|\frac{f(x+\varepsilon)-f(x)}{\varepsilon} - f'(x)\right|_p \to 0 \text{ as } |\varepsilon|_p \to 0.$$

There is an analogue between p-adic analysis and superanalysis (see Ref. [4]). p-Adic analysis is different from the usual analysis. For example the equation $f'(x) = 0$ has a solution which is not locally constant. The equation $f'(x) = 0$ has a classical solution which is not locally constant. The equation $f'(x) - f(x) = 0$ has a classical solution

$$f(x) = Ce^x = C\sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

The classical exponential series converge everywhere, thanks to the $n!$ in the denominator. But while big denominators are good things to have classically, they are not so good p-adically. Namely, this series has a disc of convergence $|x|_p < p^{1/(1-p)} < 1$. The poor convergence of e^x causes much of p-adic analysis to involve subtleties which are absent in classical analysis. Unlike the case of the field of real numbers, whose algebraic closure is only a quadratic extension, $\mathbb{Q}_p$ has algebraic extensions of arbitrary degree. The completion of its algebraic closure is not locally compact. There are two notions of p-adic global analyticity, due to Krasner and Tate. The Dwork principle states that the p-adic analogue of analytic continuation along a path is "analytic continuation along Frobenius" (see Ref. [31]). There is a strong relation between the p-adic analytic properties of the ordinary differential equations and the zeta functions of the corresponding algebraic curves. The solutions to the differential equations may be identified with the corresponding p-adic cohomology classes. It seems that there is a close relation between the p-adic theory of differential equations [31] and the Schottky problem and also Novikov's conjecture [32]. This relation could be useful in the p-adic string theory, as we will discuss later.

3.4. *p-Adic interpolation*

There is a "bridge" between the "real" world and the p-adic one. It is the integers $\mathbb{Z}$. The integers are dense in the unit disc in $\mathbb{Q}_p$. Let us consider a function $f : \mathbb{Z} \to \mathbb{Q}_p$ such that the numbers $f(n)$ have a p-adic continuity property. Then the function $f(n)$ can be extended in a unique way from the integers to the unit disc $\mathbb{Z}_p$ so that the resulting function is a continuous function of a p-adic variable with values in $\mathbb{Q}_p$. This is what is meant by p-adic interpolation. p-Adic interpolation will play the role of quantization. It will give a relation between the real macroscopic world and the p-adic sub-Planckian one. Notice that a function f on the integers can be extended in at most one way to a continuous function on $\mathbb{Z}_p$. This is because $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. In the real case, while the rational numbers are dense in the field of real numbers, the set $\mathbb{Z}$ is not. Therefore there are always infinitely many continuous real-valued functions which interpolate a given function on the integers.

There are two remarkable examples of p-adic interpolation. One is the Riemann zeta-function and the other is the gamma function. The Riemann zeta-function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \quad s > 1.$$

It is possible to show that the numbers $\zeta(2k)$ for $k = 1, 2, 3, \dots$ have a p-adic continuity property. More precisely, consider the set of numbers

$$\zeta_p(2k) = (1 - p^{2k-1})c_k\pi^{-2k}\zeta(2k).$$

where $c_k = (-1)^k(2k-1)!2^{1-2k}$. It turns out that $\zeta_p(2k)$ is a rational number. It is related to the Bernoulli number. Moreover if two such values of $2k$ are close p-adically, then the corresponding $\zeta_p(2k)$ are also p-adically close.

The classical ζ-function can be expressed as a Mellin transformation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} d\sigma(x), \quad d\sigma(x) = \frac{1}{e^x - 1}, \quad s > 1.$$

The p-adic ζ-function also can be expressed in such a form

$$\zeta_p(s) = \int_{|x|_p=1} x^{s-1} d\mu(x)$$

where $d\mu(x)$ is the Mazur measure on $|x|_p = 1$.

The classical gamma-function is a function which interpolates $n!$,

$$\Gamma(n) = \prod_{j<n} j.$$

To find the p-adic interpolation we need to modify the factorial function in the following way [33]:

$$\Gamma_p(n) = \prod_{j<n; p\nmid j} j.$$

Here $p \nmid j$ means that j is not divisible by p. Then Γ_p extends uniquely to a continuous function $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p$.

The p-adic gamma-function has the following basic properties:

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = -x \text{ if } x \in \mathbb{Z}_p \backslash \{0\}, \qquad \frac{\Gamma_p(x+1)}{\Gamma_p(x)} = -1 \text{ if } x \in p\mathbb{Z}_p.$$

For $x \in \mathbb{Z}_p$, write $x = x_0 + px_1$, where $x_0 \in \{1, \ldots, p\}$ is the first digit in x, unless $x \in p\mathbb{Z}_p$, in which case $x_0 = p$. Then we have

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{x_0}.$$

Gauss multiplication formula: for any positive integer $m, p \nmid m$, we have

$$\frac{\prod_{n=0}^{m-1} \Gamma_p\left(\frac{x+n}{m}\right)}{\Gamma_p(x) \prod_{n=1}^{m-1} \Gamma_p\left(\frac{n}{m}\right)} = m^{1-x_0} m^{-(p-1)x_1}.$$

4. p-ADIC DUAL MODEL AND STRINGS

Recall that the Veneziano amplitude was the starting point for the string theory. It is possible to construct a natural p-adic analogue of the Veneziano amplitude. As is known, the Veneziano amplitude has the form

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}$$

where $\alpha(s) = \alpha_0 + \alpha' s$, $s = (k_1 + k_2)^2$, $t = (k_1 + k_3)^2$, $k_i \in M^D$, $i = 1, 2, 3$. As a p-adic Veneziano amplitude we consider [3]

$$A_p(s,t) = \frac{\Gamma_p(-\alpha(s))\Gamma_p(-\alpha(t))}{\Gamma_p(-\alpha(s) - \alpha(t))}$$

where Γ_p is the p-adic gamma function. Here $\alpha(s) = \alpha_0 + \alpha' s$, α_0 and α' are p-adic numbers and $s = \langle k_1 + k_2, k_1 + k_2 \rangle$, $\langle \cdot, \cdot \rangle$ is a bilinear form on $\mathbb{Q}_p^D$ and $k_i \in \mathbb{Q}_p^D$.

Notice that we have the correspondence principle in the following sense

$$\lim_{p \to \infty} A_p(m,n) = A(m,n)$$

where m and n are integers.

What about the N-particle amplitudes and loop diagrams? As is known, the dual integrands can be expressed in terms of theta-functions (see Refs. [34] and [35]). There is a p-adic uniformization theory and a theory of theta-functions [36]. p-Adic Schottky groups are considered in Ref. [37]. The problem here is to find the p-adic analogue of the Koba—Nielsen measure.

Recently an interesting relation between string theories, soliton equations and infinite Grassmanians [38] - [40] has been found. A new algebraic method of computing the string amplitudes was proposed in Ref. [41]. It seems that these approaches will be useful for investigation of the p-adic string theory.

Let us consider now the theory on a finite Galois field. What is the dual theory in this case? Recall that the Veneziano amplitude has the integral representation

$$A(s,t) = \int_0^1 x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1}dx.$$

Here the function x^a, $a = -\alpha(s) - 1$ is the multiplicative character on the real axis, i.e., the Veneziano amplitude is the convolution of two characters. Therefore the natural analogue of the Veneziano amplitude in the finite world is the convolution of two characters on the Galois field

$$J(\chi_a, \chi_b) = \sum_{x \in \mathbb{F}_p} \chi_a(x)\chi_b(1-x).$$

This is the Jacobi sum, which is well known in number theory. The Gross-Koblitz formula [25] gives a close connection between the p-adic dual amplitude and the Jacobi sum. In Ref. [3] , the operator formulation for the Jacobi sum with Dirichlet characters was given.

5. QUANTUM FLUCTUATIONS OF THE NUMBER FIELD

Up to now we have considered the cases of two important fields, the p-adic field and the finite field. Let us consider the problem from the point of view of the general principles of quantum mechanics. Currently, it is the common view that at small distances or in cosmology [42], [29], we have to consider fluctuations of topology [2], of the dimensionality of space-time, of the metric and the signature of space-time [43]. So, as a general principle one can consider the statement that all physical parameters undergo quantum fluctuations. According to this general principle it is natural to believe that the number field on which a theory is developed is also the subject of quantum fluctuations. This means that apparently the usual field of real numbers is realized only with some probability and in principle the probability exists of the occurrence of any other field.

In order for this at first sight crazy statement to acquire some real sense it is necessary that a physical theory could be constructed over any number field. In this context the following proposal looks rather appropriate. It seems that the general principle may hold according to which the fundamental physical laws should be invariant under the change of the number field. This principle has some analogy with the Einstein principle of general relativity invariance, which means that the fundamental physical laws should be invariant under general transformations of co-ordinates.

In fact the principle of invariance under the change of the field of real numbers to the field of complex numbers is implicitly used in physics. It means analyticity. The principle of independence from the number field puts strong restrictions on the possible forms of equations. These restrictions occur even if we consider only a field with a norm; for analysis over a field with a norm, see Ref. [4]. For example, the usual Schrodinger equation is not invariant under a change of number field because the square root of minus one does not exist in an arbitrary field. However, the Klein-Gordon equation continues to have a meaning under a change of number field.

Let us stress that a fundamental law is not necessarily formulated in the language of differential equations or the usual differential geometry. There is a likelihood that this language is appropriate only to the description of physical phenomena in the usual space-time over real numbers. In principle, other formulations of the underlying physical laws may be possible. For example, for dual models the language of the automorphic functions is more appropriate. It can turn out that a proper language would be, for example, the language of cohomology theory.

It is appropriate to recall the famous Einstein programme to reduce all physics to geometry. It is a promising programme but let us ask the question about which geometry we would like to speak of? Why should we pick Riemannian geometry? Are there the reasons in favor of Riemannian geometry, or can one also use non-Archimedean geometry? One can go farther and ask the question why geometry over the field of real numbers but not over an arbitrary field is the proper geometry for physics. We believe that the contemporary version of Einstein's programme should look like a proposal to reduce all physics to geometry over arbitrary number fields. In fact this means the reduction of the physics to number theory. One can agree with the Pythagoreans according to whom we have to understand number, in order to understand the Universe.

If these ideas are true then number theory and the corresponding branches of algebraic geometry are nothing else than the ultimate and unified physical theory.

Of course, it is possible to generalize the above general principle and to consider some algebras instead of fields. In superanalysis [4], we exchange the field of real numbers for superalgebra with a norm. But in this paper we restrict ourselves to the case of the field.

We will discuss now an appropriate modification of quantum mechanics. In the usual quantum mechanics, the principle of superposition of probability amplitudes plays the main role and it can be written in the form

$$\langle a|c\rangle = \sum_b \langle a|b\rangle\langle b|c\rangle$$

where the probability has to be calculated as follows:

$$P_{ab} = |\langle a|b\rangle|^2. \tag{1}$$

In the construction of the quantum mechanics on an arbitrary field K, one can follow two ways. The first way consists in considering complex valued functions depending on variables belonging to K, or more generally, functions taking values in a complex Hilbert space. Here the results of representation theory on the local compact fields may be useful (see Ref. [44]) where these constructions are actually considered.

The second way one deals with wave functions taking values in the field K, i.e., $\langle a|b\rangle \in K$. Then in the case of the field K with a norm, Eq. (1) has sense if $|\cdot|$ means the norm in K. In our opinion it is difficult at present to tell which of these two ways is more favorable, so that it is reasonable to develop both simultaneously. Note that the first way is more closer to traditional quantum mechanics. However, there are important differences. Namely, it seems rather inappropriate to formulate dynamics in this case using the Schrödinger equation. A more appropriate way is to deal with unitary representation of the translation group.

If the above-mentioned hypothesis on the fluctuations of the number field is indeed realized then it is possible to suggest also the following hypothesis. It is common wisdom that in the Big Bang or in the final collapse, time and space do not have their usual meanings. But this is a purely negative answer to the question about the meaning of the time and space co-ordinates in such circumstances. What is a positive answer? Our proposal is as follows. The space and time co-ordinates would be, for example, p-adic. Of course this is an unusual world. For example, p-adic variables are not ordered. In this case, there is no meaning to the words "greater" or "less". But nevertheless we can write differential equations in such variables and we can try to understand processes in the Big Bang in a constructive way.

Then the strongest fluctuations take place in the Big Bang and a newly born Universe can have non-Archimedean or finite or other geometry over non-standard number fields. It may be that the corresponding exotic domain exists at present. An analogous hypothesis can also be considered in the context of the gravitational collapse. By this we mean that in the process of the collapse as a result of quantum effects, matter can collapse into a space with non-Archimedean geometry.

Of course, many problems are open in the approach which was discussed in the paper. But the involved mathematics is so beautiful that one can hope that it has a relation to the reality. As Dirac said: "I learnt to distrust all physical concepts as the basis for a theory. Instead one should put one's trust in a mathematical scheme, even if the scheme does not appear at first sight to be connected with physics. The physical meaning had to follow behind the mathematics" [45].

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