7. Second Theorem of Speiser.

We present here Speiser's proof, and, as we have already said, his methods are between the proved and the acceptable. Everybody quotes him but nobody reproduces his theorems. His methods do not seem convincing to me, either, though I think his proof is essentially sound. We present it more like a challenge: to turn it into a proof, filling its gaps. In any case, a flawless proof of a stronger result can be found in Levinson and Montgomery [12].

Theorem 4 (Speiser)**.** *The Riemann Hypothesis is equivalent to the fact that the non trivial zeros of the derivative $\zeta'(s)$ have a real part $\geq 1/2$, that is, that they are on the right of the critical line.*

Proof. Let us assume that there is a zero a of $\zeta(s)$ on the left of the critical line. Let us consider the lines of constant argument $\arg\zeta(s) = \text{cte}$ which come from the point a. In all of them the modulus $|\zeta(s)|$ is increasing. Thus, these lines can not cross the critical line, because at crossing it from left to right, the absolute value of $\zeta(s)$ ought to decrease.

They also, can not be tangent at the critical line, because the tangency point would be a point in the critical line where $\arg\zeta(1/2+it)$ would be stationary, and this is possible only if it is a zero. But, in the line, $|\zeta(s)|$ increases, starting from zero, so it can not go through a zero of the function.

Thus all these lines come back to the left. Some of them go back leaving the point a below, others leave it above. The line which separates both kinds of lines must reach a zero of the derivative, which will allow it to go back. This would be a zero of the derivative on the left of the critical line.

Consequently, if the Riemann Hypothesis is false, we see that there must exist a zero of the derivative on the left of the critical line.

Now let us suppose that $\zeta'(a) = 0$, where $\text{Re}(a) < 1/2$. We have to find a zero of the function on the left of the critical line. We can assume that $\zeta(a) \neq 0$, because if this were the case we would have already finished. Since the derivative vanishes, there exist two opposite lines, of constant argument and along which $|\zeta(s)|$ decreases. We follow these two lines, and we must reach a zero, because $|\zeta(s)|$ decreases. If it is on the left of the critical line, we have finished, while, in other case, it is clear that we will reach the critical line.

Our two paths, till they reach the critical line, and the segment from the critical line they determine, enclose a region Ω. From the point a, two opposite paths also set off, along which $|\zeta(s)|$ increases, and the argument of $\zeta(s)$ is constant. One of them enters our region Ω, (because in the point a, the borderline of the region has a well-defined tangent).

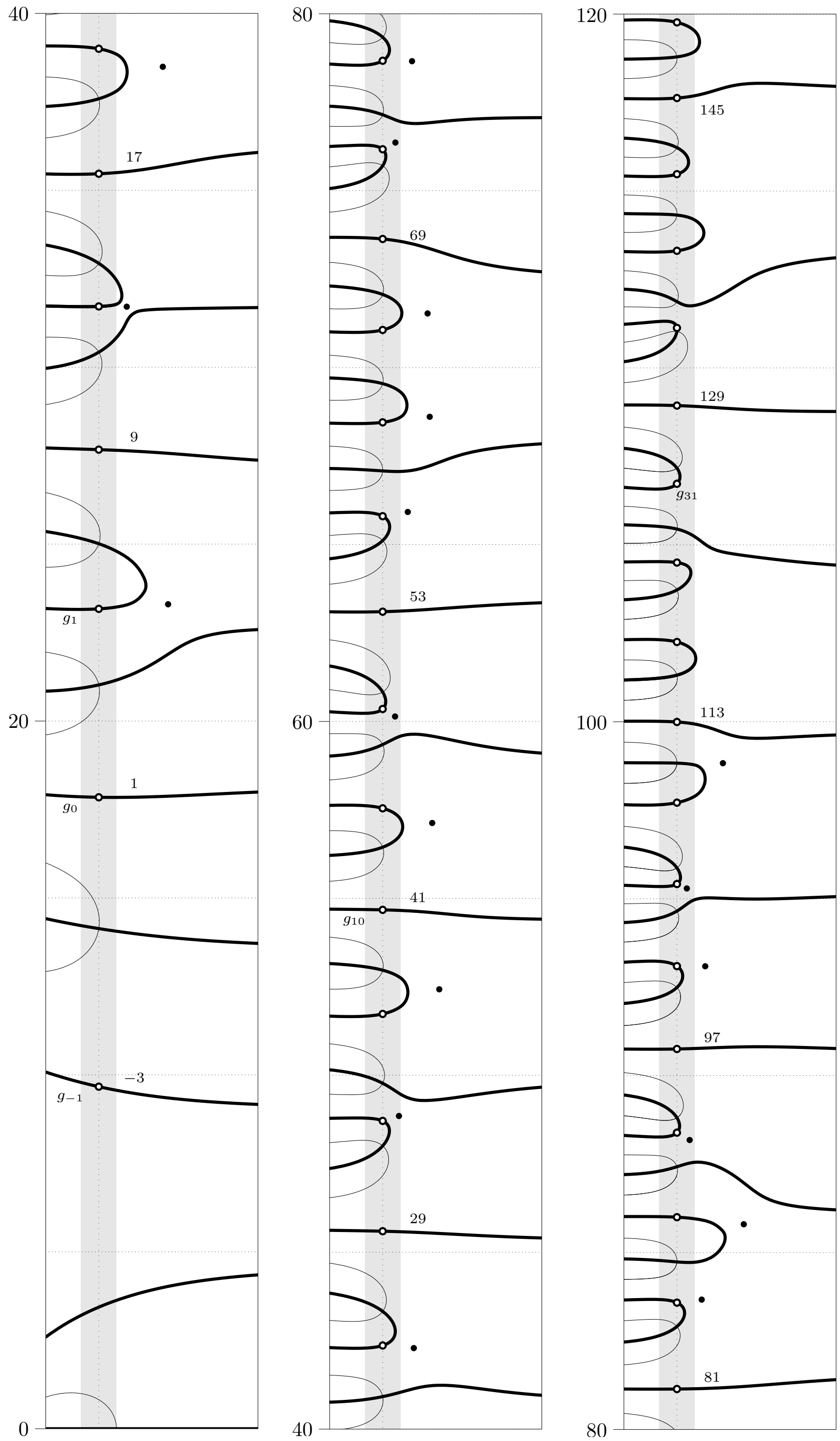


FIGURE 15. Zeros $\zeta'(s)$.

We will follow this path. Since $|\zeta(s)|$ increases to infinity along this curve, it must leave the region Ω, but it can not do it across the curves which we used to define it since along them $|\zeta(s)| < |\zeta(a)|$, and we are considering a curve where the values of $|\zeta(s)|$ are greater than $|\zeta(a)|$. It also can not leave Ω across the segment from the critical line, because to cross it from left to right it ought to do it with $|\zeta(s)|$ decreasing.

Thus supposing that there was no zero of $\zeta(s)$ on the left of the critical line has lead us to a contradiction. □

Speiser's theorem makes the zeros of $\zeta'(s)$ far more interesting. R. Spira who has given a complete proof of half of Speiser Theorem [17] has calculated the first ones (those which have an abscissa less than 100), which are represented in the figure.

We see that the real curves (thick ones) seem to be attracted by the zeros, and each sheet seem to have one zero associated with it, which would justify their crossing the critical line to approach their corresponding zero. In this way we have an insight about the place where the zeros which are beyond line 113 (that we have not draw in the figure) are situated.

We see that a zero of the derivative does explain the behaviour of line 11. If the derivative vanished at a point in which the function is $\zeta(s)$ is real, at this point two thick lines would meet perpendiculary. What happens here is that the function is almost real in the zero and the curves resemble the meeting we have described. This is also what happens in line 1187. If we see the graphics with the dots we had to do in order to see the path this line follows, we can verify that the derivative at this point has a zero with an abscissa slightly greater than 1/2.

Because of Speiser's reasoning we know that a line which comes parallel from the right at a height T does not rise or fall of level until it crosses the critical strip at a height greater than $\mathcal{O}(\log T)$. Thus, the number of parallel lines below it is $T \log 2/\pi + \mathcal{O}(\log T)$. This lines, alternatively, contain a zero of $\zeta(s)$ which is not associated with a sheet, or do not contain it. Thus the zeros which are not associated with a sheet up till a height T is approximately equal to $T \log 2/2\pi$. According to this, the number of sheets below a height T is

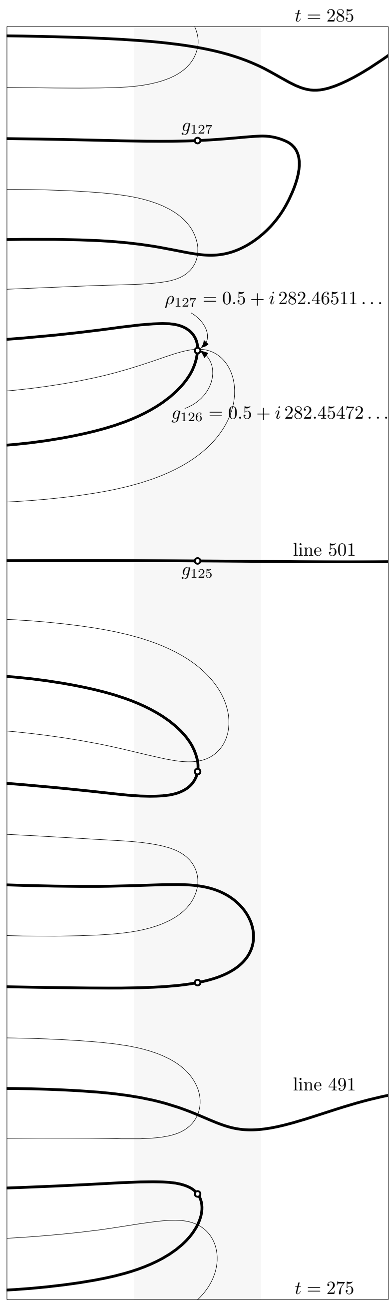
$$\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} - \Big(\frac{T\log 2}{2\pi} - \frac{1}{2}\Big) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + \frac{11}{8},$$

with an error of the order of $\log T$.

If the remarks we have made concerning the zeros of the derivative $\zeta'(s)$ are true this is the number of zeros of the derivative up till a height T. Spira conjectured this claim and Berndt [3] has proved it.

8. Looking higher. A Counterexample to Gram's Law

We will now see that most of the regularities in the behaviour of the function break at a great enough height.



Gram's law claims that between every two consecutive Gram points there exists a zero of the function $\zeta(s)$. The remark is Gram's [7], but it was Hutchinson who named it *Gram's law*, although it was him who found the first counterexample, which can be seen in the figure. Gram [7] only claimed that this would happen for the first values of n. The interval (g_{125}, g_{126}) does not contain zeros of the zeta-function. On the contrary, the next interval contains two zeros thus reestablishing the total count.
Later Lehmer [11] finds out that the exceptions grow more frequent as n increases. He also notices that, in general, these exceptions consist of a Gram interval in which there are no zeros, next to another which has two.

Possibly the only valid rule was the one formulated by Speiser: The number of thick lines crossing the line $\sigma = -1$ below a height T is

$$\frac{T}{\pi}\log\frac{T}{2\pi} - \frac{T}{\pi} + \frac{1}{4}$$

and the number of Gram points below this height is only half this number. A thick line which crosses the critical line has to do it through a Gram point or a zero. The parallel lines, alternatively, contain a Gram point or a zero. Speiser believes that each sheet uses a Gram point and a zero to enter and exit the area on the right of the critical line.

This is true, there exists a bijectiv map between the zeros and Gram points, but it is the one established by the fact that they are on the same sheet. It may be convenient to call also sheet to a parallel line which does not contain zeros and the parallel line inmediately above it.

This way, another sequence of natural numbers associated with the graphics of the zeta-function arises. In fact, Gram points g_{-1}, g_0, g_1, $\dots$ are associated with the zeros numbered

$$1, 2, 3, 4, 5, 7, 6, 8, 10, 9, 11, 13, 12, 14, 16, 15, 17, 18, 20, 19, 21, 23, 24, 22, 26, \\ 25, 27, 28, 30, 31, 29, 32, 34, 33, 35, 36, \dots$$

It is a permutation σ of the natural numbers, so that $|\sigma(n) - n| \le C \log n$. But, actually, it is well defined only if the Riemann Hypothesis is valid.

9. Almost Counterexample to Riemann Hypothesis (Lehmer)

An important landmark in the numerical study of the zeros of the zeta-function is Lehmer's paper [11] in the year 1956. In it, he proves that the first 10000 zeros of the function have a real part exactly equal to 1/2, so that the Riemann Hypothesis is valid at least for $t \le 9878.910$.

He establishes that, at this height, one out of ten Gram interval does not satisfy Gram's law. The number of exceptions increases continuously. He also finds out that in many occasions, in order to separate a zero, he must turn to Euler-MacLaurin formula because Riemann-Siegel's one does not have enough precission. This happens because the zeros of the zeta-function are very close. He studies specifically a particularly difficult case: it is an area near $t = 1114, 89$, situated in the Gram interval (g_{6707}, g_{6708}), where the function has two extremely close zeros,

$$\frac{1}{2} + i\ 7005.0629 \qquad \frac{1}{2} + i\ 7005.1006.$$

We will not repeat the graphics that Lehmer made about the behaviour of the function $Z(t)$ in a neighborhood of these points. Between these two zeros Hardy's function has the lowest relative maximum. This maximum is only 0.0039675 and it occurs at the point $t = 7005.0819$. Looking at the terms of $Z(t)$ in this point we see that a few of the first terms quickly increasing are counteracted by conspiracy of lots of small terms which sum up, thus the maximum turns out almost negative. A negative relative maximum would imply, it can be proved, a counterexample to the Riemann Hypothesis. So we call this situation an almost counterexample to the Riemann Hypothesis.

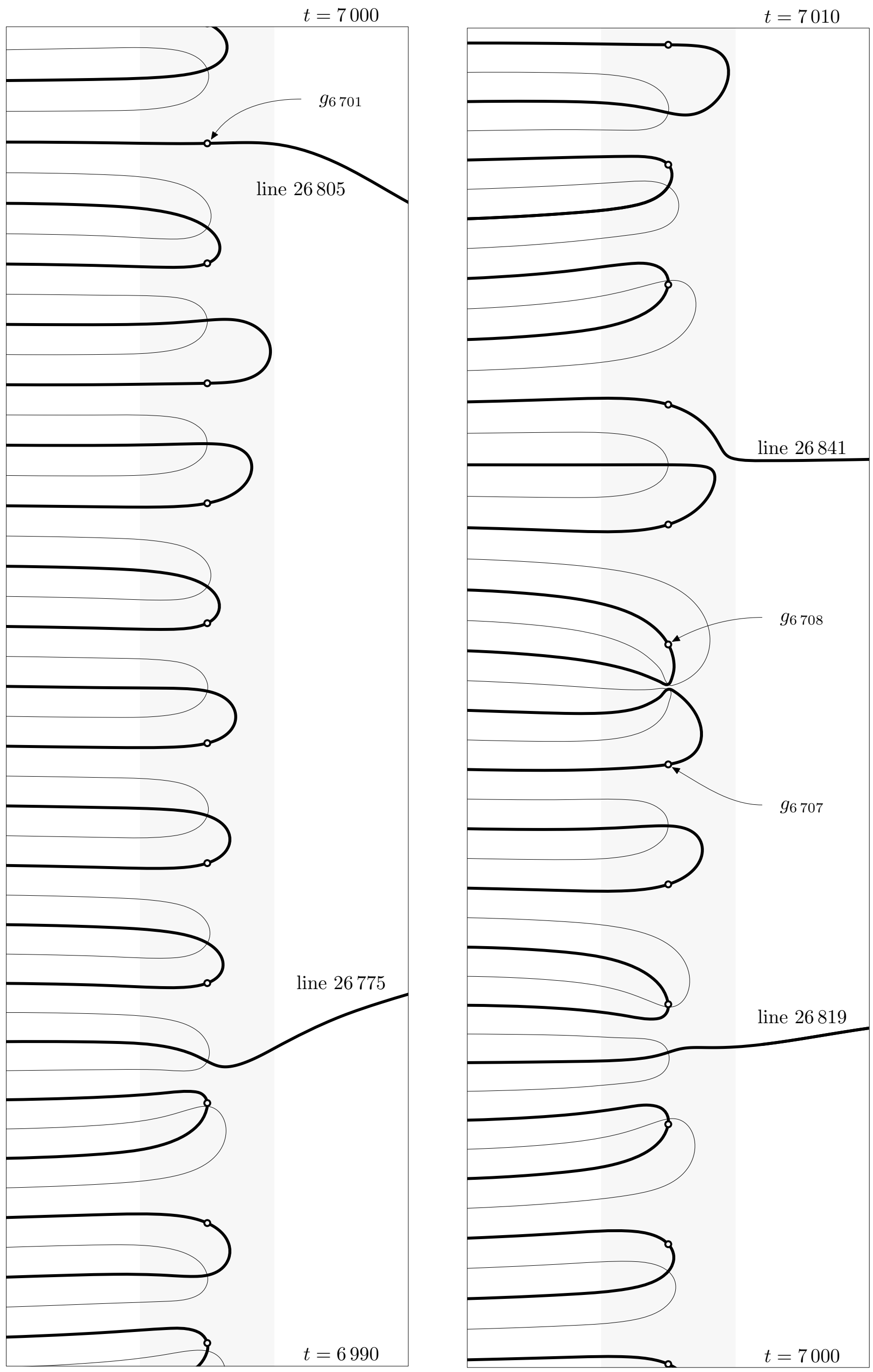


Figure 17. Almost counterexample of Lehmer

In the figures included in this page we can see the case analyzed by Lehmer. We notice that, from our point of view, it consists of two very close zeros, so that, viewing it from a far distance, it seems to be a double zero. We see that the lines seem to be continued more smoothly by the ones which are not actually joining them. Instead, the lines turn abruptly. Every time this is the case, that is, there are drastic changes in the direction of the lines, there is a zero of the derivative hanging about.

It would be easy to modify the lines artificially so that the two lines containing the Gram points would join and the other two thick lines would join each other too. So, slightly modifying the path of the thin lines, we could generate two zeros outside the critical line, which are symmetrical with respect to each other.

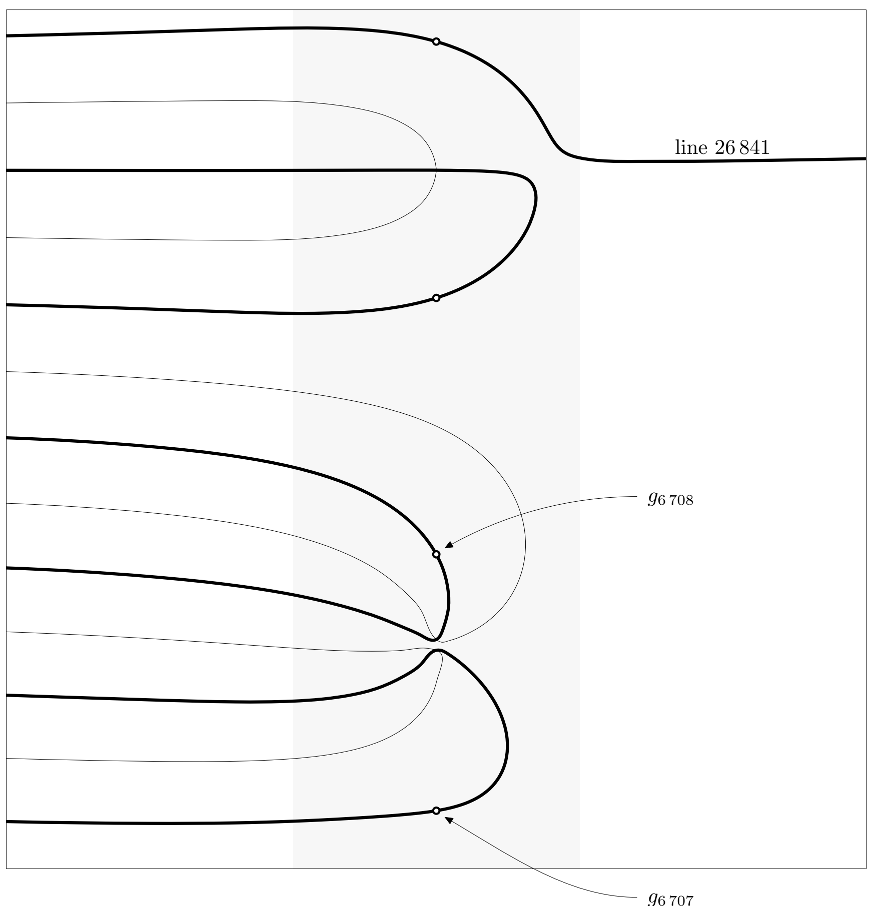


FIGURE 18. Detail of the last figure.

10. Rosser Law

Rosser, Yohe and Schoenfeld (1968) expand Lehmer's calculations and prove that the first 3 500 000 zeros are simple and situated on the critical line. These authors find out a certain regularity in the failures of Gram's law.

They distinguish between *good* and *bad* Gram points. Good points are those at which $\zeta(1/2 + ig_n) > 0$ holds, and bad ones are the rest of them. We know how to tell these points apart with the nacked eye in the X ray. Bad Gram points are surrounded by a thin line.

They call **Gram block** to a consecutive set of bad Gram points surrounded by two good ones. For example, in the precedent figure points g_{6707}, g_{6708} and g_{6709} form a Gram block.

Rosser's law claims that in a Gram block there are as many zeros as the number of Gram's intervals.

10.1. The function $S(t)$.

Let $N(T)$ be the number of zeros $\rho = \beta + i\gamma$ with $0 \le \gamma \le T$. An approximation to $N(T)$ is $\pi^{-1}\theta(T) + 1$, so that

$$N(T) = \pi^{-1}\theta(T) + 1 + S(T).$$

The value $\pi S(T)$ is also the variation of the argument of $\zeta(s)$ when s goes from $+\infty + iT$ to $1/2 + iT$.

Von Mangoldt proved that, as Riemann says, $S(T) = \boldsymbol{\mathcal{O}}(\log T)$. Later Littlewood proved

$$\int_0^T S(t)\,dt = \boldsymbol{\mathcal{O}}(\log T).$$

Selberg proves that

$$S(t) = \Omega_\pm\big((\log t)^{1/3}(\log\log t)^{-7/3}\big).$$

Thus, there exist values of t at which $S(t)$ is as high as we want it to be.

10.2. First counterexample to Rosser's law.

The first counterexample to Rosser's law (see Figure 9) is in the Gram block

$$(g_{13999525}, g_{13999527})$$

in which there is no zero of the function. In the following interval $J = (g_{13999527}, g_{13999528})$ there are three zeros which balance the total count.

In the graphics we see how the function $S(T)$ takes a value greater than 2 in a point which is situated between point $g_{13999527}$ and the first zero contained in the interval J.

This is a general rule: Gram's law is satisfied as long as $|S| < 1$ and Rosser's law as long as $|S| < 2$.

We have already said that there exist points on which S takes values as high as desired.

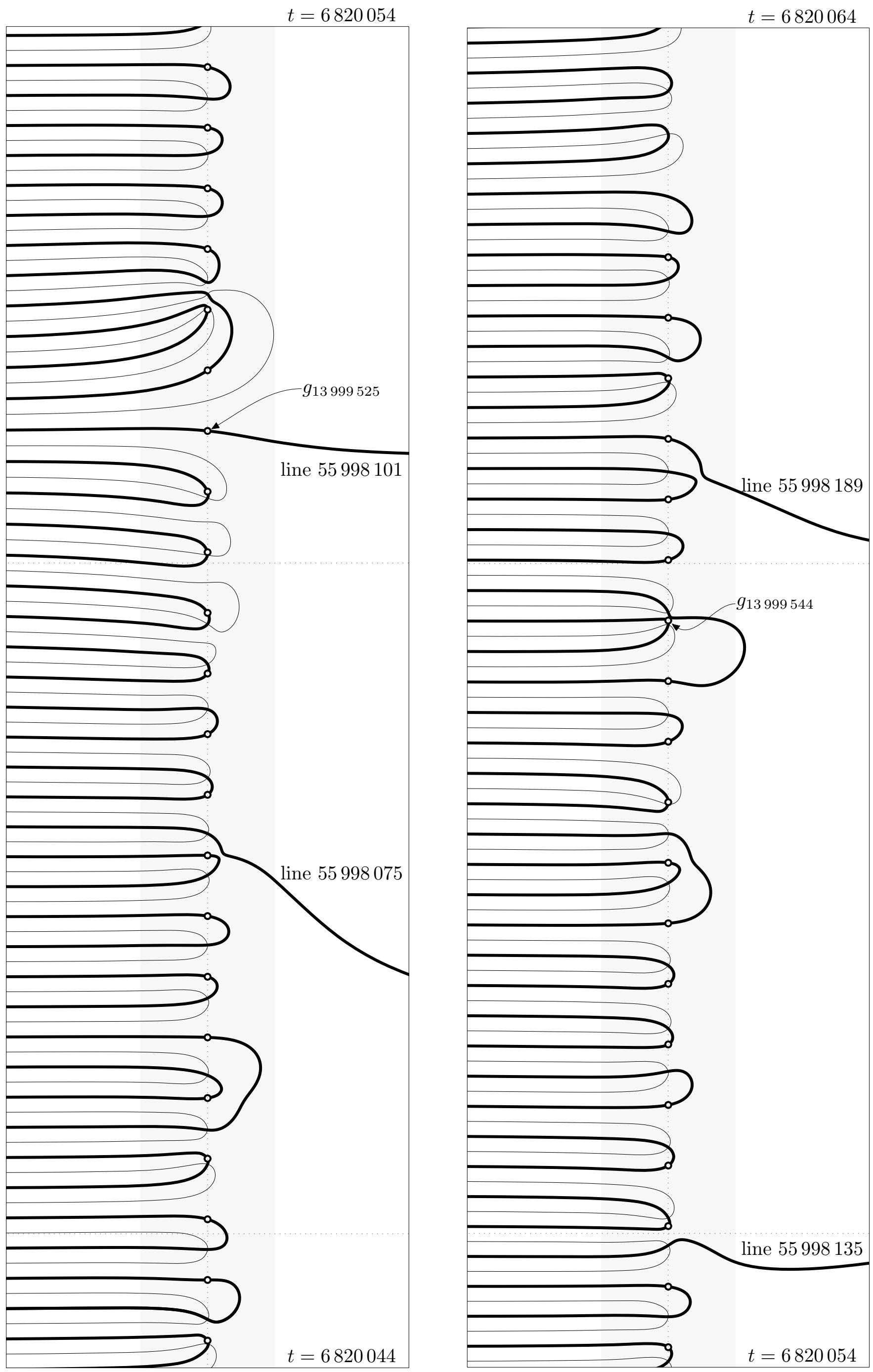


FIGURE 19. First counterexample to Rosser's law.

A high value of $S(t)$ corresponds to a point in the critical line, $1/2+it$, such that the segment from $1/2+it$ to $+\infty+it$ meets a high number of our lines.

In the tables from [4] and [5] we see that the extreme values of S which are known hardly surpass an absolute value of 2.

In Brent's table is already quoted the first counterexample to Rosser's law associated to the Gram point number 13999525, where $S(t)$ reaches a value of -2.004138. In fact, we see that there is a point $1/2+it_0$ imperceptibly above $g_{13999527}$, but before the next zero of $\zeta(s)$, such that the segment from this point to $1+it_0$ meets firstly a thick line, then a thin line, followed by another thick and another thin line. In the point $+\infty+it_0$ we start from a value equal to 1, enter the fourth quadrant, cross a thin line and thus we reach the third quadrant, cross a thick line entering the second quadrant, cross another thin line and find ourselves in the first quadrant, and finally another thick line and so we end up in the fourth quadrant. Consequently, the argument has changed in a quantity between 2π and $5\pi/2$, that, as we can see, agrees with the value given by Brent.

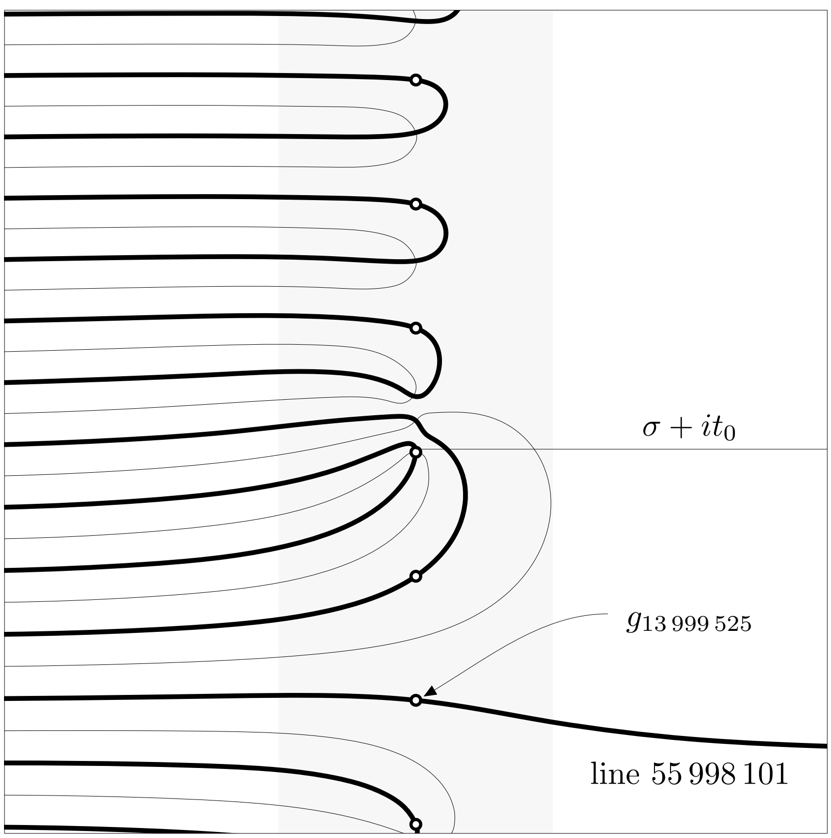


FIGURE 20. Detail of the last figure

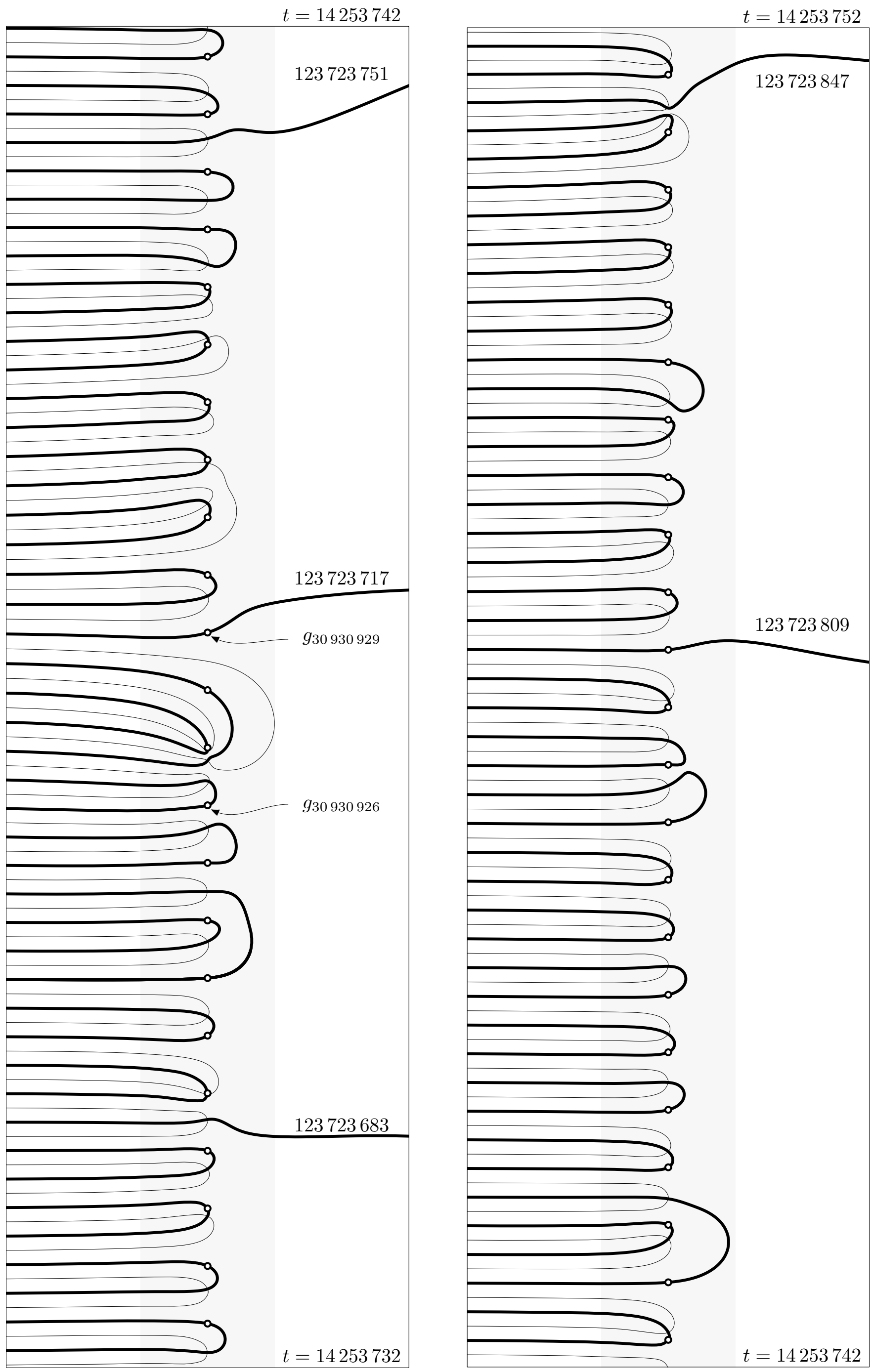


FIGURE 21. Another counterexample to Rosser's law.

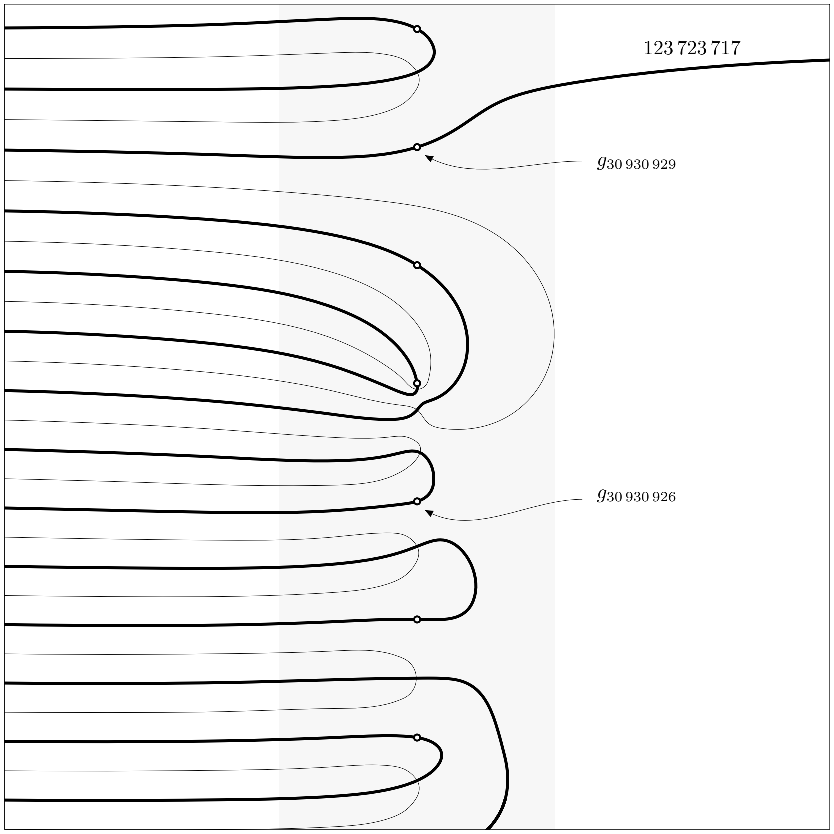


FIGURE 22. Detail of the last figure.

In the example presented in this page, S reaches a value 2.0506 and so the existence of the point $1/2 + it_0$ is a little clearer. In this case, S is positive, while in the precedent it was negative.

The previously quoted result of Selberg assures us that there are points at which $S(t)$ is as high as desired. Thus we can assume that, in a higher level, we will see coils which are analogous to Figures 22 and 20, but in which an arbitary number of lines gets involved.

He who watches attentively the preceding figures will not fail to notice that thin lines seem to be tangent at the line $\sigma = 1$. This leads me to formulate the following conjecture:

Conjecture 1. *For $Re(s) > 1$ it holds $Re\,\zeta(s) > 0$.*

A stronger version of the conjecture, for $\mathrm{Re}(s) \geq 1$, it holds $\mathrm{Re}\,\zeta(s) > 0$, would imply the prime number theorem, and thin lines would not be tangent at the critical line.

We finish with three figures of the zeta-function near a thousand millions, to show a ramdomly chosen area.

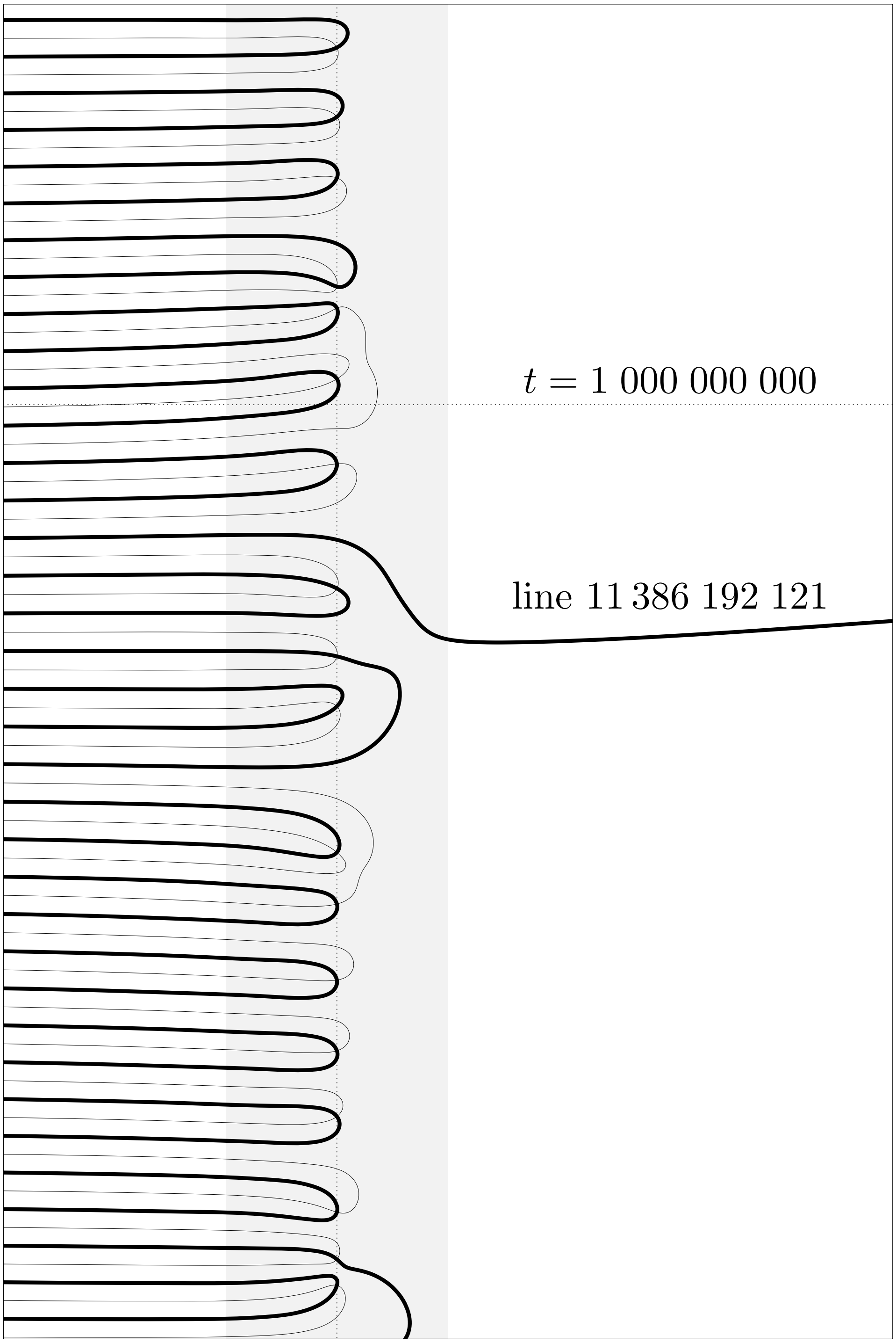


FIGURE 23. $\zeta(s)$ near $t = 1\,000\,000\,000$.

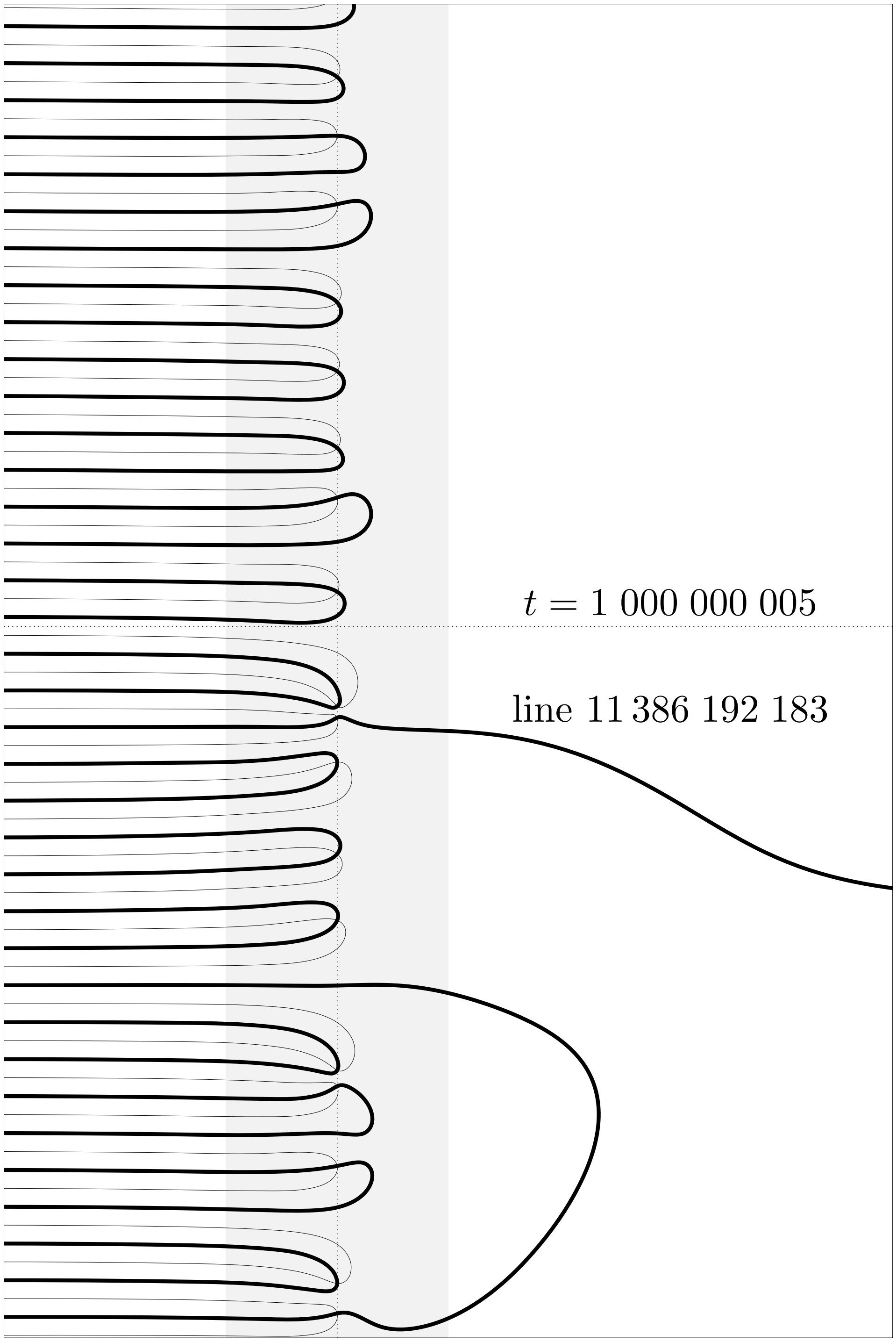


FIGURE 24 . $\zeta(s)$ near $t = 1\,000\,000\,000$.

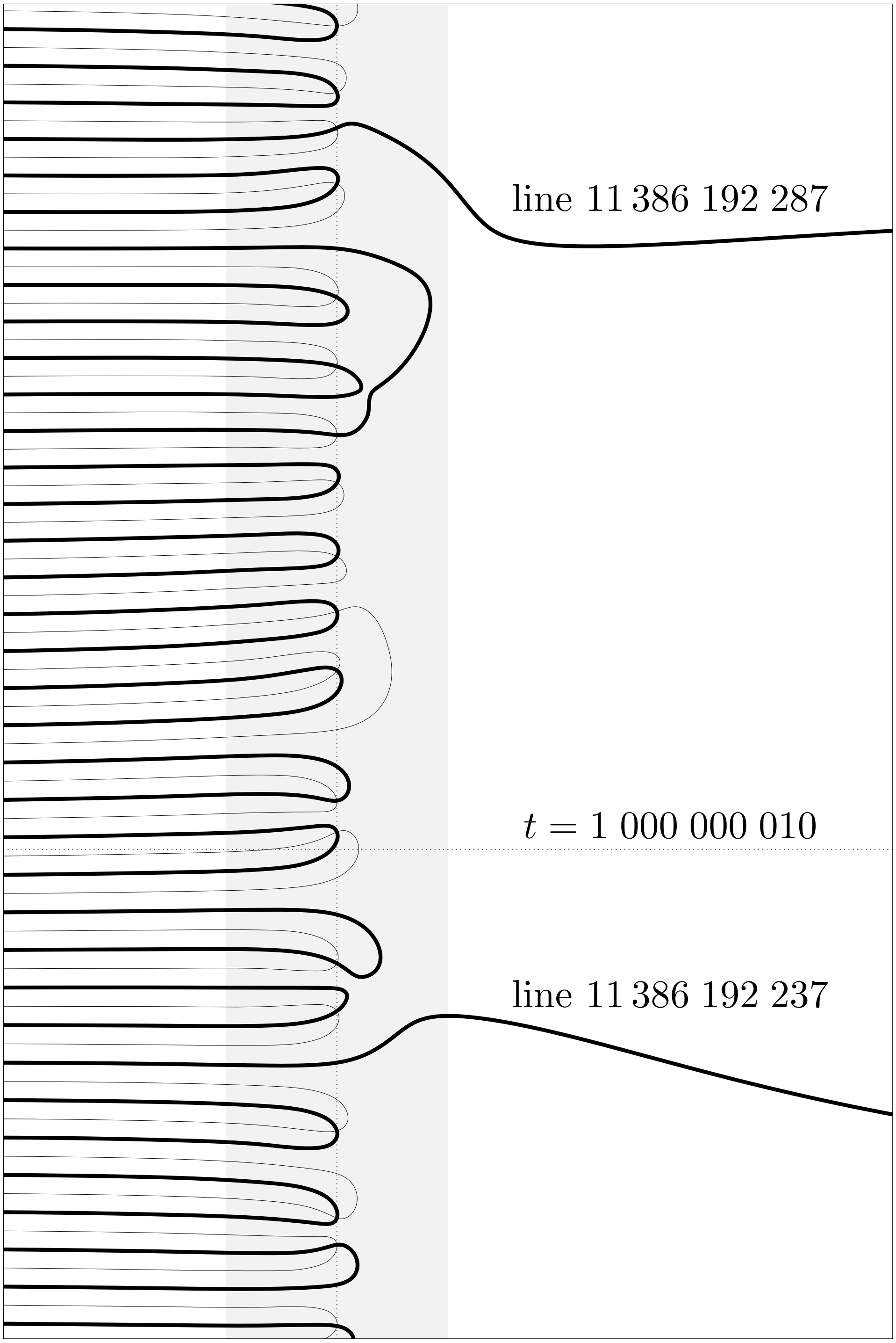


FIGURE 25. $\zeta(s)$ near $t = 1\,000\,000\,000$.

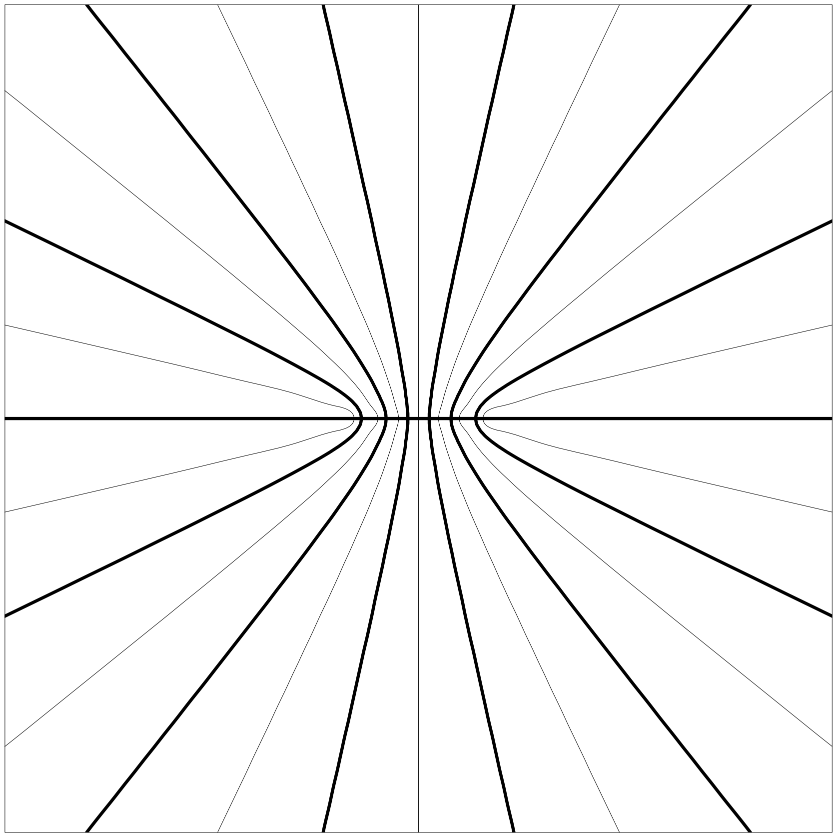


FIGURE 26. Hermite polynomial $H_7(z)$

To show how boring life can be outside Number Theory, we include the graphics of some functions.

The first one is Hermite's polynomial $H_7(z)$, that is,

$$128z^7 - 1344z^5 + 3360z^3 - 1680z$$

It has a degree equal to seven and all its roots are real. Here we represent it in the rectangle $(-17, 17)^2$, which is enough to get a clear idea of how the graphic is.

We can see the seven zeros of the function and the six ceros of the derivative.

The graphics of all Hermite's polynomials are analogous. It also looks like the graphics of other orthogonal polynomial families. But we must point out that a general polynomial can have a very complicated graphics. The regularity in this case is due to the fact that it is a very particular polynomial.

In this page we have the X rays corresponding to the Bessel function $J_7(z)$ and the Airy function $Ai(z)$ defined by

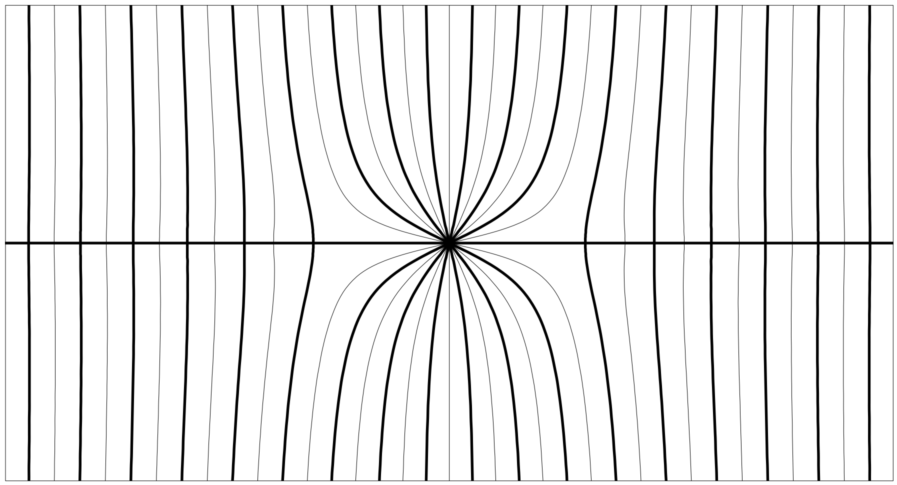


FIGURE 27. Bessel function $J_7(z)$

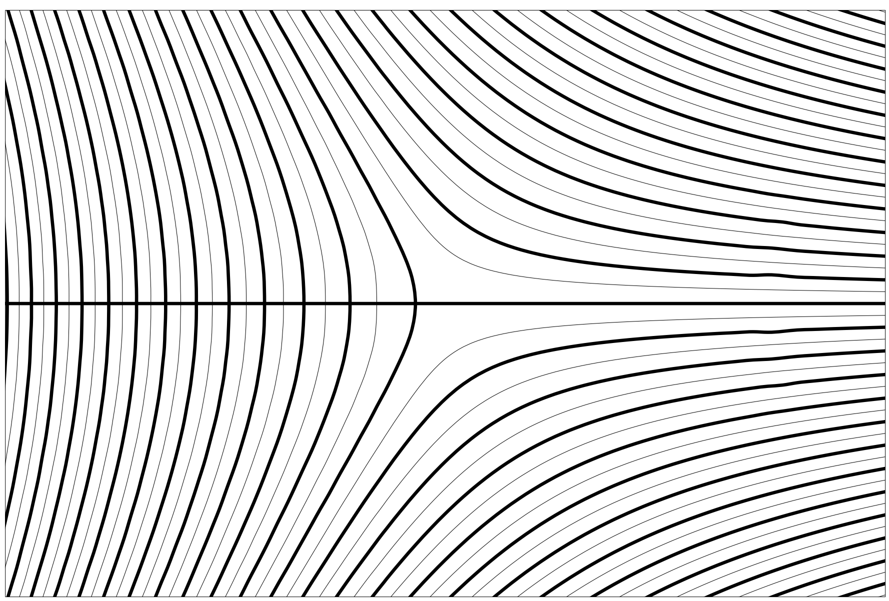


FIGURE 28. Airy function $Ai(z)$

$$J_7(z) = \Big(\frac{z}{2}\Big)^7 \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^k}{k!(k+7)!}, Ai(z) = \frac{3^{-2/3}}{\pi} \sum_{k=0}^{\infty} \frac{\Gamma\big((k+1)/3\big) \sin \frac{2\pi}{3}(k+1)}{k!} (3^{1/3} z)^k$$

The Bessel function is represented on the rectangle $(-28, 28) \times (-20, 20)$ and the Airy function on $(-15, 15) \times (-10, 10)$.

The Bessel function has a zero of order 7 in the origen. Its other zeros are real and, appart from the obvious zeros of the derivative, which are real, the derivative does not vanish.

The Airy function is surprising because of the likeness of its X ray with that of the Gamma function.

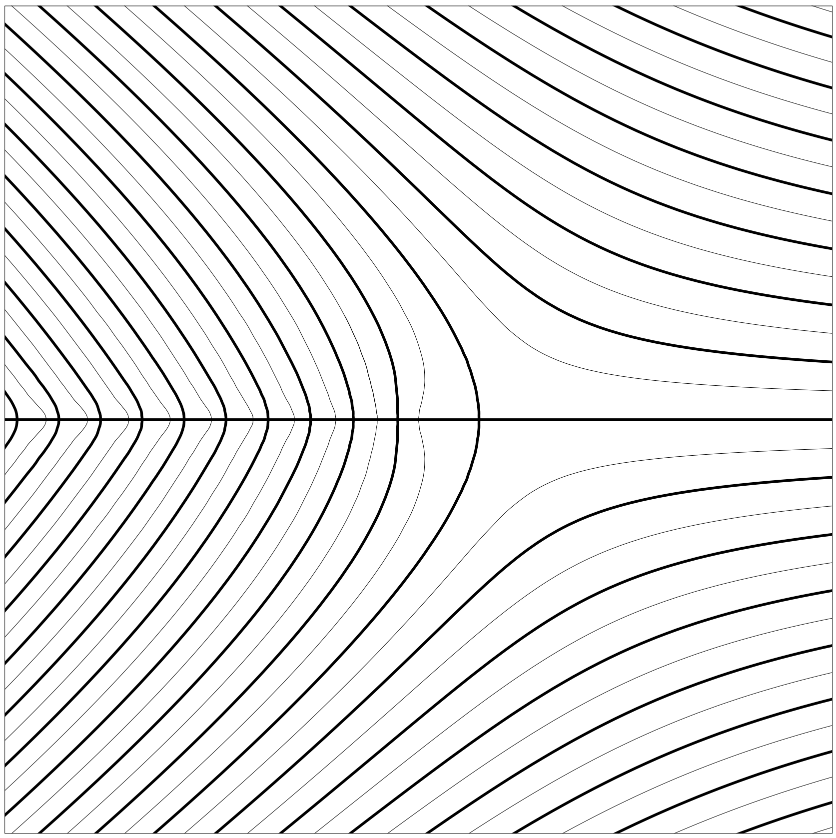


FIGURE 29. Function $\Gamma(s)$

The graphic of the function $\Gamma(s)$ shows that its derivative vanishes only at the obvious zeros. The graphics of a function and its inverse do always coincide. The figure shows the rectangle $(-10, 10)^2$.

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Facultad de Matemáticas, Universidad de Sevilla, P. O. box 1160, 41080-Seville, Spain

E-mail address: arias@us.es