manuscripta math. 118, 425–441 (2005)

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On complete families of curves with a given fundamental group in positive characteristic

Received: 28 March 2004 / Revised version: 30 May 2005
Published online: 5 December 2005

Abstract. In this paper we prove that complete families of smooth and projective curves of genus $g \geq 2$ in characteristic $p > 0$ with a constant geometric fundamental group are isotrivial.

Introduction

Let k be an algebraically closed field and let X be a complete irreducible and smooth curve over k of genus g. The structure of the étale fundamental group $\pi_1(X)$ of X is well understood, if $\mathrm{char}(k) = 0$, thanks to the Riemann existence Theorem. Namely it is isomorphic to the profinite completion Γ_g of the topological fundamental group of a compact orientable topological surface of genus g. In particular the structure of $\pi_1(X)$ depends only on g in this case. In the case where $\mathrm{char}(k) = p > 0$ the structure of the full $\pi_1(X)$ is far from being understood. However we understand the structure of some quotients of $\pi_1(X)$. Assume $\mathrm{char}(k) = p > 0$. Let $\pi_1^p(X)$ (resp. $\pi_1(X)^{p'}$) be the maximal pro-p (resp. maximal prime-to-p) quotient of $\pi_1(X)$. The following results are well known:

(1) The fundamental group $\pi_1(X)$ is a quotient of the group Γ_g. In particular $\pi_1(X)$ is topologically finitely generated.
(2) The structure of $\pi_1(X)^{p'}$ is well known by Grothendieck's specialization theory for fundamental groups ([16], X). Namely it is isomorphic to the maximal prime-to-p quotient of Γ_g.
(3) The structure of $\pi_1^p(X)$ is well known by Shafarevich theorem [19]. Namely it is a free pro-p-group on $r := r_X$ generators where r_X is the p-rank of the curve X, which is then "encoded" in the isomorphy type of the curve.

Apart from these results very little is known about the structure of the (geometric) fundamental group of curves in positive characteristic.

The anabelian geometry (or philosophy), as initiated by Grothendieck [5], predicted that the structure of the arithmetic fundamental group of hyperbolic curves over number fields should depend upon the isomorphy type of the curve in discussion. It came as a surprise when Tamagawa proved, in [24], such an anabelian

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DOI: 10.1007/s00229-005-0592-z

statement for hyperbolic affine curves defined over a finite field of characteristic $p > 0$. In this paper we investigate the question whether such anabelian phenomena hold for the geometric fundamental group of complete curves over arbitrary algebraically closed fields of characteristic $p > 0$, which would explain to some extent the complexity of π_1 in positive characteristic. In this paper we give a new evidence that this is indeed the case.

In order to get an idea about the complexity of π_1 of proper curves in positive characteristic we introduce the notion of fundamental group for points in the moduli space of curves. Let $\mathcal{M}_g \to \mathbb{F}_p$ be the coarse moduli space of proper and smooth curves of genus $g \geq 2$ in charactersitic $p > 0$. It is well known that $\mathcal{M}_g$ is a quasi-projective and geometrically irreducible variety. Let L be an algebraically closed field of characteristic p. Then $\mathcal{M}_g(L)$ is the set of isomorphism classes of irreducible proper and smooth curves of genus g over L. For a point $\overline{x} \in \mathcal{M}_g(L)$ let $C_{\overline{x}} \to \mathrm{Spec} L$ be a curve classified by $\overline{x}$ and let $x \in \mathcal{M}_g$ be a point such that $\overline{x} : \mathrm{Spec} L \to \mathcal{M}_g$ factors through x. We define the geometric fundamental group of the point x to be the fundamental group $\pi_1(C_{\overline{x}})$ of the curve $C_{\overline{x}}$ (cf. 4.1 for more details).

A key tool in the study of π_1 is Grothendieck's specialization theory for fundamental groups ([16], X). Let $y \in \mathcal{M}_g$ be a point which specializes in $x \in \mathcal{M}_g$. Then by Grothendieck's specialization theorem there exists a surjective continuous homomorphism $\mathrm{Sp}_{y,x} : \pi_1(C_{\bar{y}}) \to \pi_1(C_{\bar{x}})$. Concerning this specialization homomorphism we have the following result:

Theorem (Saïdi, Pop, Raynaud, Tamagawa:**).** *Let $x \in \mathcal{M}_g$ be a* **closed** *point and let $y \in \mathcal{M}_g$ be a point, distinct from x, and which specializes in x. Then the specialization homomorphism $Sp_{y,x} : \pi_1(C_{\bar{y}}) \to \pi_1(C_{\bar{x}})$ is not an isomorphism.*

The above theorem was proven by Pop and Saïdi [12] in the special case where the point x corresponds to a curve having an absolutely simple jacobian with p-rank equal to g or $g - 1$ and by Raynaud [15] in the case $g = 2$ and the case of supersingular curves of arbitrary genus $g > 2$ (i.e. curves whose jacobian is isogenous to a product of supersingular elliptic curves) and finally by Tamagawa [22] in the general case.

This result suggests that the structure of the geometric fundamental group π_1 is far from being constant on the moduli space $\mathcal{M}_g$ in characteristic $p > 0$ much contrary to the characteritic 0 case. Let k be an algebraically closed field of characteristic $p > 0$. Let $\mathcal{S} \subset \mathcal{M}_g \times_{\mathbb{F}_p} k$ be a k-subvariety. We say that the geometric fundamental group π_1 is *constant* on $\mathcal{S}$ if for any two points x and y of $\mathcal{S}$ such that y specializes in x the corresponding specialization homomorphism $\mathrm{Sp}_{y,x} : \pi_1(C_{\bar{y}}) \to \pi_1(C_{\bar{x}})$ is an isomorphism. This in particular would imply that all points of $\mathcal{S}$ have isomorphic geometric fundamental groups. We say that π_1 is *not constant* on $\mathcal{S}$ if the contrary holds namely: there exists two points x and y of $\mathcal{S}$ such that y specializes in x and such that the corresponding specialization homomorphism $\mathrm{Sp}_{y,x} : \pi_1(C_{\bar{y}}) \to \pi_1(C_{\bar{x}})$ is not an isomorphism. The above Theorem implies in particular that in the case $k = \overline{\mathbb{F}}_p$ the moduli space $\mathcal{M}_g \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ does not contain positive dimensional $\overline{\mathbb{F}}_p$-subvarieties on which π_1 is constant. It is thus natural to ask the following question:

Question 4.3. Let k be an algebraically closed field of characteristic $p > 0$. Does $\mathcal{M}_g \times_{\mathbb{F}_p} k$ contain k-subvarieties, of positive dimension > 0, on which π_1 is constant?

Our main result answering the above question is the following:

Theorem 4.4. *Let k be an algebraically closed field of characteristic p. Let $\mathcal{S} \subset \mathcal{M}_g \times_{\mathbb{F}_p} k$ be a* **complete** *k-subvariety of $\mathcal{M}_g \times_{\mathbb{F}_p} k$. Then the fundamental group π_1 is not constant on $\mathcal{S}$.*

Note that it is well known that $\mathcal{M}_g \times_{\mathbb{F}_p} k$ contains complete subvarieties ([11] for example). In the case where the generic points of the subvariety $\mathcal{S}$ are contained in the locus of ordinary curves (i.e. curves having maximal p-rank equal to g) this result is well known and due to Szpiro, Raynaud, and Moret-Bailly ([20], and [7]). Here one uses the fact that the p-rank is encoded in the isomorphy type of the fundamental group. It can also be deduced from Oort's result on complete families of abelian varieties of dimension g with constant p-rank equal to $g-1$ ([10], 6.2), if the generic points of the subvariety $\mathcal{S}$ correspond to curves with p-rank equal to $g-1$. One may wonder whether there exists non-isotrivial complete smooth families of curves of genus g with constant p-rank. It turns out that such families exist. In [11] Oort constructed an example of a non-isotrivial complete smooth family of curves of genus 3 having constant p-rank equal to 0. In the appendix we extend Oort's argument in order to construct such examples for any genus $g \geq 3$ (cf. B.4).

For the proof of Theorem 4.4 it is easy to reduce to the case where $\mathcal{S}$ is a complete curve. In this case we prove the following more precise result:

Theorem 4.6. *Let k be an algebraically closed field of characteristic $p > 0$. Let S be a smooth complete and irreducible k-curve. Let $f : X \to S$ be a* **non-isotrivial** *proper and smooth family of curves of genus $g \geq 2$. Then there exists a finite étale cover $S' \to S$ of S, a finite étale cover $Y' \to X' := X \times_S S'$ of degree prime to p, and a closed point $s_0 \in S'$, such that the p-rank of the geometric fibre $Y'_{k(\bar{s}_0)} \to k(\bar{s}_0)$ of Y' above the point s_0 is strictly smaller that the p-rank of the generic geometric fibre $Y'_{k(\bar{\eta})} \to k(\bar{\eta})$ of Y' above the generic point η of S'.*

The main ingredients we use in order to prove Theorem 4.6 are: first Raynaud's theory of theta divisors in positive characteristic. Secondly the Theorem of Szpiro, Raynaud, and Moret-Bailly, on the isotriviality of complete families of ordinary abelian varieties. And finally a recent result of Tamagawa on the equi-characteristic deformation of generalized Prym varieties. Finally the statement of Theorem 4.4 can be easily generalized to the case where we consider the (geometric) tame fundamental group (cf. theorem 4.10).

This paper is organized as follows. In sections 1 and 2 we review Raynaud's theory of theta divisors in characteristic $p > 0$ and its application to the study of the p-rank of cyclic with order prime-to-p étale covers of curves. In section 3 we explain Tamagawa's result on the equi-characteristic deformation of generalized Prym varieties. In appendix A we recall the results of Szpiro, Raynaud, Moret-Bailly, on complete families of abelian varieties with constant maximal p-rank. In appendix B we extend an argument of Oort in order to construct a complete family of smooth curves for every genus $g \geq 2$ which has constant p-rank equal to 0. In section 4 we prove our main result.

I would like very much to thank M. Raynaud and A. Tamagawa for their comments on an earlier version of this paper. I would like also to thank the referee for his (or her) careful reading of the paper. The author holds an EPSRC advanced research fellowship GR/R75861/02 and would like to thank EPSRC for its financial support.

1. The sheaf of locally exact differentials in characteristic $p > 0$ and its theta divisor

In this section we review, mainly following Raynaud, the definition of the sheaf of locally exact differentials associated to a smooth projective algebraic curve in positive characteristic and its theta divisor ([14], 4, and [21], 1, for futher generalisations). Let X be a proper smooth and connected algebraic curve of genus $g_X := g \geq 2$ over an algebraically closed field k of characteristic $p > 0$. Consider the following cartesian diagram:

$$\begin{array}{ccc} X^1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{F} & \operatorname{Spec} k \end{array}$$

where F denotes the absolute Frobenius morphism. The projection $X^1 \to X$ is a scheme isomorphism. In particular X^1 is a smooth and proper curve of genus g. The absolute Frobenius morphism $F : X \to X$ induces in a canonical way a k-morphism $\pi : X \to X^1$ called the *relative Frobenius* which is a radicial morphism of degree p. The canonical differential $\pi_* d : \pi_* \mathcal{O}_X \to \pi_* \Omega^1_X$ is a morphism of $\mathcal{O}_{X^1}$-modules. Its image $B_X := B := \operatorname{Im}(\pi_* d)$ is the *sheaf of locally exact differentials*. One has the following exact sequence:

$$0 \to \mathcal{O}_{X^1} \to \pi_* \mathcal{O}_X \to B \to 0$$

and B is a vector bundle on X^1 of rank $p - 1$.

Consider the *Cartier operator* $c : \pi_*(\Omega^1_X) \to \Omega^1_{X^1}$ which is a morphism of $\mathcal{O}_{X^1}$-modules. The kernel $\ker(c)$ of c is equal to B and the following sequence of $\mathcal{O}_{X^1}$-modules is exact ([17], 10):

$$0 \to B \to \pi_*(\Omega^1_X) \to \Omega^1_{X^1} \to 0$$

Let $\mathcal{L}$ be a *universal Poincaré bundle* on $X^1 \times_k J^1$, where $J^1 := \operatorname{Pic}^0(X^1)$ is the Jacobian variety of X^1. The restriction of $\mathcal{L}$ to $X^1 \times \{a\}$, for any $a \in J^1(k)$, is isomorphic to the degree zero line bundle $\mathcal{L}_a$ which is the image of a under the natural isomorphism $J^1(k) \simeq \operatorname{Pic}^0(X^1)$. Let $h : X^1 \times_k J^1 \to X^1$ and $f : X^1 \times_k J^1 \to J^1$ be the canonical projections. As $R^i f_*(h^* B \otimes \mathcal{L}) = 0$ for $i \geq 2$ the total direct image $Rf_*(h^* B \otimes \mathcal{L})$ of $(h^* B \otimes \mathcal{L})$ by f can be realized by a complex $u : \mathcal{M}^0 \to \mathcal{M}^1$ of length 1 where $\mathcal{M}^0$ and $\mathcal{M}^1$ are vector bundles on J^1. We have $\ker u = R^0 f_*(h^* B \otimes \mathcal{L})$ and $\operatorname{coker} u = R^1 f_*(h^* B \otimes \mathcal{L})$. Moreover as the Euler-Poincaré characteritic $\chi(B \otimes \mathcal{L}_a) = 0$, for all $a \in J^1(k)$, the vector bundles $\mathcal{M}^0$ and $\mathcal{M}^1$ have the same rank. In [14], Théorème 4.1.1, Raynaud proved the following theorem:

Theorem 1.1 (Raynaud:). *The determinant $\det u$ of u is not identically zero on J^1.*

In particular one can consider the divisor $\theta := \theta_B$ on J^1 which is the positive Cartier divisor locally generated by $\det u$. This is the *theta divisor* associated to the vector bundle B. By definition a point $a \in J^1(k)$ lies on the support of θ if and only if $H^0(X^1, B \otimes \mathcal{L}_a) \neq 0$.

2. p-Rank of cyclic étale covers with degree prime to p

We use the same notation as in 1. We will only discuss in this section the p-rank and the notion of *new-ordinariness* for cyclic covers of degree l := a prime integer distinct from p. This is the only case we use in this paper. For the general case of any integer prime to p see [15], 2, and [21], 3.

The absolute Frobenius morphism $F : X \to X$ induces a semi-linear map $F : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ and we have a canonical decomposition:

$$H^1(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X)^{\mathrm{ss}} \oplus H^1(X, \mathcal{O}_X)^{\mathrm{n}}$$

where $H^1(X, \mathcal{O}_X)^{\mathrm{ss}}$ is the semi-simple part on which F is bijective and $H^1(X, \mathcal{O}_X)^{\mathrm{n}}$ is the nilpotent part on which F is nilpotent. The *p-rank* $r_X := r$ of X is the dimension of the k-vector space $H^1(X, \mathcal{O}_X)^{\mathrm{ss}}$. By duality it is also the dimension of the subspace of $H^0(X, \Omega^1_X)$ on which the Cartier operator c is bijective ([17], 10). The p-rank r_X of X is also the rank of the maximal pro-p-quotient $\pi_1^p(X)$ of the fundamental group $\pi_1(X)$ of X, which is known to be a finitely generated free pro-p-group [19]. If A is an abelian variety of dimension d over k then the order of the étale part of the kernel of the morphism $[p] : A \to A$ of multiplication by p is p^h, where $0 \le h \le d$ is the p-rank of A. The abelian variety A is said to be *ordinary* if it has maximal p-rank equal to d, which is also equivalent to the fact that the Frobenius F is bijective on $H^1(A, \mathcal{O}_A)$. With the above notation if $J = \mathrm{Pic}^0(X)$ is the jacobian variety of X then it is well known that the p-rank of X equals the p-rank of J.

The relative Frobenius morphism $\pi : X \to X_1$ induces (because it is a radicial morphism) a "canonical" isomorphism $\pi_1(X) \to \pi_1(X_1)$ between fundamental groups [16], IX, Théorème 4. 10). In particular for any **prime** integer l, which is distinct from p, one has a one-to-one correspondence between μ_l-torsors of X and those of X^1. More precisely the canonical homomorphism $H^1_{\mathrm{et}}(X^1, \mu_l) \to H^1_{\mathrm{et}}(X, \mu_l)$ induced by π is an isomorphism. Consider a μ_l-torsor $f : Y \to X$ with Y connected. By Kummer theory, the torsor f is given by an invertible sheaf $\mathcal{L}$ of order l on X and $Y = \mathrm{Spec}(\oplus_{i=0}^{l-1} \mathcal{L}^{\otimes i})$. There exists then an invertible sheaf $\mathcal{L}^1$ on X^1, of order l, such that if $f' : Y^1 \to X^1$ is the associated μ_l-torsor we have a cartesian diagram:

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ Y^1 & \xrightarrow{f'} & X^1 \end{array}$$

Let $J_Y := \text{Pic}^0(Y)$ (resp. $J_X := \text{Pic}^0(X)$) denote the Jacobian variety of Y (resp. the Jacobian of X). The morphism $f : Y \to X$ induces a natural homomorphism $f^* : J_X \to J_Y$ between Jacobians which has a finite kernel (f^* is given by the pull-back of degree zero invertible sheaves). Let $J^{\text{new}} := J_{Y/X}$ denote the quotient of J_Y by the image $f^*(J_X)$ of J_X. The variety J^{new} is an abelian variety of dimension $g_Y - g_X$ and p-rank equal to $r_Y - r_X$. It is called the *new part* of the Jacobian J_Y of Y with respect to the morphism f.

Definition 2.1. *The μ_l-torsor $f : Y \to X$ is said to be* new-ordinary *if the new part J^{new} of the Jacobian of Y with respect to the morphism f is an ordinary abelian variety, i.e. if the equality $g_Y - g_X = r_Y - r_X$ holds.*

Raynaud's theory of theta divisors allows another important geometric interpretation of new-ordinariness which we explain below. This interpretation has allowed significant recent progress in the study of fundamental groups of curves in positive characteristics.

There exists an isomorphism $H^1(J_Y, \mathcal{O}_{J_Y}) \simeq H^1(Y, \mathcal{O}_Y)$ ([18], VII, théorème 9) and $H^1(Y, \mathcal{O}_Y) = H^1(X, f^*\mathcal{O}_Y) = H^1(X, \oplus_{i=0}^{l-1}\mathcal{L}^{\otimes i})$. From this we deduce that $H^1(J^{\text{new}}, \mathcal{O}_{J^{\text{new}}}) \simeq H^1(X, \oplus_{i=1}^{l-1}(\mathcal{L})^{\otimes i})$. Note that the above identifications are compatible with the action of Frobenius. Hence the kernel of Frobenius on $H^1(J^{\text{new}}, \mathcal{O}_{J^{\text{new}}})$ is isomorphic to the kernel of Frobenius acting on $H^1(X, \oplus_{i=1}^{l-1}\mathcal{L}^{\otimes i})$. On the other hand as f' is étale we have $(f')^*(B_X) = B_Y$. Thus also $(f')_*(B_Y) = B_X \otimes (f')_*(\mathcal{O}_{Y^1}) = \oplus_{i=0}^{l-1}(B_X \otimes (\mathcal{L}^1)^{\otimes i})$. Now by duality the kernel of the Frobenius acting on $H^1(X^1, \oplus_{i=1}^{l-1}\mathcal{L}^{1^{\otimes i}})$ is (non-canonically) isomorphic to the kernel of the Cartier operator acting on $H^0(X^1, \pi_*\Omega_X^1 \otimes (\oplus_{i=1}^{l-1}(\mathcal{L}^1)^{\otimes i}))$ which is $\oplus_{i=1}^{l-1}H^0(X^1, B_X \otimes (\mathcal{L}^1)^{\otimes i})$. Thus we see that the above μ_l-torsor $f : Y \to X$ is new ordinary if and only if the Frobenius F is injective (hence bijective) on $H^1(J^{\text{new}}, \mathcal{O}_{J^{\text{new}}})$, i.e. if and only if $H^0(X^1, B \otimes (\mathcal{L}^1)^{\otimes i}) = 0$ for all $i \in \{1, \ldots, l-1\}$. Finally this last statement is equivalent, by the very definition of the theta divisor θ_X associated to the vector bundle B_X, to the following:

Proposition 2.2. *The μ_l-torsor $f : Y \to X$ is new-ordinary if and only if the subgroup $< \mathcal{L}^1 >$ generated by $\mathcal{L}^1$ in J^1 intersects the support of the theta divisor θ_X at most at the zero point 0_{J^1} of J^1.*

Using the above interpretation of new-ordinariness and with an input from intersection theory one can prove that for $l >> 0$ "most" μ_l-torsors are new-ordinary. More precisely one has the following result which is essencially due to Serre and Raynaud (see [14], théorème 4.3.1, and [21], corollary 3.10, for a proof):

Theorem 2.3. *There exists a constant c depending only on g and p such that for each prime integer $l \neq p$ the set of elements of $J[l](k)$ whose corresponding μ_l-torsor is not new-ordinary has cardinality $\leq c(l-1)l^{2g-2}$ (Here $J[l]$ denotes the kernel of multiplication by l in J). Moreover one can take $c = (p-1)3^{g-1}g!$.*

In particular if $l >> 0$ we can find an element of $J[l](k)$ such that the corresponding μ_l-torsor is new-ordinary since Card $J[l](k) = l^{2g}$.

3. Equi-characteristic deformation of generalized Prym varieties

In this section we state the theorem of Tamagawa on the local infinitesimal Torelli problem for generalized Prym varieties. This theorem is an essential tool in the proof of the main result of this paper.

Let k be an algebraically closed field of arbitrary characteristic. Denote by $\mathcal{C}_k$ the category of artinian local rings with residue field k. For a proper and smooth k-variety X_0 one defines the (equi-characteristic) deformation functor M_{X_0} of X_0 to be the functor:

$$M_{X_0} : \mathcal{C}_k \to \text{(Sets)}$$

which to an element R of $\mathcal{C}_k$ associates the set of isomorphism classes of pairs (X, φ) where X is a proper and smooth R-scheme and φ is an isomorphism $X \times_R k \simeq X_0$. The functor M_{X_0} is well understood in the case where X_0 has dimension 1 and genus $g \geq 2$ (resp. if X_0 is an abelian variety of dimension d). In this case the functor M_{X_0} is pro-representable by a ring of formal power series of $3g-3$ (resp. d^2) variables over k. We will be mainly interested in these two cases.

Now assume that X_0 is a proper connected and smooth algebraic curve over k with genus $g \geq 2$. Let l be a prime integer distinct from the characterisic of k and let $f_0 : Y_0 \to X_0$ be a μ_l-torsor with Y_0 connected. The torsor f_0 corresponds to an element $\mathcal{L}_0 \in J_0[l](k)$ where $J_0[l](k)$ denotes the k-subgroup of l-torsion points in the jacobian J_0 of X_0. Let J_0^{new} be the new part of the jacobian of Y_0 with respect to the morphism f_0. Then for any element R of $\mathcal{C}_k$ there is a natural map:

$$T_{\mathcal{L}_0}(R) : M_{X_0}(R) \to M_{J_0^{\text{new}}}(R)$$

defined as follows: let (X, φ) be an element of $M_{X_0}(R)$. The μ_l-torsor $f_0 : Y_0 \to X_0$ lifts uniquely, by the theorems of lifting of étale covers ([16], I, 8), to a μ_l-torsor $f : Y \to X$. Let $J_X := \text{Pic}^0(X)$ (resp. $J_Y := \text{Pic}^0(Y)$) be the relative jacobian of X (resp. the relative jacobian of Y) which is an abelian scheme over R and let $f^* : J_X \to J_Y$ be the natural homomorphism which is induced by the pull back of invertible sheaves. Define $J^{\text{new}} := J_Y/f^*(J_X)$ to be the quotient of J_Y by the image $f^*(J_X)$ of J_X. Then J^{new} is an abelian R-scheme and there exists a natural isomorphism $\psi : J^{\text{new}} \times_R k \simeq J_0^{\text{new}}$ which is induced by φ. Thus the pair (J^{new}, ψ) is an element of $M_{J_0^{\text{new}}}(R)$ which we define to be the image under $T_{\mathcal{L}_0}(R)$ of (X, φ) in $M_{J_0^{\text{new}}}(R)$. The infinitesimal Torelli problem asks whether or not the above natural map:

$$T_{\mathcal{L}_0} : M_{X_0} \to M_{J_0^{\text{new}}}$$

is an immersion. More precisely let $k[\epsilon]$ ($\epsilon^2 = 0$) be the ring of dual numbers on k. Then the question is whether the natural map: $T_{\mathcal{L}_0}(k[\epsilon]) : M_{X_0}(k[\epsilon]) \to M_{J_0^{\mathrm{new}}}(k[\epsilon])$ between tangent spaces is injective. This is also equivalent asking whether if A (resp. B) is the pro-representing object of the functor M_{X_0} (resp. of the functor $M_{J_0^{\mathrm{new}}}$) then the natural homomorphism $B \to A$ induced by $T_{\mathcal{L}_0}$ is surjective. If this is the case it would in particular imply the following: for every element $R \in \mathcal{C}_k$ and $(X, \varphi) \in M_{X_0}(R)$ if the image (J^{new}, ψ) of (X, φ) in $M_{J_0^{\mathrm{new}}}(R)$ via the map $T_{\mathcal{L}_0}(R)$ is a trivial deformation of J_0^{new} then X is a trivial deformation of X_0. This is a generalization of the classical infinitesimal Torelli problem which asks whether the natural map from the space of deformations of the curve X_0 to the space of deformations of its jacobian is an immersion in which case one knows that the answer is yes, if the curve X_0 is not hyperelliptic. Concerning the above generalization Tamagawa proves the following:

Theorem 3.1 (Tamagawa:)**.** *Let X_0 be a proper connected and smooth algebraic curve over an algebraically closed field k with genus $g \geq 2$. Let $d_{X_0} := min\{deg(f) \,/\, f : X_0 \to \mathbb{P}^1_k$ non constant$\}$ be the gonality of X_0 ([22], 1). Assume that $d_{X_0} \geq 5$. Then there exists a constant c_1 which depends only on X_0 and such that for each prime integer $l \neq char(k)$ the subset of the elements $\mathcal{L}_0$ of $J[l](k)$ such that the corresponding natural map $T_{\mathcal{L}_0} : M_{X_0} \to M_{J_0^{\mathrm{new}}}$ is not an immersion has cardinality $\leq c_1 l^{2g-2}$.*

In particular if $l >> 0$ then one can find an element $\mathcal{L}_0 \in J[l](k)$ such that the corresponding map: $T_{\mathcal{L}_0} : M_{X_0} \to M_{J_0^{\mathrm{new}}}$ is an immersion. For a proof of the above result see [22], corollary 4.16. Concerning the gonality of curves Tamagawa also proves the following:

Theorem 3.2 (Tamagawa:)**.** *Let X be a proper connected and smooth algebraic curve over an algebraically closed field k, with genus $g \geq 2$. Then there exists an étale cover $f : Y \to X$ such that the gonality d_Y of Y satisfies $d_Y \geq 5$. Moreover the cover f can be chosen to be a composition of two cyclic étale covers of (suitable) degree prime to the characteristic of k.*

For the proof of Theorem 3.2 combine Theorem 2.7, Proposition 2.14, and Corollary 2.19 from [22].

Combining both the Theorems 3.1 and 2.3 above we obtain the following result which we will use in the proof of our main Theorem in section 6. This result was also used by Tamagawa, in [22], in order to prove Theorem 6.1 in that paper.

Theorem 3.3. *Let X be a proper connected and smooth algebraic curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$. Assume that the gonality d_X of X is ≥ 5. Then if $l \neq p$ is a prime integer such that $l > 1 + c_1 + (p-1)3^{g-1}g!$, where c_1 is the constant in Theorem 3.1, there exists a non zero element $\mathcal{L} \in J[l](k)$ such that the following two conditions are satisfied:*

(i) The μ_l-torsor $f : Y \to X$ corresponding to $\mathcal{L}$ is new ordinary.
(ii) The natural map $T_{\mathcal{L}} : M_X \to M_{J^{\mathrm{new}}}$ is an immersion.

4. Complete families of curves with a given fundamental group in characteristic $p > 0$

This is our main section in which we prove the main result of this paper which asserts that complete families of curves with a constant geometric fundamental group are isotrivial. In all what follows we fix a prime integer $p > 0$.

First we will explain how to define the fundamental group of points in the moduli space of curves. Let $\mathcal{M}_g \to \operatorname{Spec} \mathbb{F}_p$ be the coarse moduli scheme of smooth and projective curves of genus g in characteristic $p > 0$. Let k be an algebraically closed field of characteristic $p > 0$. For a geometric point $\overline{x} \in \mathcal{M}_g(k)$ let $C_{\overline{x}} \to \operatorname{Spec} k$ be a smooth and projective curve of genus g which is classified by $\overline{x}$ and let $x \in \mathcal{M}_g$ be the point such that $\overline{x} : \operatorname{Spec} k \to \mathcal{M}_g$ factors through x. We define the *geometric fundamental group* of the point x to be the geometric fundamental group $\pi_1(C_{\overline{x}})$ of the curve $C_{\overline{x}}$ (we assume of course the choice of a base point). We remark that the structure of $\pi_1(C_{\overline{x}})$, as a profinite group depends only on the point x and not on the concrete geometric point $\overline{x} \in \mathcal{M}_g(k)$ used to define it. Indeed, first if $\overline{k(x)}$ is an algebraic closure of the residue field $k(x)$ at x and C_x is the curve classified by $\operatorname{Spec} \overline{k(x)} \to \mathcal{M}_g$ then $C_{\overline{x}} \simeq C_x \times_{\overline{k(x)}} k$ is the base change of C_x to k. Hence $\pi_1(C_{\overline{x}}) \simeq \pi_1(C_x)$ by the geometric invariance of the fundamental group for proper varieties ([16], X, Corollaire 1.8). Second the isomorphy type of C_x as an $\mathbb{F}_p$-scheme does not depend on the choice of $\overline{k(x)}$: a geometric point of M_g dominating x.

Next we recall the specialization theory of Grothendieck for fundamental groups. Let $y \in \mathcal{M}_g$ be a point which specializes into the point $x \in \mathcal{M}_g$. Then Grothendieck's specialization theorem shows the existence of a surjective continuous homomorphism $\operatorname{Sp} : \pi_1(C_{\overline{y}}) \to \pi_1(C_{\overline{x}})$ ([16], X). In particular if η is the generic point of $\mathcal{M}_g$ then $C_{\bar{\eta}}$ is the generic curve of genus g and every point x of $\mathcal{M}_g$ is a specialization of η. Hence for every $x \in \mathcal{M}_g$ there exists a surjective homomorphism $\operatorname{Sp}_x : \pi_1(C_{\overline{\eta}}) \to \pi_1(C_{\overline{x}})$. For every such an x we fix such a map once for all. In particular if y specializes to x we also fix a surjective homomorphism $\operatorname{Sp}_{y,x} : \pi_1(C_{\overline{y}}) \to \pi_1(C_{\overline{x}})$, such that $\operatorname{Sp}_{y,x} \circ \operatorname{Sp}_y = \operatorname{Sp}_x$.

Definition 4.1. *Let $\mathcal{S} \subset \mathcal{M}_g$ be a subscheme of $\mathcal{M}_g$. We say that the (geometric) fundamental group π_1 is* constant *on $\mathcal{S}$ if for any two points x and y of $\mathcal{S}$ such that y specializes in x the corresponding specialization homomorphism $Sp_{y,x}$: $\pi_1(C_{\overline{y}}) \to \pi_1(C_{\overline{x}})$ is an isomorphism. We say that π_1 is* not constant *on $\mathcal{S}$ if the contrary holds namely: there exist two points x and y of $\mathcal{S}$ such that y specializes in x and such that the corresponding specialization homomorphism $Sp_{y,x}$: $\pi_1(C_{\bar{y}}) \to \pi_1(C_{\bar{x}})$ is not an isomorphism.*

For every field k of characteristic p we define in a similar way the geometric fundamental group of points in $\mathcal{M}_g \times_{\mathbb{F}_p} k$ as well as the notion of a subvariety $\mathcal{S} \subset \mathcal{M}_g \times_{\mathbb{F}_p} k$ on which the geometric fundamental group π_1 is constant.

Definition 4.2. *Let S be a connected scheme of characteristic p and let $f : X \to S$ be a relative smooth S-curve of genus g. We say that the (geometric) fundamental group π_1 is* constant *on the family f if for any two points t and s of S, such that t*

specializes in s, the corresponding specialization homomorphism $Sp_{t,s} : \pi_1(X_{\bar{t}}) \to \pi_1(X_{\bar{s}})$ is an isomorphism, where $X_{\bar{t}} := X \times_S k(\bar{t})$ (resp. $X_{\bar{s}} := X \times_S k(\bar{s})$) is the geometric fibre of X over the point t, (resp. the geometric fibre of X over the point s). If the above condition doesn't hold we say that the fundamental group π_1 is not constant on the family f.

Let S be a scheme of characteristic p and let $f : X \to S$ be a relative smooth S-curve of genus g. It is clear that the fundamental group is constant on the family f if and only if the fundamental group is constant on the image of the map $S \to \mathcal{M}_g$ induced by the family f. It is quite natural to ask the following question:

Question 4.3. Let k be an algebraically closed field of characteristic p. Does $\mathcal{M}_g \times_{\mathbb{F}_p} k$ contain k-subvarieties of positive dimension > 0 on which π_1 is constant?

Our main result is the following:

Theorem 4.4 (Main Result:**).** *Let k be an algebraically closed field of characteristic p. Let $\mathcal{S} \subset \mathcal{M}_g \times_{\mathbb{F}_p} k$ be a* **complete** *k-subvariety of $\mathcal{M}_g \times_{\mathbb{F}_p} k$. Then the fundamental group π_1 is not constant on $\mathcal{S}$.*

For the proof of Theorem 4.4 it is clear that one can reduce to the case where S is a complete and irreducible curve. The proof of Theorem 4.4 then follows easily by using B.1 (appendix) from the following Theorem 4.5:

Theorem 4.5. *Let k be an algebraically closed field of characteristic p. Let S be a smooth complete and irreducible k-curve. Let $f : X \to S$ be a* **non-isotrivial** *proper and smooth family of curves of genus $g \geq 2$. Then the fundamental group π_1 is not constant on the family f. Or equivalently if $h : S \to \mathcal{M}_g \times_{\mathbb{F}_p} k$ is the map defined by f then the fundamental group π_1 is not constant on the image $h(S)$ of S in $\mathcal{M}_g \times_{\mathbb{F}_p} k$.*

In the process of proving Theorem 4.5 we prove in fact the following more precise result:

Theorem 4.6. *Let k be an algebraically closed field of characteristic p. Let S be a smooth complete and irreducible k-curve. Let $f : X \to S$ be a* **non-isotrivial** *proper and smooth family of curves of genus $g \geq 2$. Then there exist a finite étale cover $S' \to S$, a finite étale cover $Y' \to X' := X \times_S S'$ of degree prime to p, and a closed point $s_0 \in S'$ such that the p-rank of the geometric fibre $Y'_{k(\bar{s}_0)} \to k(\bar{s}_0)$ of Y' above the point s_0 is strictly smaller that the p-rank of the generic geometric fibre $Y'_{k(\bar{\eta})} \to k(\bar{\eta})$ of Y' above the generic point η of S'.*

First we start with the following lemmas:

Lemma/Definition 4.7. *Let k be an algebraically closed field of characteristic p. Let S be a smooth complete and irreducible k-curve. Let $f : X \to S$ be a smooth family of curves. Let s be a closed point of S and let $f_s : Y_s \to X_s := X \times_S k(s)$ be a μ_n-torsor over the fibre of X above the point s, where n is coprime to p. Then there exist a positive integer d, such that if n is coprime to d, then there exist a finite étale* *cover $h : S' \to S$, a μ_n-torsor $f' : Y' \to X' := X \times_S S'$, and a closed point $s' \in S'$ with $h(s') = s$ such that the fibre $f'_{s'} : Y'_{s'} := Y' \times_{S'} k(s') \to X'_{s'} := X' \times_{S'} k(s')$ of the torsor f' above the point $s' \in S'$ coincides with the given cover $f_s : Y_s \to X_s$. We call such a pair (f', h) a* **good lifting** *of the cover $f_s : Y_s \to X_s$.*

Proof. In the case where the morphism f has a section the above lemma follows easily from the homotopy exact sequence of fundamental groups in [16], XIII, Proposition 4.3 (see Lemma 4.3.1 in loc. cit). In the general case, let x be a closed point of the generic fibre X_η of X over S and let Z be the schematic closure of x in X. Denote by Y be the normalization of Z. The canonical morphism $Y \to S$ is finite of degree d, it is a "multisection" of f of degree d. Assume further that the integer n is coprime to d. In the following we may and shall identify μ_n with $\mathbb{Z}/n\mathbb{Z}$. The sheaf $R^1 f_*(\mathbb{Z}/n\mathbb{Z})$ is locally constant on S_{et}. In particular there exists a finite étale cover $h : S' \to S$ such that $R^1 f_*(\mathbb{Z}/n\mathbb{Z})/S'$ is constant. We denote by $Y' \to X' := X \times_S S' \to S'$ a multisection of $f' : X' \to S'$ above Y. The Leray spectral sequence in étale cohomology with respect to the morphism $f' : X' := X \times_S S' \to S'$ and the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ gives rise to an exact sequence of terms of low degree: $0 \to H^1(S', f'_*(\mathbb{Z}/n\mathbb{Z})) \to H^1(X', \mathbb{Z}/n\mathbb{Z}) \to H^0(S', R^1 f'_*(\mathbb{Z}/n\mathbb{Z})) \to H^2(S', f'_*(\mathbb{Z}/n\mathbb{Z})) \to H^2(X', \mathbb{Z}/n\mathbb{Z})$. Note that $f'_*(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. Let s' be a closed point of S' such that $h(s') = s$. The fibre of the sheaf $R^1 f'_*(\mathbb{Z}/n\mathbb{Z})$ at s' is isomorphic to $H^1(X_s, \mathbb{Z}/n\mathbb{Z})$. Let $c_s \in H^1(X_s, \mathbb{Z}/n\mathbb{Z})$ be the class corresponding to the torsor $f_s : Y_s \to X_s := X \times_S k(s)$. Then c_s can be lifted to a global section $c \in H^0(S', R^1 f'_*(\mathbb{Z}/n\mathbb{Z}))$ since $R^1 f_*(\mathbb{Z}/n\mathbb{Z})/S'$ is constant. The element c is the image of a class $\tilde{c} \in H^1(X', \mathbb{Z}/n\mathbb{Z})$ via the above sequence if and only if its image c' in $H^2(S', f'_*(\mathbb{Z}/n\mathbb{Z}))$ vanishes. The element c' is thus the obstruction to lift the μ_n-torsor $f_s : Y_s \to X_s := X \times_S k(s)$ to a μ_n-torosr $f' : Y' \to X'$. We will show that $c' = 0$. The image of c' in $H^2(X', \mathbb{Z}/n\mathbb{Z})$ vanishes thus it also vanishes in $H^2(Y', \mathbb{Z}/n\mathbb{Z})$ via the canonical map $H^2(X', \mathbb{Z}/n\mathbb{Z}) \to H^2(Y', \mathbb{Z}/n\mathbb{Z})$. We have a canonical map $H^2(S', \mathbb{Z}/n\mathbb{Z}) \to H^2(Y', \mathbb{Z}/n\mathbb{Z})$. We also have a norm map $H^2(Y', \mathbb{Z}/n\mathbb{Z}) \to H^2(S', \mathbb{Z}/n\mathbb{Z})$ and the composite map $H^2(S', \mathbb{Z}/n\mathbb{Z}) \to H^2(Y', \mathbb{Z}/n\mathbb{Z}) \to H^2(S', \mathbb{Z}/n\mathbb{Z})$ is multiplication by d. Hence we deduce that the class c' is annihilated by d. Since it is also annihilated by n, the group $H^2(S', \mathbb{Z}/n\mathbb{Z})$ being n-torsion, we deduce that $c' = 0$. □

Lemma 1. *Let k be an algebraically closed field of characteristic p. Let S be a smooth complete and irreducible k-curve. Let $f : X \to S$ be a proper and smooth family of curves of genus $g \geq 2$. Assume that the fundamental group π_1 is constant on the family f. Then for every finite cover $S' \to S$, and every finite étale cover $Y' \to X' := X \times_S S'$, the fundamental group π_1 is also constant on the family $Y' \to S'$.*

Proof. Standard using the functorial properties of fundamental groups. □

Lemma 2. *Let k be an algebraically closed field of characteristic p. Let S be a smooth complete and irreducible k-curve. Let $f : X \to S$ be a proper and smooth family of curves of genus $g \geq 2$. Let $S' \to S$ be a finite étale cover and let $Y' \to X' := X \times_S S'$ be an étale cover. Assume that the smooth relative S'-curve $Y' \to S'$ is isotrivial. Then the smooth relative S-curve $X \to S$ is also isotrivial.*

Proof. Use Lemma 1.32 in [23]. □

Next we will prove the main Theorem 4.5.

Proof of Theorems 4.5 and 4.6. Fix a closed point s of S and let $X_s := X \times_S k(s)$ be the fibre of X above the point $s \in S$. By Tamagawa's result, Theorem 3.2, we can find an étale cover $Y_s \to X_s$ such that the gonality d_{Y_s} of Y_s is ≥ 5. Moreover the cover $Y_s \to X_s$ can be chosen to be a composition of two cyclic of order prime to p covers and such that these degrees are coprime to the degree of a given multisection of $X \to S$. In particular we can find by Lemma 4.7 a finite étale cover $h : S_1 \to S$ and an étale cover $f_1 : Y \to X_1 := X \times_S S_1$ such that the pair (f_1, h) is a good lifting of the cover $Y_s \to X_s$. Now consider the smooth family of curves $Y \to S_1$. Let s_1 be a closed point of S_1 such that $h(s_1) = s$. Then by construction the gonality of the fibre $Y_{s_1} := Y \times_S k(s_1)$ of Y above s_1 is ≥ 5. In particular by using Theorem 3.3 we can find a prime integer $l >> 0$ distinct from p and a non trivial μ_l torsor $Z_{s_1} \to Y_{s_1}$ such that the following two conditions hold:

i) The μ_l-torsor $Z_{s_1} \to Y_{s_1}$ is new ordinary.
ii) The natural map $T : M_{Y_{s_1}} \to M_{J^{\text{new}}_{s_1}}$ induced by the μ_l-torsor $Z_{s_1} \to Y_{s_1}$, where $J^{\text{new}}_{s_1}$ is the new part of the jacobian of Z_{s_1} with respect to the above torsor, is an immersion (cf. 3).

Also by applying Lemme 4.7 we can find a finite étale cover $h' : S_2 \to S_1$ and a μ_l-torsor $f_2 : Z' \to Y' := Y \times_{S_1} S_2$ such that the pair (f_2, h') is a good lifting of the μ_l-torsor $Z_{s_1} \to Y_{s_1}$. Let $J_{Y'}$ (resp. $J_{Z'}$) be the relative jacobian of Y' over S_2 (resp. the relative jacobian of Z' over S_2) which is an S_2-abelian scheme. Let $f_2^* : J_{Y'} \to J_{Z'}$ be the canonical homomorphism which is induced by the pull back of invertible sheaves. Let $J^{\text{new}} := J_{Z'}/f_2^*(J_{Y'})$ be the new part of the jacobian $J_{Z'}$ with respect to the μ_l-torsor $Z' \to Y'$. Let η' be the generic point of S_2 and let s_2 be a point of S_2 such that $h'(s_2) = s_1$. The fibre $J^{\text{new}}_{s_2} := J^{\text{new}} \times_{S_2} k(s_2)$ of J^{new} above the point s_2 is by construction an ordinary abelian variety. This implies a fortiori that the generic fibre $J^{\text{new}}_{\eta'} := J^{\text{new}} \times_{S_2} k(\eta')$ of J^{new} where η' is the generic point of S_2 is also ordinary since $J^{\text{new}}_{\eta'}$ specializes to $J^{\text{new}}_{s_2}$. The following two cases can occur:

Case 1: The abelian scheme $J^{\text{new}} \to S_2$ has constant p-rank, i.e. all fibres of J^{new} over S_2 are ordinary abelian varieties. Then since S_2 is complete we deduce from Theorem A.3 that the abelian scheme $J^{\text{new}} \to S_2$ is isotrivial. Note that the deformation J^{new} of $J^{\text{new}}_{s_2}$ induces infinitesimal deformations of $J^{\text{new}}_{s_2}$ which are trivial deformations since J^{new} is isotrivial. Since the map T is an immersion we conclude that the deformation $Y' \to S_2$ is isotrivial, as well. A fortiori the family $X \to S$ is also isotrivial by Lemma 4.9. But this contradicts our hypothesis that the family $X \to S$ is not isotrivial. So case 1 can not occur.

Case 2: The abelian scheme $J^{\text{new}} \to S_2$ does not have constant p-rank, i.e. there exists a closed point $\tilde{s} \in S_2$ such that the p-rank of the fibre $J^{\text{new}}_{\tilde{s}} := J^{\text{new}} \times_{S_2} \tilde{s}$ of J^{new} over the point $\tilde{s}$ is strictly smaller than the p-rank of the generic fibre $J^{\text{new}}_{\eta'}$ of J^{new}. This in particular implies that the p-rank of the fibre $Z'_{\tilde{s}} := Z' \times_{S_2} \tilde{s}$ of Z' above the point $\tilde{s}$ is strictly smaller than the p-rank of the generic fibre

$Z'_{\eta'} := Z' \times_{S_2} \eta'$ of Z'. This already proves Theorem 4.6. Now this implies in particular that the geometric fundamental group π_1 is not constant on the family $Z' \to S_2$. Thus by Lemma 4.8 we deduce that the geometric fundamental group π_1 is not constant on the family $X \to S$. This finishes the proof of Theorem 4.5. □

Theorem 4.5 can be generalized to the situation where we consider the full tame fundamental group. More precisely we have the following theorem:

Theorem 4.8. *Let k be an algebraically closed field of characteristic p. Let S be a smooth complete and irreducible k-curve with generic point η (resp. geometric generic point $\bar{\eta}$ above η). Let $f : X \to S$ be a proper and smooth family of curves of genus $g \geq 2$. Let $\{s_1, s_2, \dots, s_n\}$ be n sections of f with disjoint support such that $2 - 2g - n < 0$. Assume that the family f is not isotrivial. Then the tame fundamental group is not constant on the pair $(f, \{s_1, s_2, \dots, s_n\})$ i.e. there exists a closed point $s \in S$ such that the specialization homomorphism $Sp : \pi_1^t(X_{\bar{\eta}} - \{s_1(\bar{\eta}), s_2(\bar{\eta}), \dots, s_n(\bar{\eta})\}) \to \pi_1^t(X_s - \{s_1(s), s_2(s), \dots, s_n(s)\})$ between tame fundamental groups, where $X_{\bar{\eta}} := X \times_S \bar{\eta}$ (resp. $X_s := X \times_S s$), is not an isomorphism. Here $\bar{\eta}$ is a geometric point above η.*

The proof of 4.10 follows from the fact (easily seen) that if the geometric tame fundamental group π_1^t is constant on the fibres of f then this implies that the geometric fundamental group π_1 (which is the quotient of π_1^t by the normal subgroup generated by inertia) is also constant on the fibres of f (one has to use the fact that inertia specialises to inertia via the specialisation map) and we are then reduced to 4.5.

Appendix A: Families of abelian varieties with constant *p*-rank in characteristic *p*

Let $p > 0$ be a fixed prime integer. In this section we would like to explain the startification of the moduli space of polarized abelian varieties in characteristic p by the p-rank, and state the theorem of Raynaud, Szpiro and Moret-Bailly on the isotriviality of complete families of ordinary abelian varieties.

Let $g \geq 1$ be an integer. Let $\mathcal{A}_g \to \operatorname{Spec} \mathbb{F}_p$ denotes the coarse moduli scheme of principally polarized abelian varieties of dimension g in characteristic p. The scheme $\mathcal{A}_g$ is a quasi-projective variety of dimension $g(g+1)/2$ and has the following property: for every scheme S of characteristic p and $X \to S$ a principally polarized abelian S-scheme of relative dimension g there exists a unique natural map $S \to \mathcal{A}_g$ defined by the family $X \to S$. This map sends a point $s \in S$ to the moduli point corresponding to the fibre $X_s \to \operatorname{Spec} k(s)$ of X above the point s.

Let k be an algebraically closed field of characteristic p. For each fixed integer $0 \leq f \leq g$ let $\mathcal{V}_f \subset \mathcal{A}_g \times_{\mathbb{F}_p} k$ be the subset of points corresponding to abelian varieties having a p-rank $\leq f$. Concerning the subsets $\mathcal{V}_f$ we have the following:

Theorem A.1. *For each fixed integer $0 \leq f \leq g$ and any algebraically closed field k of characteristic p let $\mathcal{V}_f \subset \mathcal{A}_g \times_{\mathbb{F}_p} k$ be the subset of points corresponding to abelian varieties having a p-rank $\leq f$. Then the subset $\mathcal{V}_f$ is a closed subscheme of $\mathcal{A}_g \times_{\mathbb{F}_p} k$ and every irreducible component of $\mathcal{V}_f$ has dimension equal* *to $g(g+1)/2-g+f$. Moreover the closed subscheme $\mathcal{V}_0$ is a complete subvariety of $\mathcal{A}_g \times_{\mathbb{F}_p} k$ of dimension $g(g-1)/2$.*

For the proof of the fact that $\mathcal{V}_f$ is closed see [11], Corollary 1.5. For the statement concerning the dimension of the above strata see [9], Theorem 4.1. Finally for the fact that $\mathcal{V}_0$ is complete see [11], proof of Theorem 1.1 a).

Next we state the theorem of Raynaud, Szpiro and Moret-Bailly.

Definition A.2. *Let k be a field of characteristic $p > 0$. Let S be a normal and integral k-variety and let $X \to S$ be an abelian S-scheme of relative dimension g. The family $X \to S$ is called* isotrivial *if there exists a finite cover $S' \to S$ such that the abelian scheme $X \times_S S'$ is S'-isomorphic to the product of S' with an abelian variety defined over some finite extension of k.*

Theorem A.3 (Raynaud, Szpiro, Moret-Bailly [7], chapitre XI, 5:)**.** *Let k be a field of characteristic $p > 0$. Let S be a normal and integral k-variety which is* **projective**. *Let $X \to S$ be an abelian S-scheme of relative dimension g. Assume that all the geometric fibres $X_{\bar{s}} \to \mathrm{Spec}\, k(\bar{s})$ of X over S are ordinary abelian varieties. Then the family $X \to S$ is isotrivial.*

Appendix B: Families of curves with constant *p*-rank in characteristic *p*

Let $p > 0$ be a prime integer. In this section we want to extend the argument of Oort in [11] in order to show the existence, for every integer $g \geq 3$, of a non-isotrivial complete family of smooth and proper curves of genus g with constant p-rank equal to 0. Before stating the main result we will explain and adopt some notation.

Let $g \geq 2$ be an integer and let $\mathcal{M}_g \to \mathrm{Spec}\, \mathbb{F}_p$ be the coarse moduli scheme of projective smooth and irreducible curves of genus g in characteristic p. The scheme $\mathcal{M}_g$ is a quasi-projective irreducible variety of dimension $3g-3$. Let $\overline{\mathcal{M}}_g \to \mathrm{Spec}\, \mathbb{F}_p$ be the Deligne-Mumford compactification of $\mathcal{M}_g$ which is a coarse moduli scheme of projective and stable curves of genus g in characteristic p. The scheme $\overline{\mathcal{M}}_g$ is an irreducible projective variety which contains $\mathcal{M}_g$ as an open subscheme ([3] for more details). We will also consider $\mathcal{M}'_g \to \mathrm{Spec}\, \mathbb{F}_p$ which is the coarse moduli scheme of projective and stable curves of genus g in characteristic p whose jacobian is an abelian variety (these are exactly the projective and stable curves of genus g whose configuration of irreducible components is tree like cf. [1], 9, corollary 12).

Let S be a scheme of characteristic p. By a smooth relative (or a family of) curve(s) $f : X \to S$ over S of genus g we mean that f is a proper and smooth equidimensional morphism with relative dimension 1 whose fibres are curves of genus g. We say that the relative curve $f : X \to S$ is *complete* if S is a complete and irreducible variety in characteristic p. The moduli scheme $\mathcal{M}_g$ has the following property: if $f : X \to S$ is a smooth relative curve over S of genus g then there is a natural map $S \to \mathcal{M}_g$ which is uniquely determined by f. This map sends a point $s \in S$ to the moduli point corresponding to the fibre $X_s \to \mathrm{Spec}\, k(s)$ of X above the point s. We also have the following:

Proposition B.1. *Let k be a field of characteristic p and let $\mathcal{S}$ be a subvariety of $\mathcal{M}_g \times_{\mathbb{F}_p} k$. Then there exists a finite cover $h : \mathcal{S}' \to \mathcal{S}$ and a smooth relative $\mathcal{S}'$-curve $f' : X' \to \mathcal{S}'$ of genus g such that the natural morphism $\mathcal{S}' \to \mathcal{M}_g$, induced by f', factorizes $\mathcal{S}' \to \mathcal{S} \to \mathcal{M}_g$ through h.*

Proof. Standard by passing to the fine moduli scheme of smooth projective and irreducible curves of genus g with a symplectic level n-structure. □

Definition B.2. *Let S be a scheme of characteristic p and let $f : X \to S$ be a smooth relative S-curve of genus g. The curve $f : X \to S$ is said to be* isotrivial *if the corresponding map $S \to \mathcal{M}_g$ has an image which consists of a point.*

Definition B.3. *Let S be a scheme of characteristic p and let $f : X \to S$ be a smooth relative S-curve of genus g. Let $0 \le r \le g$ be an integer. We say that the family $f : X \to S$ has constant p-rank, equal to r, if each geometric fibre $X_{\bar{s}} \to Spec\, k(\bar{s})$ of f has a p-rank equal to r.*

In [11] Oort showed, in the proof of theorem 1.1. b), the existence over any algebraically closed field k of characteristic p of a complete curve contained in $\mathcal{M}_3 \times_{\mathbb{F}_p} k$ and which is contained in the locus of curves having p-rank equal to 0. This indeed corresponds by Proposition B.1 to a non-isotrivial complete family of smooth curves of genus 3 with constant p-rank equal to 0. Oort's argument can be easily extended, using the results in [9], in order to prove the following:

Theorem B.4. *Let $g \ge 3$ be an integer. Let k be an algebraically closed field of characteristic p. Then $\mathcal{M}_g \times_{\mathbb{F}_p} k$ contains a complete irreducible curve which is contained in the locus of curves having p-rank equal to* 0.

Proof. The proof consists in considering the Torelli map together with a dimension argument. More precisely let:

$$t : \mathcal{M}'_g \times_{\mathbb{F}_p} k \to \mathcal{A}_g \times_{\mathbb{F}_p} k$$

be the Torelli morphism which sends the class of a stable curve whose jacobian is an abelian variety to the class of its jacobian endowed with its canonical principal polarization coming from the theta divisor. Torelli's theorm ([6], 12) says that the map t is injective on geometric points. In particular the image $\mathcal{J}'_g := t(\mathcal{M}'_g \times_{\mathbb{F}_p} k)$ (resp. $\mathcal{J}_g := t(\mathcal{M}_g \times_{\mathbb{F}_p} k)$) of $\mathcal{M}'_g \times_{\mathbb{F}_p} k$ (resp, of $\mathcal{M}_g \times_{\mathbb{F}_p} k$), which is called the *jacobian locus* (resp. *open jacobian locus*), is a subvariety of $\mathcal{A}_g$ of dimension $3g - 3$. Moreover $\mathcal{J}'_g$ is closed in $\mathcal{A}_g$ ([8], lecture IV, p. 74). For each fixed integer $0 \le f \le g$ let $\mathcal{V}_f$ be the closed subscheme of $\mathcal{A}_g \times_{\mathbb{F}_p} k$ as defined in B.3. Then every irreducible component of $\mathcal{J}'_{f,g} := \mathcal{J}'_g \cap \mathcal{V}_f$ has dimension at least $3g - 3 - g + f = 2g - 3 + f$ ([11], lemma 1.6). Moreover it is well known that every irreducible component of $\mathcal{J}'_{0,g} = \mathcal{J}'_g \cap \mathcal{V}_0$ has dimension $2g - 3$ ([4] 2.3, for example). We claim that $\{\mathcal{J}'_g - \mathcal{J}_g\} \cap \mathcal{V}_0$ has codimension at least two in $\mathcal{J}'_{0,g}$. Indeed for a positive integer $g' \ge 1$ let $\mathcal{J}_{0,g'} := \mathcal{J}_{g'} \cap \mathcal{V}_0$. Then $\{\mathcal{J}'_g - \mathcal{J}_g\} \cap \mathcal{V}_0$ is contained in the images of $\mathcal{J}_{0,g_1} \times \cdots \times \mathcal{J}_{0,g_t}$ for all possible family of positive integers $\{g_1, \ldots, g_t\}$ such that $g_1 + \cdots + g_t = g$, with $t \ge 2$, via the natural morphism $\mathcal{J}'_{0,g_1} \times \cdots \times \mathcal{J}'_{0,g_t} \to \mathcal{V}_0$ (here we define $\mathcal{M}_1 \simeq \mathcal{J}_1 = \mathcal{A}_1$ to be the j-line over k parametrizing elliptic curves, in particular $\mathcal{J}'_{0,1} = \mathcal{J}_{0,1}$ has dimension 0 in this case). Now counting the dimension of $\mathcal{J}'_{0,g_1} \times \cdots \times \mathcal{J}'_{0,g_t}$ which is $\sum_i (2g_i - 3) \leq 2g - 5$, where the sum is taken over all $i \in \{1, \ldots, t\}$ such that $g_i > 1$ or otherwise this dimension equals 0, we conclude that $\{\mathcal{J}'_g - \mathcal{J}_g\} \cap \mathcal{V}_0$ has codimension at least two in $\mathcal{J}'_{0,g} = \mathcal{J}'_g \cap \mathcal{V}_0$. □

Finally since $\mathcal{V}_0$ is projective we can find a closed immersion $\mathcal{V}_0 \to \mathbb{P}^N_k$ into a projective k-space of suitable dimension. Further we can find a general linear subspace L of $\mathbb{P}^N_k$, of suitable dimension, such that $L \cap \mathcal{J}'_{0,g}$ is a (necessarily complete) curve S' and such that $L \cap \{\{\mathcal{J}'_g - \mathcal{J}_g\} \cap \mathcal{V}_0\}$ is empty ([2], II, Chapter 3, 1.2). Now the inverse image $S := t^{-1}(S')$ of S', via the Torelli map t, is a complete curve contained in $\mathcal{M}_g \times_{\mathbb{F}_p} k$ and which by construction is contained in the locus of curves having p-rank equal to 0.

Corollary B.5. *Let $g \geq 3$ be an integer. Let k be an algebraically closed field of characteristic p. Then there exists a complete and smooth algebraic k-curve S and a non isotrivial smooth S-curve $f : X \to S$ of genus g with constant p-rank equal to* 0.

Remark B.6. It is tempting to try to construct an S-curve X as in 5.5, for special values of g, by considering a Galois cover $f : X \to Y$ with group $\mathbb{Z}/p\mathbb{Z}$ where Y is a ruled surface over S and such that f is étale outside an S-section of Y. The Deuring-Shafarevich formula comparing the p-rank in Galois p-covers would then imply that all fibres of X over S have p-rank equal to 0. However, by a result of Pries, all such covers $f : X \to Y$ are necessarily isotrivial ([13], Theorem 3.3.4).

Question B.7. Let $g \geq 3$ be an integer. Let k be an algebraically closed field of characteristic p. Let $\tilde{r}$ be the maximum of the integers r such that $\mathcal{M}_g \times_{\mathbb{F}_p} k$ contains a complete curve S whose image under the Torelli morphism is contained in $\mathcal{V}_r - \mathcal{V}_{r-1}$. We have $0 \leq \tilde{r} < g$. What is the value of $\tilde{r}$? Does the value of $\tilde{r}$ depend only on g? Or does it depend on p as well?

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