Period-harmonic-tupling jumps to Chaos and Fractal-scaling in a class of Series [1]

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Abstract — Series like Riemann zeta function are found to have large jumps at harmonic periods. The route to chaos for these series is via the cascade of period-harmonic-tupling jumps in contrast to the cascade of period-doubling bifurcations for unimodal iterative maps like the logistic map. The sets of oscillations between one jump and the next also exhibits the fractal-scaling property. This gives rise to the simplest of all approximations of Riemann zeta function
$\zeta(s) \approx \left(\sum_{n=1}^{N} n^{-s}\right) + (1/2)(N+1)^{-s}$ where $N = Im(s)/\pi - 1$

Introduction

For unimodal iterative maps like the logistic map, the route to chaos is period-doubling bifurcation [**1**]. However, for series like the Riemann zeta function, there is surprisingly a counterpart — the period-harmonic-tupling jumps to chaos.

Consider an oscillating function $f(x)$, $x \in \mathbf{R}^+$, and its series $F(s = \sigma + i\,t, N) = \sum_{n=1}^{N} f(n)$, where $f(x)$ has a period which increases monotonically with x and may contain parameters $\sigma, t \in \mathbf{R}$.

For definiteness, consider $f(x) = \cos(t\ lnx)$, $t \in \mathbf{R}$. The local maxima of $f(x)$ are

$$x_m = e^{\frac{2m\pi}{|t|}}, \quad m \in \mathbf{Z}^+, \quad \text{which satisfy}$$

$$\frac{d}{dx}\cos(t\ lnx) = -\frac{t}{x}\sin(t\ lnx) = 0 \ \text{ and } \cos(t\ lnx) = 1$$

The period τ of $f(x)$ at x is then the separation between the nearest two adjacent local maxima of $\cos(t\ lnx)$,

$$\tau = x_{m+1} - x_m = e^{\frac{(2m+1)\pi}{|t|}} - e^{\frac{2m\pi}{|t|}} = e^{\frac{2m\pi}{|t|}}\left(e^{\frac{2\pi}{|t|}} - 1\right)$$

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Each term $f(x)$ in the series $F(s,N)$ samples the continuous domain of x at discrete points $x = n, n \in \mathbf{Z}^+$. As N increases, $F(s,N)$ goes through cycles of small oscillations interrupted by large jumps as illustrated in Fig. 1.

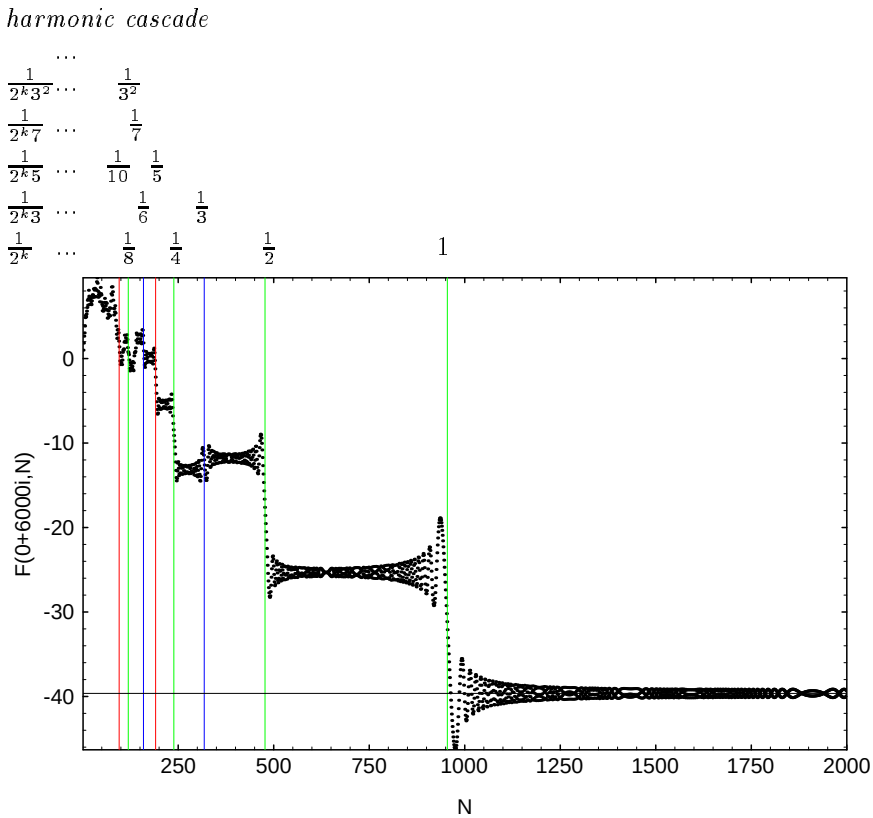

Fig. 1. Period harmonic-tupling to chaos for decreasing N, or reverse period harmonic-tupling to non-chaotic oscillations for increasing N in the series $F\,(s = \sigma + i\,t = 0 + 6000\,i, N)$. Large jumps occur at $N(\tau = 1/z) \approx (1/z)(1/2)(|t|/\pi - 1) \approx (954/z)$ where z runs through all positve integers ≥ 2. The cascade of these large jumps at harmonic periods are indicated by vertical lines. No large jump occurs at $N(\tau = 2) \approx 1909$ as expected. There is also approximate self-similarity among the envelopes of the oscillations in between one pair of jumps and the next which approaches self-similarity — becomes fractal — as $t \to \infty$. The scaling factor between neighboring envelopes is $(z/(z+2))$. The horizontal line indicates the value of the real part of Riemann zeta function $Re\,(\zeta(0 + 6000\,i))$.

This behaviour is due to the discrete sampling of $f(x)$.

The term $\cos(t\ lnn),\ n \in \mathbf{Z}^+$, can begin to approximate $\cos(t\ lnx),\ x \in \mathbf{R}^+$, well only when the period $\tau \geq 2$ — when the period τ is large enough to accommodate a local minimum at n in between two adjacent local maxima at $n-1$ and $n+1$. Hence, there will be no further large jumps interrupting small oscillations in $F(s,N)$ for $N \geq N(\tau = 2)$, the value of N when period $\tau = 2$. This makes the condition for a good approximation to be

$$\text{period}, \tau = e^{\frac{2m\pi}{|t|}} \left(e^{\frac{2\pi}{|t|}} - 1\right) \geq 2$$

To the nearest integer, the value of N when the period $\tau = 2$ is

$$N(\tau = 2) \approx x_m = e^{\frac{2m\pi}{|t|}} = \frac{2}{e^{\frac{2\pi}{|t|}} - 1} \sim \frac{|t|}{\pi} - 1$$

where the last expression is the asymptotic limit for large $|t|$. This can be verified by applying L'Hospital rule to

$$\lim_{t \to \infty} \left[\left(\frac{1}{\frac{|t|}{2\pi} - c} \right) / \left(\frac{1}{e^{\frac{2\pi}{|t|}} - 1} \right) \right]$$

and setting c to $1/2$ in the asymptotic limit. Fig. 2 shows how well the 2nd term approaches the last term asymptotically for increasing $|t|$.

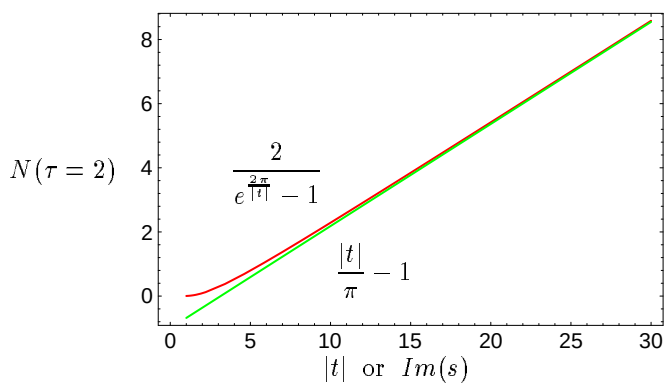

Fig. 2. The asymptotic limit of $\frac{2}{e^{\frac{2\pi}{|t|}} - 1}$ is $\frac{|t|}{\pi} - 1$

The oscillations for $N \gg N(\tau = 2)$ naturally resemble those of the integral

$$\int_0^N f(x)dx = \int_0^N \cos(t\ lnx)dx = \frac{N\left(\cos(t\ lnN) + t\sin(t\ lnN)\right)}{1+t^2}$$

Period-harmonic-tupling jumps to Chaos in Series

As the period of $f(x)$ increases monotonically with x, many of the terms $f(n)$, $n \in \mathbf{Z}^+$, in the series $F(s,N)$ will have the same sign when the discrete sampling points n are near that value of x where the period $\tau = (1/z)$, $z \in \mathbf{Z}^+$. These terms add up into a large jump away from the neighborhood of the preceding small oscillations. They form a cascade of large jumps at the rate of a reverse harmonic progression corresponding to the reverse harmonic sequence of the period,

$$\ldots, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1$$

The midpoints of these large jumps are thus at

$$N\left(\tau = \left(\frac{1}{z}\right)^k\right) \approx \left(\frac{1}{z}\right)^k \left(\frac{1}{2}\right)\left(\frac{|t|}{\pi} - 1\right), \quad z, k \in \mathbf{Z}^+$$

The period 1/3 of $f(x)$ can be thought of as analogous to the period 3 in the logistic map, and the cascade of jumps at harmonic periods as analogous to the cascade of bifuraction periods $[\mathbf{2} - \mathbf{3}]$ as illustrated in Fig. 1.

So conversely, if N is initially set to a large value, $N \gg 1$, and is then decreased towards zero, the oscillations of $F(s,N)$ becomes increasingly chaotic as the small oscillations are interrupted by more and more closely-spaced large jumps at the rate of a harmonic progression. Eventually, the large jumps are overcrowded and the discrete sampling interval is too wide to describe small oscillations. The small oscillations are overwhelmed and chaos sets in. The number of jumps in the cascade that are clearly observable increases as $t \to \infty$.

A small change in t corresponds to a large change in the intermediate path of oscillations of $F(s,N)$ for $N < N(\tau = 2)$ and therefore a large change in $F(s,N)$ for $N > N(\tau = 2)$.

So $F(s,N)$ exhibits the period-harmonic-tupling jumps to chaos for decreasing N, or, looking from the other side, the reverse period-harmonic-tupling jumps to non-chaotic oscillations.

Fractal-scaling between sets of oscillations

There is fractal-scaling between the sets of small oscillations of $F(s,N)$ in the limit of $t \to \infty$ in $f(x)$. For finite t, the self-similarity is approximate. The small oscillations are partitioned into sets of small oscillations by the cascade of large jumps. The envelope of a set of small oscillations bounded by two adjacent

large jumps is similar in shape to the envelope of any other such set of small oscillations. The width of a set bounded by the period $1/z_1$ and $1/(z_1+1)$ jumps is $N(\tau = 1/z_1) - N(\tau = 1/(z_1+1))$. The scaling factor for this set to any other set bounded by period $1/z_2$ and $1/(z_2+1)$ jumps is thus

$$\left(\frac{1/z_1 - 1/(z_1+1)}{1/z_2 - 1/(z_2+1)}\right) = \frac{z_2(z_2+1)}{z_1(z_1+1)}$$

The reverse harmonic sequence terminates at 1 and the set of oscillations bounded below by period 1 is unbounded above. So the scaling factor for this unbounded set to any other set is infinite. Fig. 1 illustrates the approximate self-similarity for a finite t.

There are interesting patterns of voids in the oscillations. The most prominent feature is the the 'wobbling bubble' at $N \approx N(\tau = 2)$ whose shape wobbles in cycles with increasing $|t|$ as shown in Fig. 3.

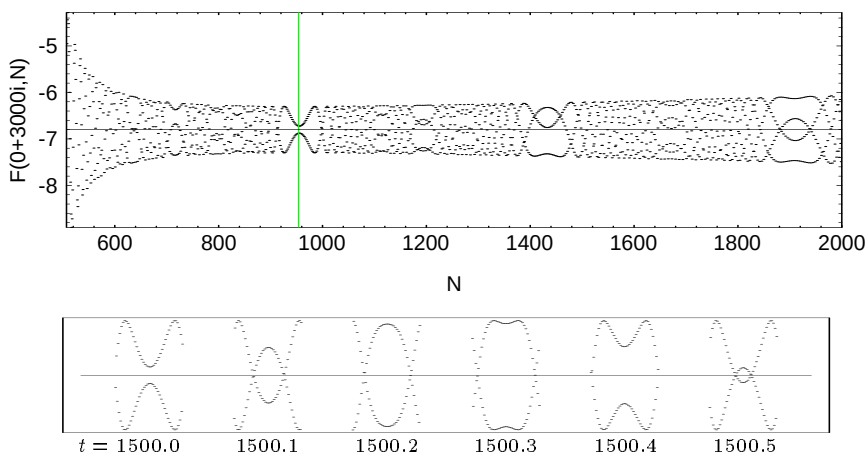

Fig. 3. The upper figure shows the interesting patterns of voids in the oscillations of $F(s = \sigma + i\,t = 0 + 1500\,i, N)$ near $N(\tau = 2) \approx 952$. The lower figure shows the cyclic wobbling behaviour of the 'wobbling bubble' at $N(\tau = 2)$ with increasing $|t|$.

The Chaotic and Fractal Dance in the series of Riemann zeta function

Now, consider the infinite series of Riemann zeta function [**4** – **5**] defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad Re(s) > 1$$

where the equivalent series up to N terms is

$$F(s,N) = \sum_{n=1}^{N} n^{-s}$$

The analogy to the above $f(x)$ is made most clearly by rewriting $F(s,N)$ as

$$F(s=\sigma+it) = \left[1+\sum_{n=2}^{N} n^{-\sigma}\cos(t\ lnn)\right] - i\left[\sum_{n=2}^{N} n^{-\sigma}\sin(t\ lnn)\right]$$

$$\text{where}\ \ \sigma = Re(s)\ \ \text{and}\ \ t = Im(s).$$

$x^{-\sigma}\cos(t\ lnx)$ has exactly the same period as $f(x) = cos(t\ lnx)$ and therefore the series of Riemann zeta function also jumps at the same harmonic period to chaos for decreasing N, or jumps at reverse harmonic period to stability and convergence for decreasing N and $\Re(s) > 1$ — an unusual way of looking at the convergence. This, together with the 'wobbling bubble' feature at $N = N(\tau = 2)$ shown in Fig. 3., gives rise to the simplest of all approximations of Riemann zeta function

$$\zeta(s) \approx \left(\sum_{n=1}^{N} n^{-s}\right) + \frac{1}{2}(N+1)^{-s}$$

$$\text{where}\quad N = N(\tau=2) = \frac{|t|}{\pi} - 1$$

The series also shares the same fractal-scaling pattern.

The analytic continuation of $\zeta(s)$ [**6**] to the negative-half plane can be obtained by applying Euler-Maclaurin Summation Formula [**7**]. Although

$$\zeta(s) = \lim_{N\to\infty} F(s,N,M) = \lim_{N\to\infty}\left(\begin{array}{c}\sum_{n=1}^{N} n^{-s} - \frac{1}{-s+1}N^{-s+1} - \frac{1}{2!}N^{-s} \\ -\sum_{k=1}^{M}\frac{B_{2k}}{(2k)!}\frac{\partial^{2k-1}}{\partial N^{2k-1}}(N^{-s}) \\ +O(N^{-s-2M-1})\end{array}\right)$$

for $Re(s) > -2M-1$, where B_{2k} is the $(2k)$-th Bernoulli Number,

converges faster for larger M, the series still jumps at reverse harmonic period to stability and actually begin to converge only after the final large jump at $N(\tau=1) \approx (1/2)(|t|/\pi - 1)$.

The Chaos Series Safari

Charted here are only the fringes of a world of wonders where the series with monotonically increasing periods, like the Riemann zeta function, roam and jump at harmonic periods. Series with decreasing periods can be jumping chaotically at harmonic periods as well. Extended here is an invitation to the adventuresome to trail the more exotic species of series with varying periods.

Conclusions

Riemann zeta function is a function full of surprises. Not only do the statistics of the distribution of its zeros exhibit chaos [$\mathbf{8-14}$], but so does the series itself because of the discrete sampling. On top of that, the sets of oscillations of the series also become fractal in the limit $t = Im(s) \to \infty$.

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