

X-RAY OF RIEMANN'S ZETA-FUNCTION

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1. INTRODUCTION

This paper is the result of the effort to give the students of the subject *Analytic Number Theory* an idea of the complexity of the behaviour of the Riemann zeta-function. I tried to make them **see** with their own eyes the mystery contained in its apparently simple definition.

There are precedents for the figures we are about to present. In the tables of Jahnke-Emde [9] we can find pictures of the zeta-function and some other graphs in which we can see some of the lines we draw. In the dissertation of A. Utzinger [19], directed by Speiser, the lines $\operatorname{Re} \zeta(s) = 0$ and $\operatorname{Im} \zeta(s) = 0$ are drawn on the rectangle $(-9, 10) \times (0, 29)$.

Besides, Speiser's paper contains some very interesting ideas. He proves that the Riemann Hypothesis is equivalent to the fact that the non trivial zeros of $\zeta'(s)$ are on the right of the critical line. He proves this claim using an entirely geometric reasoning which is on the borderline between the proved and the admissible. Afterwards rigorous proofs of this statement have been given.

Our figures arise from a simple idea. If $f(z) = u(z) + iv(z)$ is a meromorphic function, then the curves $u = 0$ and $v = 0$ meet precisely at the zeros and poles of the function. That is the reason why we mark the curves where the function is real or the curves where it is imaginary on the z -plane. In order to distinguish one from the other, we will draw with thick lines the curves where the function is real and with thin lines the curves where the function is imaginary.

When I distributed the first figure (the X ray of the zeta function) to my students, I was surprised at the amount of things one could see in the graphic. I spent a whole hour commenting on this figure.

Afterwards, I have kept thinking about these graphics. I believe they can be used to systematize the knowledge which today is scattered. The graphics give it a coherency which makes it easier to remember.

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2. THE X RAY

2.1. Remarkable Points. Recall that the Riemann zeta-function is defined for $\text{Re}(s) > 1$ by the series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

but it is possible to extend it to the whole plane as a meromorphic function with a single pole in $s = 1$, which is simple.

We already have the X ray on sight. The main thing to take into account is that the thick lines are formed by those points s in which $\zeta(s)$ is real, and the thin lines by those in which $\zeta(s)$ is imaginary.

In the figure we see that, in fact, the lines have a simpler behaviour on the right of the line $s = 1$, that is, on the right of the grey strip which represents the so called **critical strip**: $0 \leq \text{Re}(s) \leq 1$.

In the figure **the real axis, the oval** and a thick line that surrounds the oval stand specially out.

We can see that the real axis cuts lots of thin lines, first the oval in the pole $s = 1$, and in $s = -2$, which is a zero of the function, later, a line in $s = -4$, another in $s = -6$, \dots , which are the so called **trivial zeros** of the zeta-function. In the figure we can see these zeros up to $s = -28$, because the graph represents the rectangle $(-30, 10) \times (-10, 40)$. These zeros situated in the negative part of the real axis seem to draw the spine of this X ray.

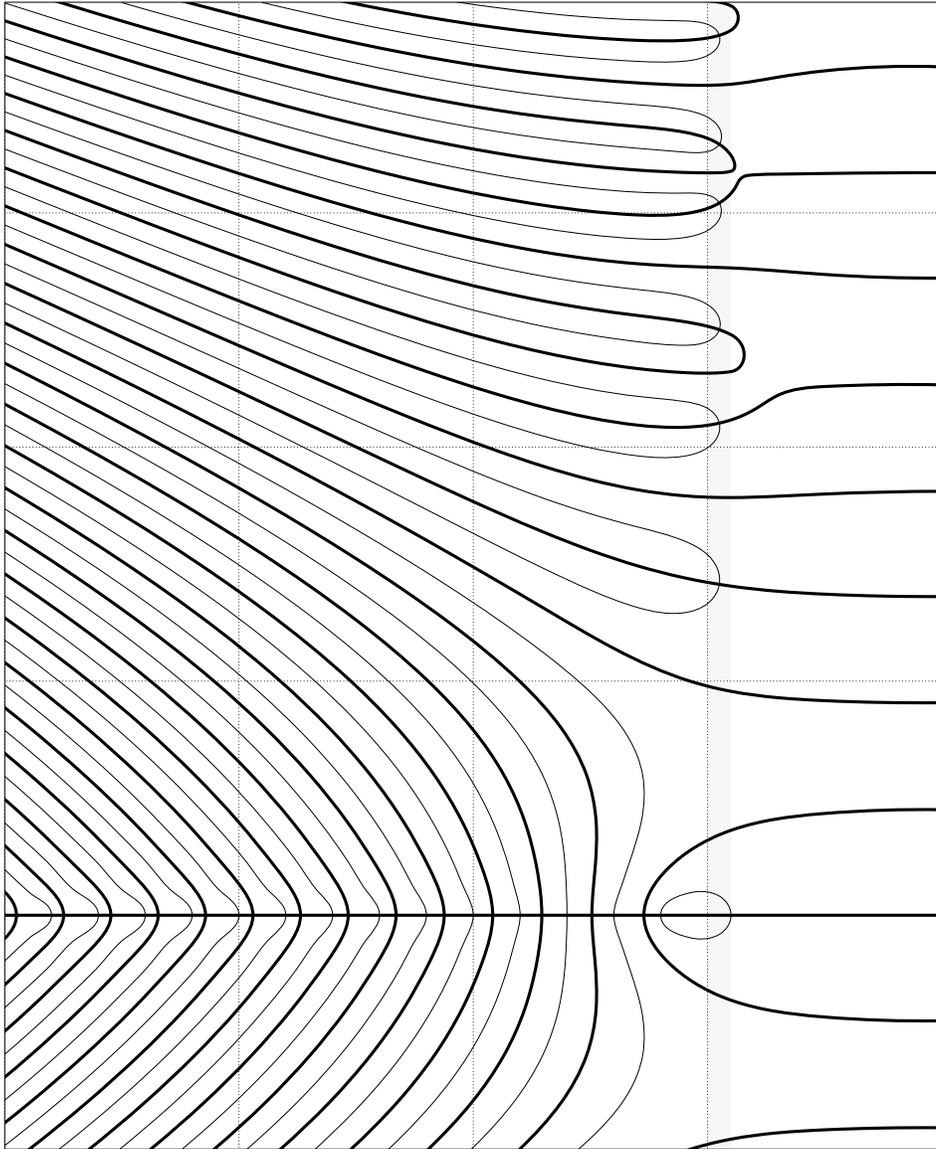
In the critical strip we see that thick and thin lines meet, that is, we detect the existence of non trivial zeros. With two decimal digits they are $0.5 + i14.13$, $0.5 + i21.02$, $0.5 + i25.01$, $0.5 + i30.42$, $0.5 + i32.93$, $0.5 + i37.58$. They seem to have a real part equal to $\frac{1}{2}$. In fact, it is possible to prove it.

The next remarkable points are the points in which the real axis meets a thick line. These are points at which the derivative $\zeta'(s) = 0$. Since the function $\zeta(s)$ is real when s is real and since it has zeros in the points $-2n$, Rolle's theorem from elemental calculus tells us that these zeros have to exist, one between each pair of consecutive even numbers.

For every meromorphic function $f(s) = u(s) + iv(s)$, the lines $u = 0$ (and the lines $v = 0$) are smooth curves, except in the points where the derivative vanishes, in which case two or more curves meet.

An important property which is valid in the general case of a meromorphic function is the **monotony** of the function along the curves.

Let us think of a thick line $v = 0$. In this curve, the function f is real. At each point of this curve, the derivative of v in the direction of the curve vanishes. If the derivative of f does not vanish in any point of the curve, neither will the derivative of $u = f$.

FIGURE 1. X ray of $\zeta(s)$

So, if on a portion of a curve the derivative of f does not vanish, then the function varies monotonously as we moves along the curve. In fact, the derivative of u in the direction of the curve can not vanish on that portion of the curve, so that it will keep having a constant sign.

When we are dealing with the $u = 0$ curves, the monotonous function is $v = f/i$.

We would like to know, for each line and each direction, whether the function increases or decreases. But before tackling this point, we must give each line a name.

2.2. The numbering of the lines. There are three special lines which we will not number: the oval, the one that surrounds the oval, and the real axis.

The rest of the lines can be numbered with integer numbers. In the first place, there are thin lines which pass through the zeros in $-4, -6, -8, \dots$. Line $-2n$ is the line which cuts the real axis at the point $-2n$.

Between these lines we can find thick lines. For example, there is one line which goes between the lines numbered -4 and -6 . This one does not go through the point -5 , but it cuts the real axis on a point which is between -4 and -6 . We will say that this one is line -5 . In the same way lines $-(2n + 1)$, $n = 2, 3, \dots$, are defined. These are thick lines which cut the real axis between $-2n$ and $-2n - 2$.

In this way, we now know which are the lines numbered n for $n = -4, -5, -6, \dots$. If we consider now how these lines cut the left border of the figure (line $x = -30$), we see that the lines already numbered are followed by other lines, so that it seems natural to call line -3 the thick line which runs parallel to line -4 , above it, until the latter turns to cut the real axis while the thick line goes on to the right, crossing the critical strip at a height near 10.

So, the even lines are always lines in which the function $\zeta(s)$ is purely imaginary, while the odd lines are thick lines in which $\zeta(s)$ is real. We must also add that, in many cases, two lines join in a zero. For example, lines -2 and 0 join in the first non trivial zero of the zeta-function. Lines 5 and 7 also join in the third zero.

This numbering does not include the symmetrical lines below the real axis.

Now we can speak of concrete lines. Do you see how line 11 turns? It seems it intended to continue with line 13 or 15, but in the end it follows another path, cutting lines 10 and 12.

2.3. Lines on the right of the critical strip. The behaviour on the right of the critical strip is governed by the fact that there are no thin lines. This is easy to understand. For $\operatorname{Re}(s) = \sigma \rightarrow +\infty$ we have $\zeta(s) \rightarrow 1$ uniformly. So, there exists an abscissa σ_0 so that for $\sigma > \sigma_0$ it holds $\operatorname{Re} \zeta(s) > 0$. It follows that the function does not take purely imaginary values in this half-plane. It is not difficult to prove that there are no thin lines passing through the half-plane $\sigma > 1.63622\dots$

On the right, the only thick lines that exist are essentially parallel to the x axis, and they are equally spaced. In order to understand the reason,

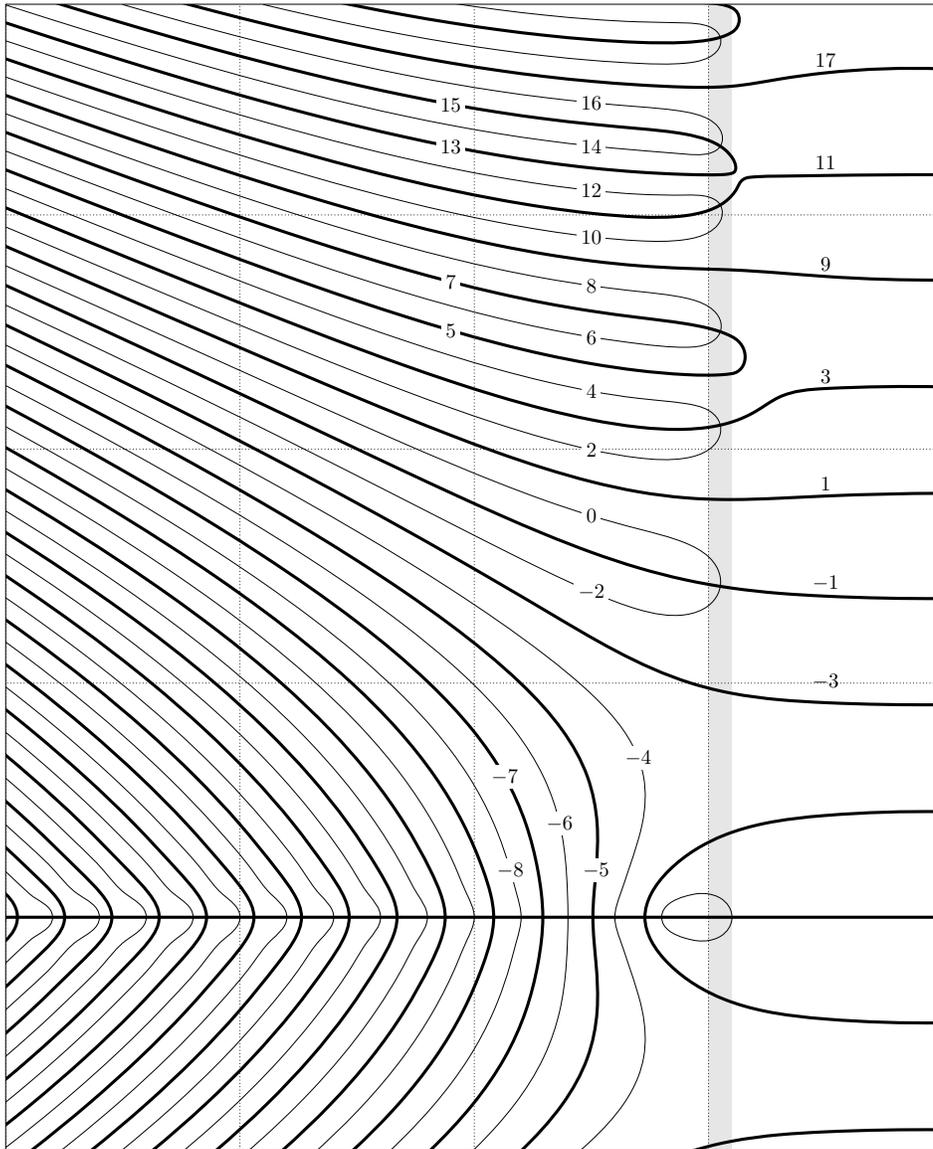


FIGURE 2. Numbering the lines

we start with

$$\operatorname{Im} \zeta(s) = - \sum_{n=2}^{\infty} \frac{\sin(t \log n)}{n^{\sigma}}.$$

For big enough values of σ , the first term of this sum dominates the rest of them. It is clear that this first term vanishes for $t = n\pi/\log 2$. So,

some parallel lines, separated by a distance which is approximately equal to $\pi/\log 2 \approx 4.53236014\dots$, exist.

We can see in the figure that these parallel lines, when crossing the critical strip, alternatively, contain a non trivial zero, or not. It is easy to explain. The derivative of $\zeta(s)$ along the line which goes at a height $n\pi/\log 2$, when $\sigma \gg 0$, is given by

$$\frac{\partial}{\partial \sigma} \zeta(s) = \frac{\partial}{\partial \sigma} \left(1 + \frac{\cos(t \log 2)}{2^\sigma} + \dots \right) \approx -\frac{\cos(t \log 2)}{2^\sigma} \log 2 \approx -\frac{(-1)^n}{2^\sigma} \log 2.$$

Thus, when the function $\zeta(s)$ runs along this curve from right to left, (parting from 1), it is increasing for even values of n . So, for even n , $\zeta(s)$ will take on this line the values in $(1, +\infty)$. On the other hand, for odd n it will take the values in $(1, -\infty)$, and, in particular, it will vanish.

Of course we are using the fact that the function is monotonous on the lines, which depends on the fact that the derivative $\zeta'(s)$ does not vanish on these lines. This is what happens to the line surrounding the oval, which ought to contain a zero but does not.

2.4. Orientating ourselves. It holds $\lim_{\sigma \rightarrow +\infty} \zeta(\sigma + it) = 1$. uniformly in t . Let us suppose that we go along line -1 , starting from the zero on the critical line, and going to the right. The function will take values in the interval $(0, 1)$, starting from zero and tending to 1. The points we leave on our left, that is, the ones which are a little above line -1 , will turn into points on the left of the segment $(0, 1)$, that means, points from the first quadrant.

If, on the contrary, we start from the zero and move to the left following line -1 between lines -2 and 0 , the zeta-function will take negative values, in the interval $(-\infty, 0)$. The points between lines -1 and 0 will turn into points from the second quadrant.

If we situate ourselves at the zero with our arms outstretched, the right arm to the right and the left one to the left, we see in front of us a thin line (line 0), on which the function will take values ix with $x > 0$. Behind us we have a line (line -2) in which it will take values ix with $x < 0$. On the northeast we have points s which turn into values $\zeta(s)$ situated on the first quadrant. On the northwest we have a region which turns into the second quadrant, etc.

In a line such as line -3 , which comes from the right and does not contain any zeros, the zeta-function takes real values greater than 1, precisely all the points in the interval $(1, +\infty)$, because it is known that, on the left, the modulus of the zeta-function tends to infinity.

2.5. The regions of the plane. The lines in the graphics divide the plane into regions. It is easy to realize that the points of one of these regions have to turn by the function $\zeta(s)$ into points of the same quadrant.

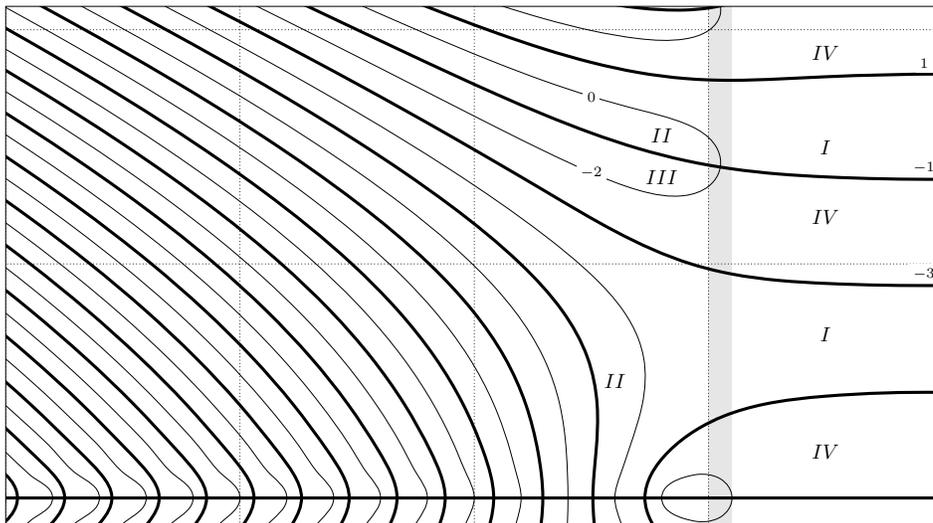


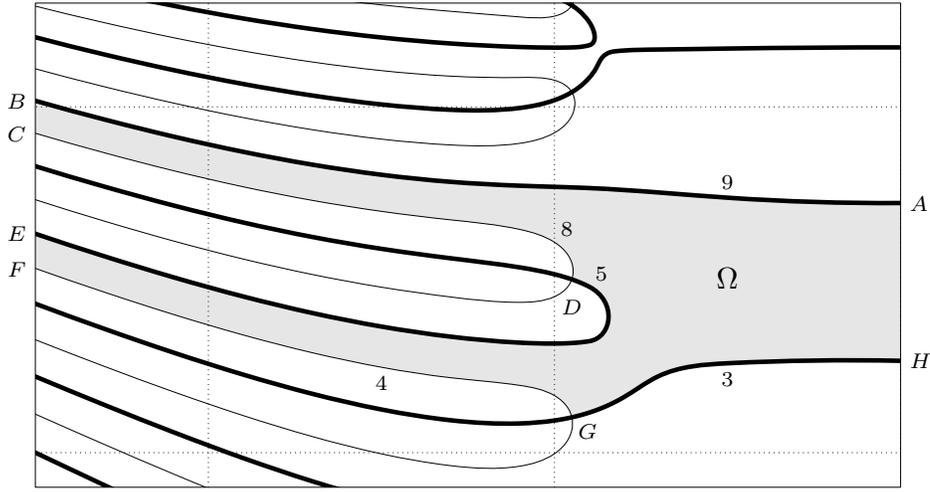
FIGURE 3. Each region transform into a quadrant.

If the points of a region Ω turn into the first quadrant, for example, and crossing a thin line we reach another region, then the points of this new region will turn into points of the second quadrant; if, on the contrary, we leave Ω crossing a thick line, then the new region will turn into the fourth quadrant.

We have various ways to know into which quadrant will a given region turn. For example, if we situate ourselves at a non trivial zero and we orientate in the way described above, then the region in the northeast will turn into the first quadrant, the one situated in the northwest in the second, and so on.

Another way to see it could be to take into account that, if we walk along a thick line, so that the function $\zeta(s)$, which is real, increases, then we know that on our left we have a region which turns into the first quadrant and on our right one which turns into the fourth quadrant. For example, let us consider the thick line number 1 (this is the line which comes from the right and goes between the first and the second non trivial zeros of the zeta-function). and let us walk through it to the left. That is, we walk through this line coming from $+\infty$. The values of the zeta-function on the points of this line are greater than 1, and, as we walk through it, the values on its points are growing. Then, the region on our right will turn into the first quadrant.

As an exercise, we frame this question: Into what do the points of the oval turn?

FIGURE 4. The region Ω .

2.6. The equation $\zeta(s) = a$. Let us consider an equation of the form $\zeta(s) = a$. Where are its solutions? How are they distributed?

If a is situated in a determined quadrant, say the first, then there will not be any solution in the region bordered by lines -2 and -1 . In fact, the points of this region turn into points from the third quadrant. Thus, the solutions have to be situated in the regions we know to turn into the first quadrant. We can say how many solutions are there in each of these regions.

For example, let us consider the region Ω , bordered by lines 9, 8, 5, 4 and part of line 3.

As s runs through the border of this region, ABCDEFGHA, the image $\zeta(s)$ runs through a closed path, according to the following table

s	$A \rightarrow B$	$C \rightarrow D$	$D \rightarrow E$	$F \rightarrow G$	$G \rightarrow H$
$\zeta(s)$	$1 \rightarrow +\infty$	$+i\infty \rightarrow 0$	$0 \rightarrow +\infty$	$+i\infty \rightarrow 0$	$0 \rightarrow 1$

so that $\zeta(s)$ goes round the first quadrant twice. Thus, if a is a point from the first quadrant, the equation $\zeta(s) = a$ will have exactly two solutions when $s \in \Omega$, according to the argument principle.

It is clear that, under the former conditions, the equation $\zeta'(s) = 0$ must have a solution in the region Ω . There are other solutions of $\zeta'(s) = 0$, one for each of the lines $-3, -5, -7, \dots$ precisely in the points at which these lines meet the real axis.

In the X ray figure we thus see 14 solutions of $\zeta'(s) = 0$, situated on the real axis, and, besides, the existence of two more can be inferred, one

in the region called Ω and the other in the region situated between lines 11 and 17.

2.7. The construction of the graphics. The first one who calculated zeros of the function $\zeta(s)$ was Riemann himself. Gram [7] calculated the first ten zeros and proved that they were the only ones satisfying $0 < \text{Im}(\rho) < 50$. The following step in this direction were the works of Backlund [2], who managed to prove that the zeros $\alpha + i\beta$, with $\alpha > 0$ and $0 < \beta < 200$ where exactly 79, every one of them with real part $\alpha = 1/2$. Later, Hutchinson [8] managed to extend these calculations up to 300. This task has been followed by many mathematicians like Titchmarsh, Comrie, Turing, Lehmer, Rosser, and so on.

We have used the same technics they used to localize certain values of σ , the solutions of $\text{Re}\zeta(\sigma + it) = 0$ or $\text{Im}\zeta(\sigma + it) = 0$. Afterwards, we have written other programs, based on these, ones which situate the points belonging to the same line in order.

These graphics owe much to the program Metafont, created by Knuth to design the fonts used by $\text{T}_{\text{E}}\text{X}$, and to its modification MetaPost, by J. D. Hobby, which allows one to obtain PostScript graphics.

For $|\sigma - 1/2| > 5/2$ I have proved that the curves follow the path I have drawn here, but for the area near the critical strip I just followed the method until I was convinced that the curves follow the path I have drawn. This has required, in some cases, the calculation of numerous points on each line. So they can not be considered proved.

2.8. The existence of non trivial zeros. One of the problems which have arised in drawing these graphics was to know the number of a line. Applying the Stirling series for $\log\Gamma(s)$ and the functional equation of $\zeta(s)$ we can prove the following:

Theorem 1. *The number of a line passing through the point $-1 + it$ (with $t > 5$) is the integer number which is nearer to*

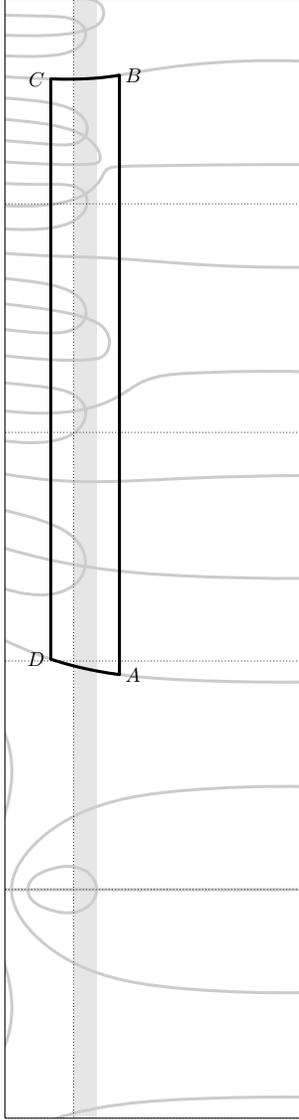
$$(2.1) \quad \frac{2t}{\pi} \log \frac{t}{2\pi} - \frac{2t}{\pi} + \frac{1}{2}.$$

Logically, the role of line $\sigma = -1$ is not important as long as we are not near the critical strip.

If we watch all the lines coming to the critical strip from the left and we know that the thin lines can not surpass a certain point, we see they must come back and they can not do it unless they cut the thick lines which accompany them. This little rigorous reasoning can be turned into a proof, by means of the argument principle, of the following theorem.

Theorem 2. *Let us suppose that $-1 + iT$ is on one of the parallel lines which do not contain any zero. The number of non trivial zeros $\rho = \beta + i\gamma$ below this line and satisfying $\beta > 0$ is equal to the integer number nearest to*

$$(2.2) \quad \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}.$$



Proof. In order to prove it, let us consider the region bordered by the lines $\sigma = -1$, $\sigma = 2$, line number 3 and the line which is referred to in the statement. In the figure we have represented the case when this line is line number 17. The number of zeros we are looking for is equal to the variation of the argument of $\zeta(s)$ along this curve. But, along the segment AB , there is no variation, because all the segment turns into points of the first and the fourth quadrant, starting and finishing on the real axis. In the portions of the curves CB and DA there is also no variation because on them the function $\zeta(s)$ is real. Thus, the variation of the argument is exactly the one which takes place in the segment CD . This segment cuts several lines. As it goes from one to the other, the argument of $\zeta(s)$ varies precisely in $\pi/2$. This way, the variation of the argument will be $(N+3)\pi/2$ if the highest line is line number N . By the previous theorem N is the integer number nearest to 2.1. It follows easily that the number of zeros is equal to $(N+3)/4$, and thus it is the integer number nearest to 2.2. \square

If the line we are dealing with is one of the parallel line which do contain zeros, an analogous reasoning proves that the number of zeros below this line, counting the one which is on line N on, is $(N+5)/4$, and so it is the integer number which is nearest to

$$(2.3) \quad \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{11}{8}.$$

So, the parallel lines which do not contain zeros have a number $N \equiv 1$ and the ones which do contain zeros, on the contrary, satisfy $N \equiv 3 \pmod{4}$.

3. TECHNICS TO CALCULATE THE FUNCTION $\zeta(s)$.

In order to construct the graphics the possibility of calculating the function is essential. There are two basic ways to calculate the function $\zeta(s)$.

3.1. Euler-MacLaurin Formula. It is the following

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{1}{2} \frac{1}{N^s} + \frac{N^{1-s}}{s-1} + \sum_{k=1}^M T_k + R(N, M),$$

where

$$T_k = \frac{B_{2k}}{(2k)!} N^{1-s-2k} \prod_{j=0}^{2k-2} (s+j),$$

B_n are the Bernoulli numbers, and there are good known bounds for the error term $R(N, M)$.

Choosing N and M conviniently as a function of s allows one to calculate $\zeta(s)$ for any s and with arbitrary precission.

3.2. Riemann-Siegel Formula. Before explaining what does it consist of, we should define Hardy's function (called so though it was known by Riemann)

$$Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right)$$

with

$$\theta(t) = \text{Im}\left(\log \Gamma\left(\frac{1}{4} + i \frac{t}{2}\right)\right) - \frac{t}{2} \log \pi.$$

The functional equation implies that the function $Z(t)$ is real for real values of t . This allows one to locate zeros $\rho = \beta + i\gamma$ with $\beta = 1/2$, because a change in the sign of $Z(t)$ implies the existence of a zero with abscissa exactly $1/2$.

The Riemann-Siegel formula allows one to calculate the function for any point s , but with a limited precission depending on who s is. We will only write it for a point of the form $s = 1/2 + it$. It is the following:

$$Z(t) = 2 \sum_{n=1}^m \frac{\cos(\theta(t) - t \log n)}{\sqrt{n}} + g(t) + R,$$

where

$$\begin{aligned} m &= \lfloor \sqrt{t/2\pi} \rfloor \\ g(t) &= (-1)^{m-1} \left(\frac{t}{2\pi}\right)^{-1/4} h(\xi) \\ h(\xi) &= (\sec 2\pi\xi) \cos 2\pi\phi \end{aligned} \quad \begin{aligned} \xi &= \left(\frac{t}{2\pi}\right)^{1/2} - m \\ \phi &= \xi - \xi^2 + 1/16 \end{aligned}$$

The error is $\mathcal{O}(t^{-3/4})$. The Riemann-Siegel formula requires, in order to achieve a fixed precision, the calculation of approximately \sqrt{t} terms of the first sum, while the Euler-MacLaurin formula requires a number of terms of the order of t . When $t \approx 10^9$, this difference turns out to be very significant.

This formula has an interesting story. In a letter directed to Weierstrass, Riemann claims that:

the two theorems I have just stated here: that between 0 and T there exist approximately $\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$ real roots of the equation $\xi(\alpha) = 0$.

that the series $\sum_{\alpha} (\text{Li}(x^{1/2+i\alpha}) + \text{Li}(x^{1/2-i\alpha}))$, when its terms are ordered according to the increasing growth of α , tends to the same limit than the expression

$$\frac{1}{2\pi i \log x} \int_{a-bi}^{a+bi} d_s \frac{1}{s} \log \frac{\xi((s-\frac{1}{2})i)}{\xi(0)} x^s ds$$

when b grows beyond boundaries.

are consequences of a new expansion ξ which I have not been able to simplify enough to communicate it.

Riemann considered proved¹ all his other claims contained in [13].

Because of this, it was clear that, among Riemann's papers should be the expansion he was talking about. Some 70 years after his death, C. L. Siegel [15] managed to figure out these papers. It would be worth for the reader to have a look at the photocopy of the sheets of Riemann's manuscript which contain the famous expansion, which can be found in Edward's book [6], to realize the difficulty of Siegel's task.

4. GRAPHICS OF THE ZETA-FUNCTION IN THE CRITICAL LINE.

In the following pages we present some graphics corresponding to the values of t between 0 and 560. We have just drawn the strip $-1 \leq \sigma \leq 2$. The part which is not represented has nothing new and our imagination can supply it without any trouble. On the left we have pointed the values of t , each time it increases in twenty unities, and we have written the number of some lines (the parallel ones which do not contain any zeros). We leave till later the explanations about some points we have marked in the graphics with little circles, situated on the critical line.

¹It seems that the Riemann's capability to prove his claims has been doubted more than once without proper support. To get more information on this point you can consult my paper[1].

In the usual proof about the number of non trivial zeros up till a given height T it is proved that a horizontal segment cutting the critical strip between $\sigma = -1$ and $\sigma = 2$, at a height of, say, t , can only cut a number of lines of the order $\log t$. This, and the fact that the lines coming from the right can not cut themselves, because that would generate a region bordered by points where the function $\zeta(s)$ is real and bounded, allows Speiser to infer that the lines coming from the right have to cross the critical strip and move away to the right towards the infinity.

In fact, if one of these lines went up towards the infinity through the critical strip it would force the others to go above it, also through the critical line, to infinity. As there are a number of them of the order of t , we would reach a contradiction with the results already proved about the horizontal segment in the critical strip.

As the number of lines on the left has an order of $t \log t$, we see that it is necessary for some of them to go back to the right, joining others. Speiser calls these figures, formed by two lines, when the function is real, *sheets*.

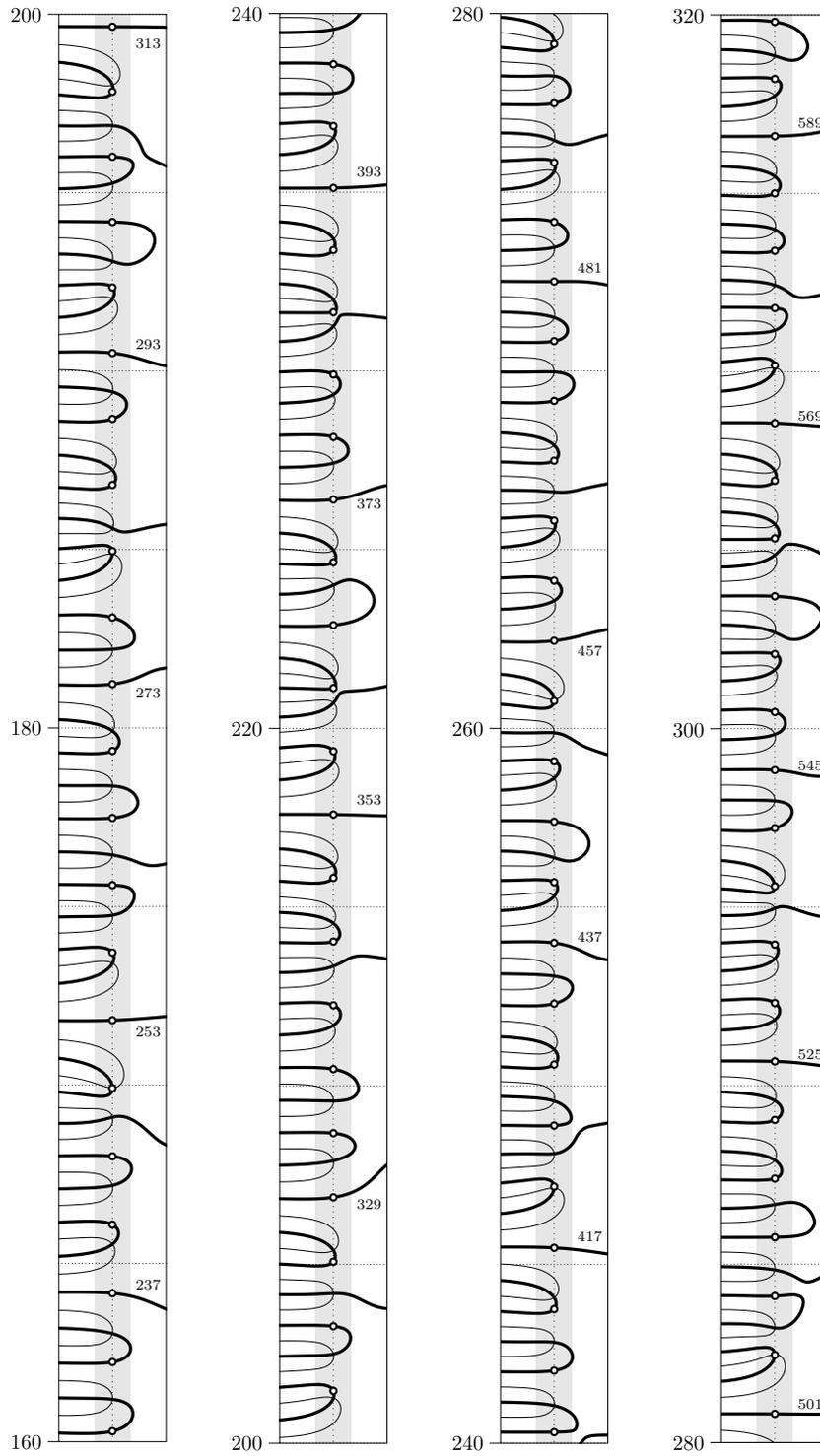
The graphics give rise to some integer sequences, the most obvious is the sequence formed by the number of lines which escape to the right. That is,

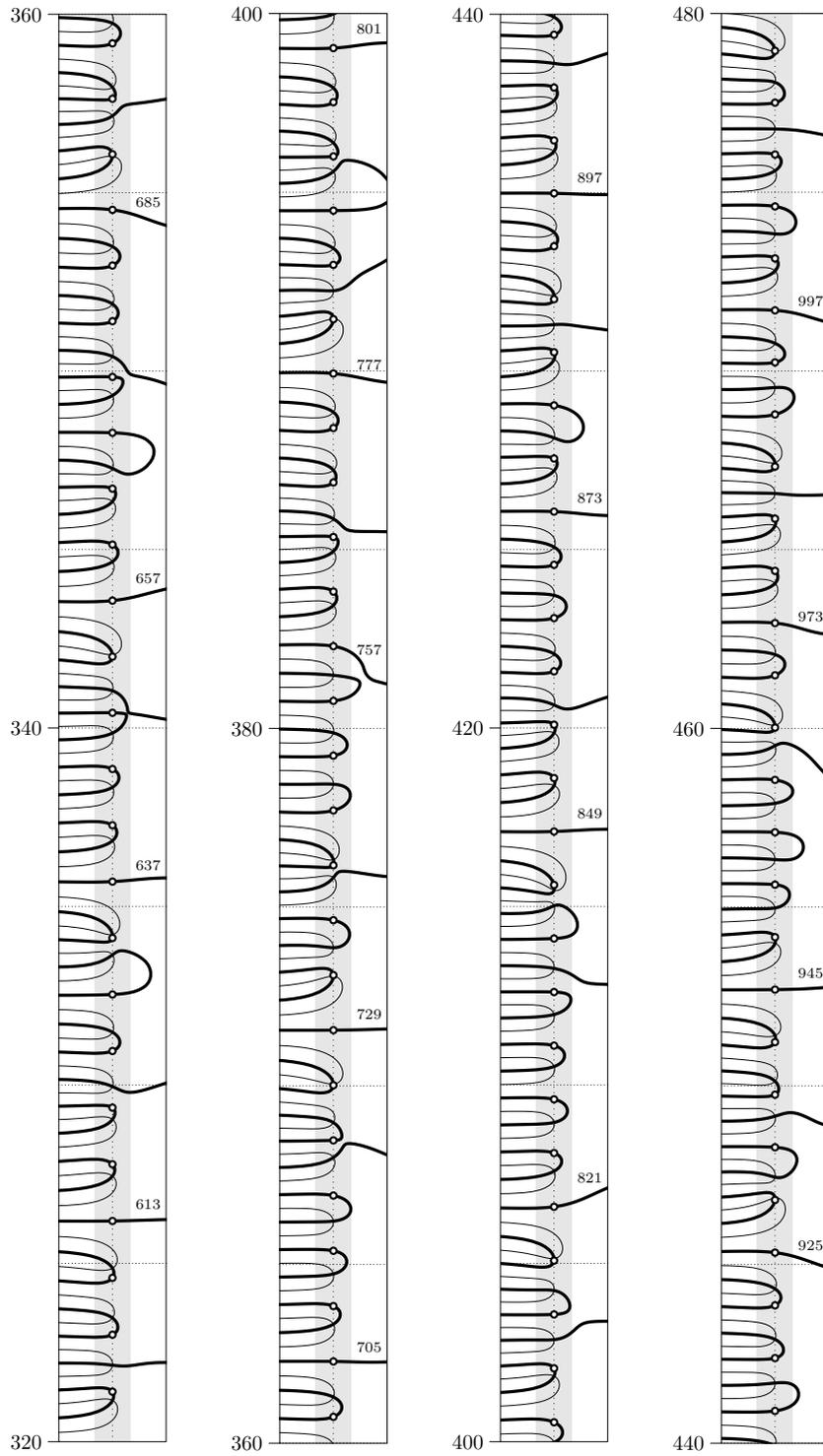
$-3, -1, 1, 3, 9, 11, 17, 23, 29, 35, 41, 47, 53, 59, 69, 75, 81, 91, 97, 103, 113,$
 $123, 129, 135, 145, 155, 161, 171, 181, 187, 197, 207, 217, 223, 237, 247, 253$
 $263, 273, 283, 293, 307, 313, 323, 329, 343, 353, 359, 373, 383, 393, 403, 417$
 $423, 437, 451, 457, 467, 481, 491, 501, 511, 525, 535, 545, 559, 569, 579, \dots$

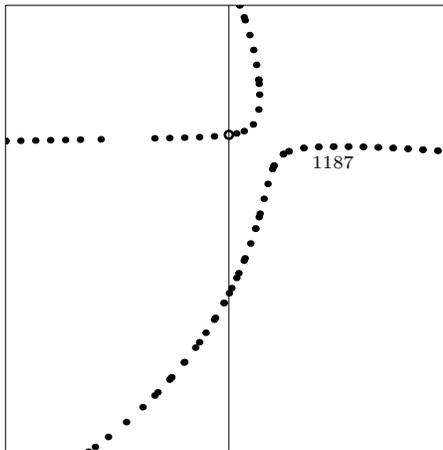
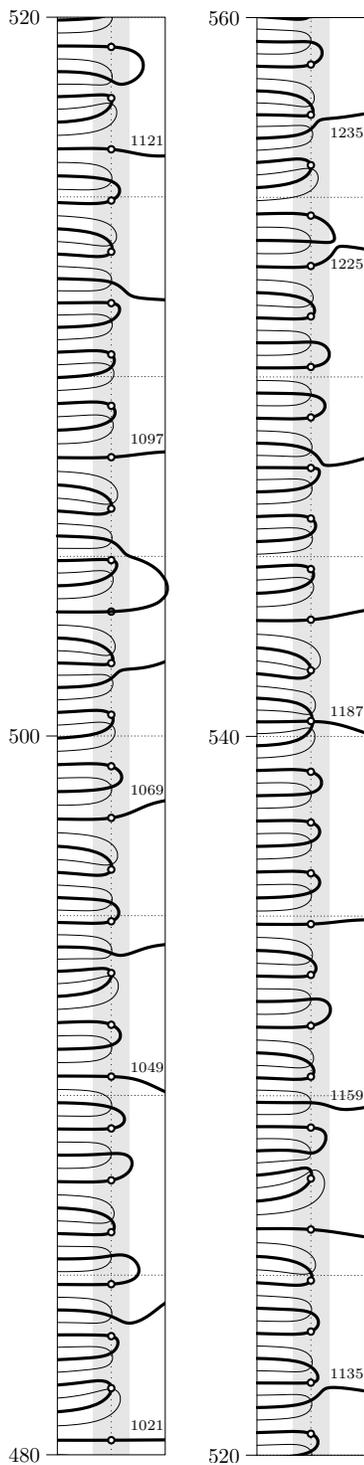
Speiser suggests that it could be connected with the distribution of the prime numbers. I could not see any conection.

Looking at these graphics, a general scheme seems to appear. Between two parallel lines which do not contain zeros there are an even number of thin lines, joining by pairs, a thick line coming parallel from the right cuts one of these loops and the rest are inserted with loops of thick lines. Later we will see how these simple ideas about the function break.

The thin lines almost do not cross over the critical line, while the thick lines sometimes reach $\text{Re}(s) = 2$. In fact, we see that at a high height they do cross over this line, this happens for the first time in the sheet formed by lines 789 and 791. But what is really surprising happens in line 1085. This forms a sheet with line 1091. So this sheet surrounds completely the one formed by lines 1087 and 1089. For higher values of t this becomes quite frequent.







Sometimes, as in line 1187, can happen that the path followed by the line is not clear. In such cases we draw the confusing area, for example, in this case we can see here a square which represents the rectangle $(0.2, 0.8) \times (540, 540.8)$, marking the points where the function $\zeta(s)$ is real. This way it is clear that line 1187 turns down when it gets to the critical line, and the pair of lines 1189 and 1191 form a sheet.

The monotony of the function on the curves is valid except in the poles, provided we define suitably the prolongation of the curve in the points where the derivative vanishes. In fact, in one of these points several curves meet, and there are as many on which the function increases as on which the function decreases, thus it always is possible, when we reach a zero of order n of the derivative, to go back with an angle of $2\pi/n$, so that the monotony is maintained all through the curve.

Another particularity of the curves we would like to emphasize is that, except possibly the ones which go through the poles, the curves have to leave every compact. Thus they come from and go to the infinity.

5. FIRST THEOREM OF SPEISER

Theorem 3 (Speiser). *The Riemann Hypothesis is equivalent to the fact that all the sheets meet the critical line.*

Proof. Let us recall that the sheets are the thick lines in which $\zeta(s)$ is real and which, coming from the left, go back to the left joining another thick line.

In the first place, on each sheet, the zeta-function takes all the real values. In fact, the function is real and monotonous and it tends to infinity as $\sigma \rightarrow -\infty$. Thus in a sheet there is always one (and only one) zero of the function.

If some sheet would not touch the critical line, the corresponding zero would be an exception to the Riemann Hypothesis.

The proof of the other implication is a little more intricate. In the first place, we must notice that if a real line crosses the critical line through a point α , then it holds that $\zeta(s) = 0$ or else the modulus $|\zeta(s)|$ decreases as the curve goes through this point from left to right. To see this, we notice that, since the function of Hardy $Z(t)$ is real for real values of t , it follows that $\theta(t) + \arg \zeta(1/2 + it) = \text{cte}$, except at one zero of the zeta-function. Since $\theta(t)$ is an increasing function it follows that $\partial_t \arg \zeta(1/2 + it) < 0$ unless $1/2 + it$ is a zero of the zeta-function. Let us consider then the analytic function

$$\log \zeta(1/2 + it) = \log |\zeta(1/2 + it)| + i \arg \zeta(1/2 + it).$$

From the Cauchy-Riemann equations it now follows that the derivative of $|\zeta(s)|$ with respect to σ in the point $\sigma = 1/2$ must be negative.

If a line (of a sheet) surpasses the critical line from left to right it must go through it again in order to come back to the left. If the zero which is on the sheet is not one of the points where the line cuts the critical line, we can apply the results obtained above to both of them. It follows that in one of the points it must be satisfied that $\zeta(s) > 0$, and in the other, $\zeta(s') < 0$, so that the absolute value can be decreasing in both. Since the function is monotonous along the sheet, it follows that the zero is situated on the right of the critical line.

Thus there are two kinds of sheets cutting the critical line. In some of them, the zero is located in one of the two cuts between the critical line and the sheet. In others, the zero is situated on the right of the critical line.

Consequently, if the Riemann Hypothesis is false, there would be a zero situated on the left of the critical line, which can be situated only on a sheet which does not cut the critical line. If the Riemann Hypothesis is true, then all the sheets are of the kind which have a zero exactly on the line and so they cross it (or are tangent at it).

This way, we see that if the Riemann Hypothesis is true all the sheets would meet the critical line. \square

In the figures above we can see that, in fact, all the sheets do contain a zero and another point on which it cuts the critical line. We have marked the latter with a

little circle. These points are the so called Gram points, and we will see some interesting results about these points below.

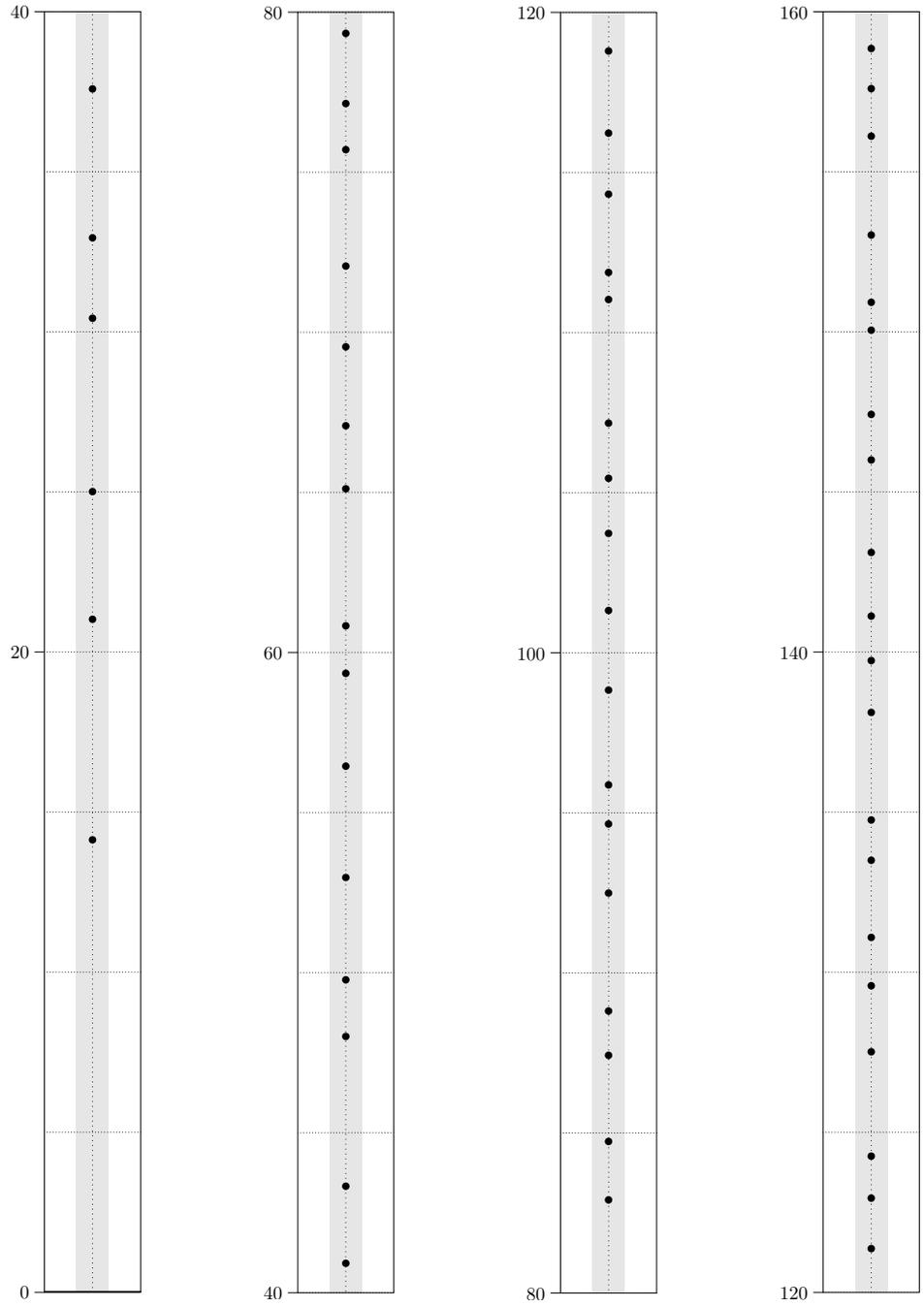


FIGURE 12. Zeros of $\zeta(s)$

6. SEPARATING THE ZEROS. GRAM POINTS.

In the last page we marked the zeros of the function $\zeta(s)$. We see that there is some randomness in their distribution.

To try to understand their distribution the Gram points are brought in. The Gram points, which we have marked with a little circle in our figures, are points on the critical line where the function $\zeta(s)$ is real and does not vanish. In the graphics of the following page we have marked Gram points. It is remarkable how regularly are these points distributed.

The Gram point g_n is defined as the solution of the following equation:

$$\theta(g_n) = n\pi$$

Since Hardy's function is real,

$$\zeta(1/2 + i g_n) = e^{-i\theta(g_n)} Z(g_n) = (-1)^n Z(g_n)$$

is real. Thus these points are situated on thick lines.

Looking at the preceding figures, we verify that, in most of the cases represented there, it holds:

$$\zeta(1/2 + i g_n) > 0.$$

We will see later that this inequality has exceptions for greater values of t .

In the Riemann-Siegel formula, the first term, which is the most important, has a value of $(-1)^n$ for $t = g_n$, which partly explains the tendency of $Z(g_n)$ to have the sign $(-1)^n$.

An important consequence of the preceding is that $Z(g_n)$ and $Z(g_{n+1})$ have oposed signs, so that the function $\zeta(1/2 + it)$ will vanish in the interval (g_n, g_{n+1}) . In Figure 14 we can see this graphically.

When one intends to calculate the real zeros of a polynomial, a first task is to separate the zeros, that is, to find an increasing sequence of values for t in which the polynomial takes alternatively values of different signs. An analogous aim is achieved with Gram points.

Gram noticed that these points seemed to separate the zeros of the zeta-function and claimed that this would be true for not too high values of t . Hutchinson named the fact that in each interval (g_n, g_{n+1}) would be a zero of the zeta-function Gram's law.

Titchmarsh uses this idea to prove that there is an infinity of zeros on the critical line, proving that the mean value of $Z(g_{2n})$ is positive and that of $Z(g_{2n+1})$ is negative.

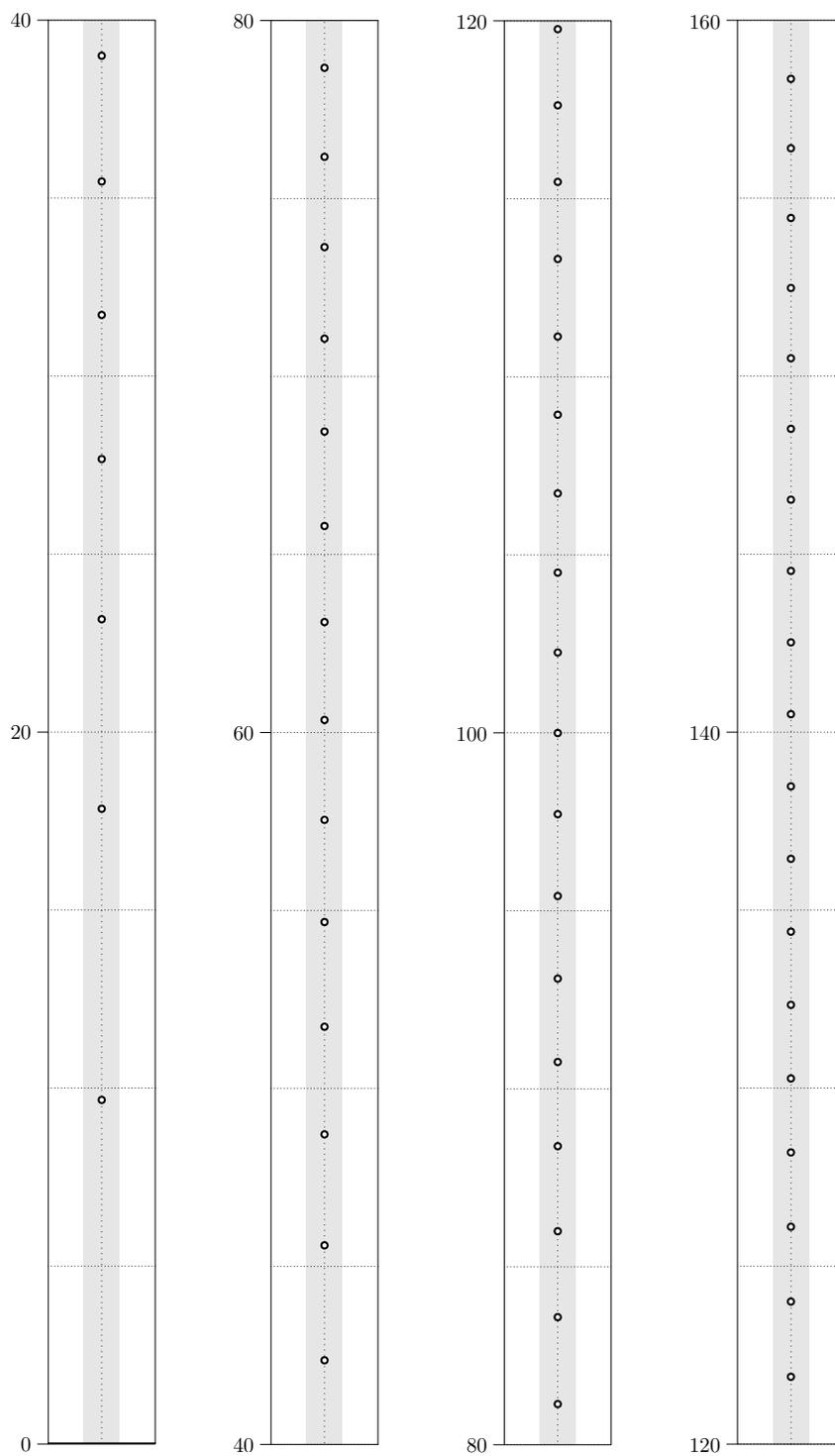


FIGURE 13. Gram Points.

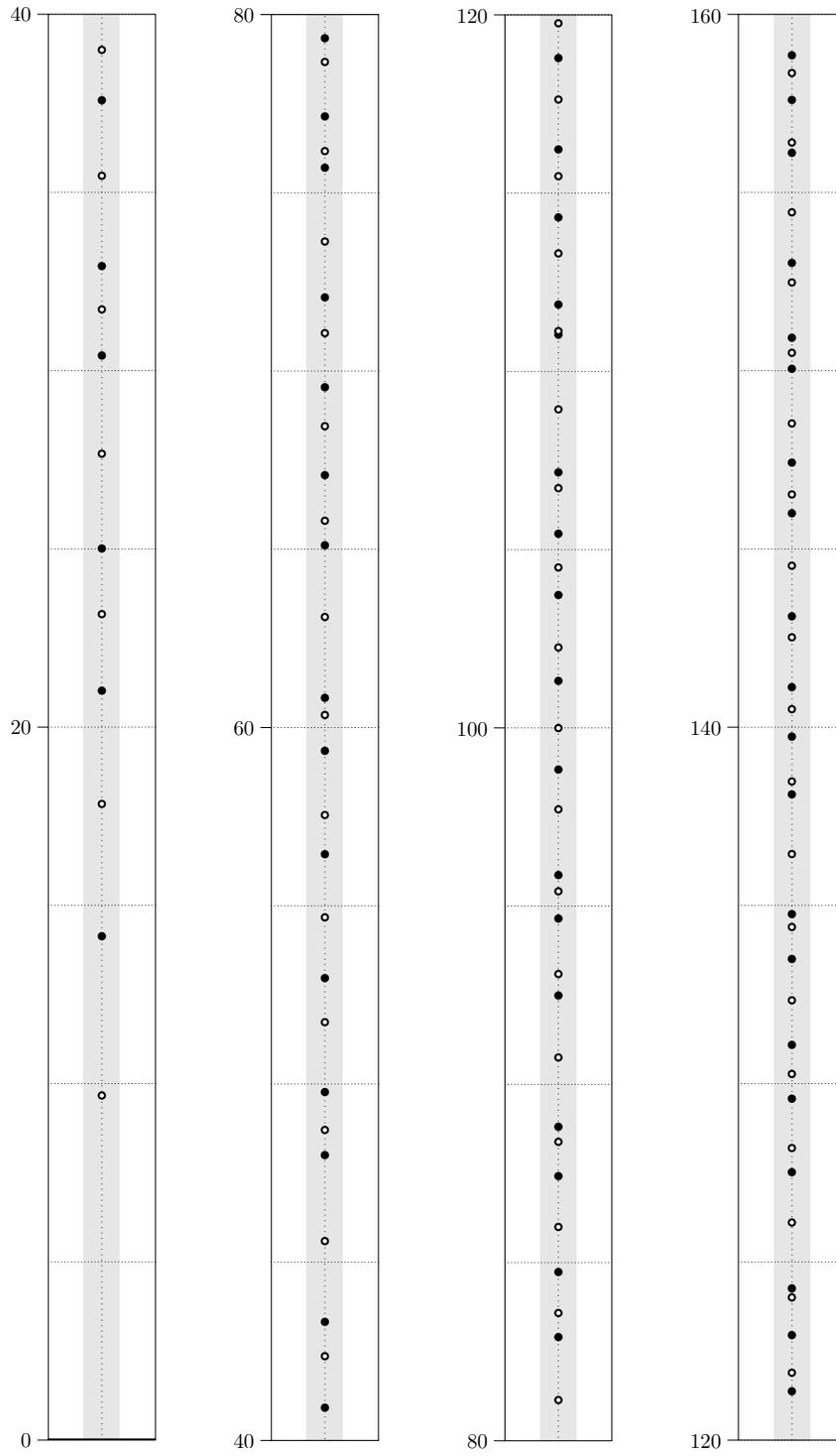


FIGURE 14. Gram's points separate the zeros.