ON PLANAR PIECEWISE AND TWO-TORUS PARABOLIC MAPS

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We investigate properties and examples of iterated planar piecewise parabolic (PWP) maps, including those induced by two-torus parabolic maps and their inverses. PWP maps are area-preserving maps that have constant linearizations but only one eigenvector with eigenvalue one. We obtain necessary and sufficient conditions for a two-torus parabolic map to be invertible. For noninvertible PWP maps, there are a number of questions about the existence and structure of attractors for such maps. We introduce the simplest example of a nontrivial PWP map, defined by a parabolic linear map with a translation that is a different constant on each side of a given line in the plane. For this “half plane parabolic map” we obtain sufficient conditions for a bounded attractor to exist, discuss their dynamical properties and give some examples of these and related maps.

Keywords: Piecewise linear map; parabolic map; global attractor.

1. Introduction

Many applications require the study of iterated discontinuous maps that may be well-modeled by linear maps on a number of different regions [Wu & Chua, 1994]. For some cases of such maps, there is a good understanding, e.g. for piecewise contracting maps [Bruin & Deane, 2009], while a number of authors have been working on understanding the fascinating patterns that appear for piecewise isometric maps [Ashwin & Fu, 2002; Goetz, 2000a, 2000b]. In the area-preserving case, parabolic maps can be viewed as a transitional case between piecewise isometric and piecewise nonisometric maps, and because of applications, they are of interest in their own right [Ashwin et al., 2000; Gunturk & Thao, 2005; Życzkowski & Nishikawa, 1999].

We say a two-by-two matrix $M$ is parabolic if it has two eigenvalues 1 and properly parabolic if it is parabolic and only has one eigenvector. A planar piecewise parabolic (PWP) map on a region $P \subseteq \mathbb{R}^2$ is a map $f : P \to P$ of the form

$$f(x) = M_i x + b_i, \quad \text{for } x \in P_i, \ i = 1, 2, \ldots, N,$$

where each $M_i$ is parabolic, and $P_i, \ i = 1, \ldots, N$, is a partition of $P$ into disjoint (possibly unbounded) convex polygons such that

$$P = \bigcup_{i=1}^N P_i.$$

In the case that $M_i = M$ independent of $i$ we say the map $f$ is a simple PWP map and we primarily consider such maps in this paper. In the case that
$P$ is unbounded we say $f$ has a global attractor if there is a bounded set $X \subset P$ such that all orbits eventually enter $X$ and remain therein.

In this paper we address in Sec. 2 questions of invertibility of two-torus parabolic maps (defined below) and obtain Proposition 1 giving an optimal result characterizing the invertibility of two-torus parabolic maps. Section 3 discusses a particular PWP map of the plane defined on two half-planes. Proposition 2 gives surprisingly nontrivial conditions for the simplest of nontrivial PWP maps (5) to have a global attractor, and gives some examples of global attractors for such PWP maps. The paper finishes with a discussion of general properties and some more examples in Sec. 4.

1.1. Two-torus parabolic maps

Consider a two-torus map $f(x,y) = (x',y')$ of the form

$$\begin{align*}
x' &= ax + by \pmod{1} \\
y' &= cx + dy \pmod{1}
\end{align*}$$

(1)

where $(x,y) \in [0,1)^2$, with $ad - bc = 1$. In matrix form we write this as $f = g \circ M$, where $M = (a,b,c,d)$ is a matrix and $g(x,y) = (x - \lfloor x \rfloor, y - \lfloor y \rfloor)$ is a map taking modulo 1 in each component. Maps of the form (1) have been called two-torus parabolic maps [Ashwin et al., 2000; Fu et al., 2007; Życzkowski & Nishikawa, 1999] in the case that the eigenvalues $\lambda_1, \lambda_2$ of the matrix $M$ are both equal to 1; if $\lambda_1 = \lambda_2$, then the map can be referred to as a two-torus elliptic map, and it can be transformed into a piecewise isometry [Ashwin & Fu, 2002; Buzzi, 2001; Deane, 2002; Goetz, 2000b]; or if $\lambda_1 > 1 > \lambda_2$ then it is a two-torus hyperbolic map.

By a change of parameters [Ashwin et al., 2000], in the parabolic case $M$ can either be written in the form

$$M = M_{A,\alpha} := \begin{pmatrix} 1 + A & A \\ \alpha & 1 + A \end{pmatrix},$$

where $A, \alpha$ are real parameters (in which case, we denote the map $f_{A,\alpha} = g \circ M_{A,\alpha}$) or $M$ is one of the horocyclic cases $h_y := (1,0; k,1)$ or $k^y := (1,b;0,1)$. Although the inverse (if it exists) of a two-torus parabolic map (1) cannot typically be written in a similar form, both (1) and its inverse can be seen as examples of planar piecewise parabolic maps.

In Sec. 2 we improve a result in [Ashwin et al., 2000] by showing that the necessary conditions for a two-torus parabolic map to be invertible can be refined to give necessary and sufficient conditions.

2. Inverses for Two-Torus Parabolic Maps

The paper [Ashwin et al., 2000] gives a necessary condition for a parabolic two-torus map $f = g \circ M$ to be invertible. Our first result is that this necessary condition can be made sufficient by the addition of one condition (note that $(k,l)$ denotes the highest common factor of $k$ and $l$).

Proposition 1. A two-torus parabolic map $f = g \circ M$ is invertible if and only if $M$ is equal to one of the following forms:

$$\begin{pmatrix} 2 + l & -k \\ (1 + l)^2 & -l \\ k & 1 \\ b & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} k & (k-1)^2 \\ l & 2 - k \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $k,l \in \mathbb{Z}\backslash\{0\}$, $(k,l) = 1$, and $b$ is real.

Proof. For the horocyclic cases, it is easy to show that the map is invertible for all $b$. Here we consider the case $a = 2 + l$, $b = -k$, $c = (1 + l)^2/k$, $d = -l$. It has been shown in [Ashwin et al., 2000] that $f$ is noninvertible if and only if there exist $K, L \in \mathbb{Z}\backslash\{0\}$ such that

$$|Kd - Lb| < 1, \quad |La - Kc| < 1.$$  \hspace{1cm} (3)

Suppose $(k,l) = 1$, if $f$ is noninvertible, then there exist $K, L \in \mathbb{Z}\backslash\{0\}$ such that

$$|Kl - Lk| < 1, \quad |L(2 + l) - K(1 + l)^2| < 1.$$  \hspace{1cm} (4)

Then we have $KL = Lk$ from the first inequality, and by substituting $Lk/k$ for $K$ into the second inequality, we have

$$\left|L(2 + l) - \frac{K(1 + l)^2}{k}\right| = \left|\frac{L}{\frac{k}{k}}\right| < 1.$$  \hspace{1cm} (5)

But since $(k,l) = 1$ and $KL = Lk$, we have $|L/k| \leq 1$. This gives a contradiction. On the other hand, if $(k,l) = p > 1$, then let $K = k/p, L = l/p$ which will satisfy the two inequalities (4) and it follows that $f$ is noninvertible.

A consequence of this (already noted in [Ashwin et al., 2000]) is that the set of noninvertible
two-torus parabolic maps is open and dense within the set of two-torus parabolic maps. As discussed in [Fu et al., 2007], it is known that for invertible two-torus parabolic maps, some of their inverses are still two-torus parabolic, but in fact most are not. More precisely, we call a parabolic map integral parabolic map if it has an integral matrix. It is shown in [Fu et al., 2007] that if \( f \) is an invertible two-torus parabolic map, then \( f^{-1} \) is a two-torus parabolic map if and only if \( f \) is integral.

If \( f \) is not integral but invertible, then its inverse cannot be written in such a form \( f = g \circ M_{\alpha} \).

It turns out that trajectories within each half-plane will move along parabolic paths (except for some extreme cases where plane will move along parabolic paths (except for where \( L \) is a parabola plane, where they switch to a different parabola. Move on its line of slope \( s \)).

Results hold for the map on the right half-plane. (invertible can not be written in such a form parabolic map if and only if \( f \) is integral. (Show in [Fu et al., 2007], it is known that for invertible parabolic map if and only if \( f \) is an integral matrix. It is shown in [Fu et al., 2007] that if \( f \) is an invertible parabolic map, then \( f^{-1} \) is a parabolic map if and only if \( f \) is integral. (In case (a) we note that there are either unbounded orbits that remain on the left \( (p_L < 0) \) or unbounded orbits that remain on the right \( (p_R > 0) \).)

3. Boundedness for a Simple PWP Map on Two Half-Planes

We consider a simple PWP map defined on the plane \((x, y) \in \mathbb{R}^2\), analogous to the map of Goetz [Goetz, 2007] for piecewise isometries:

\[
\begin{align*}
&f(x, y) = \begin{pmatrix} 1 + A & \frac{A}{\alpha} \\ -A & 1 - A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_L \\ v_L \end{pmatrix} & x < 0 \\
&f(x, y) = \begin{pmatrix} 1 + A & \frac{A}{\alpha} \\ -A & 1 - A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_R \\ v_R \end{pmatrix} & x \geq 0.
\end{align*}
\]

If we define

\[
\begin{align*}
p_L &= \frac{A(1 + \alpha^2)}{2s_L}, & p_R &= \frac{A(1 + \alpha^2)}{2s_R} \\
s_L &= \alpha u_L + v_L, & s_R &= \alpha u_R + v_R
\end{align*}
\]

then the parabolae for the map on the left are bounded to the left if \( p_L > 0 \); parabolae for the map on the right are bounded to the right if \( p_R > 0 \). Points move on a line with slope \(-\alpha\) if \( s_L = 0 \), jump above that line if \( s_L > 0 \) and jump below that line if \( s_L < 0 \). Finally, let

\[
\sigma = 3(s_L - s_R) + \frac{2\alpha}{A}(s_R - u_L).
\]

Proposition 2. We can classify the behavior of the iterated map (5) as follows:

(a) If \( p_L < 0 \) or \( p_R > 0 \) then there are initial conditions whose trajectories diverge to infinity.
(b) If \( p_L > 0 \) and \( p_R < 0 \) then the following results hold:

(b1) If \( A > 0 \) and \( \sigma < 0 \) then there is a global attractor for the map.
(b2) If \( A < 0 \) and \( \sigma > 0 \) then there is a global attractor for the map.

Proof. In case (a) we note that there are either unbounded orbits that remain on the left \( (p_L < 0) \) or unbounded orbits that remain on the right \( (p_R > 0) \).
For (b) we proceed by constructing maps of first return to one side of the plane. In case (b1), suppose
\[ p_L > 0, \quad p_R < 0, \quad A > 0 \]
and note that \( s_L > 0, s_R < 0 \). The map will move points down on parabolae on the right and up on parabolae on the left.

Suppose the point \((0, Y_1)\) goes to the right first, then the parabola on the right through \((0, Y_1)\) will intersect the \(y\)-axis again at \((0, y)\) given by
\[ L_R(x, y) = Ay^2 + e_Ry = Ay_1^2 + e_Ry_1 \]
meaning that
\[ y = y_1 \quad \text{or} \quad y = y_2 = \frac{e_R}{A} - y_1. \]

Hence the parabola on the right intersects the \(y\)-axis at \((0, Y_2) = (0, Q_R - Y_1)\) where \(Q_R = -e_R/A\). Starting on the axis at some large positive \(Y_1 > Q_R/2\), the furthest we can go into the left half-plane is the image of the point \((0, Y_2)\) under the map, i.e.
\[ (X_3, Y_3) = \left( \frac{4Y_2}{u_R} + u_R(1 - A)Y_2 + v_R \right). \]
This lies on a parabola on the left given by
\[ L_L(x, y) = L_L(X_3, Y_3) \tag{9} \]
which intersects the \(y\)-axis at points \((0, Y_1)\) and \((0, Y_5)\) with \(Y_5 < Y_1\). The quadratic (9) can be expressed as
\[ y^2 + R_1y = Y_1^2 + P_1Y_1 + Q_1 \]
for
\[ P_1 = \frac{4u_R + 3Au_R + 3Av_R}{A}, \quad R_1 = \frac{e_L}{A}, \quad Q_1 = \frac{-4u_R^2 - 2u_R^2 - 4Av_R}{A} \]
and \(Q_1\) is an expression involving only constants from the map. This has the largest solution
\[ Y_5 = \frac{R_1}{2} + \sqrt{\frac{R_1^2}{4} + Y_1P_1 + Q_1 + \frac{R_2^2}{4}} \]
\[ = \frac{R_1}{2} + Y_1\left[ 1 + \frac{P_1}{Y_1} + \frac{4Q_1 + R_2^2}{4Y_1^2} \right] \]
Noting that \(\sqrt{1 + x} \leq 1 + x/2\) for small enough \(x\), by taking large enough \(Y_1\) we have
\[ Y_5 \leq Y_1 + \frac{P_1 - R_1}{2} \]
Similarly, the furthest we can go into the right half-plane is the image of the point \((0, Y_5)\) under the map, i.e. \((X_6, Y_6) = (AY_5/A + u_L(1 - A)Y_5 + v_L)\) which lies on the parabola on the right given by
\[ L_R(x, y) = L_R(X_6, Y_6), \tag{10} \]
By a similar argument, we can find intersections of this with the \(y\)-axis at \((0, Y_2)\) and \((0, Y_4)\) with \(Y_7 < Y_6\), where
\[ Y_6 = \frac{e_R}{2A} + \sqrt{Y_2^2 + P_2Y_2 + Q_2} \]
\[ = \frac{e_R}{2A} + Y_1\sqrt{1 + \frac{P_2}{Y_1} + \frac{Q_2}{Y_1^2}} \]
where \(Q_2\) is a constant and
\[ P_2 = 2Av_L - 3Av_R + 2u_R - A - 3u_R - 34u_R\alpha. \]
Hence for large enough \(Y_1\) we have
\[ Y_8 < Y_5 - \frac{e_R}{2A} + \frac{P_5}{2} < Y_1 + \sigma \]
where
\[ \sigma = 3(s_L - s_R) + \frac{2a}{A}(u_R - u_L). \]

For the other case \((Y_1\) large and negative) we first go left, but otherwise the proof is similar to the argument above. The parabola on the left through \((0, Y_1)\) will intersect the \(y\)-axis again at \((0, Y_2) = (0, -c_L/A - Y_1)\) given by
\[ L_L(x, y) = Ay^2 + e_Ly = Ay_1^2 + e_Ly_1, \]
and the furthest we can go into the right half-plane is \((X_3, Y_3) = (AY_2/A + u_L(1 - A)Y_2 + v_L)\) lying on the parabola
\[ L_R(x, y) = L_R(X_3, Y_3) \tag{11} \]
which intersects the \(y\)-axis at points \((0, Y_4)\) and \((0, Y_5)\) with \(Y_4 > Y_5\). Hence the furthest we can go into the left half-plane is the image of the point \((0, Y_5)\) under the map, i.e. \((X_6, Y_6) = (AY_5/A + u_R(1 - A)Y_5 + v_R)\) which lies on the parabola on the left given by
\[ L_L(x, y) = L_L(X_6, Y_6), \]
with its intersections with the \(y\)-axis at \((0, Y_2)\) and \((0, Y_4)\) with \(Y_7 < Y_6\), and
\[ Y_6 = \frac{e_L}{2A} - \sqrt{Y_2^2 + P_2Y_2 + Q_2} \]
\[ = \frac{e_L}{2A} + Y_1\sqrt{1 + \frac{P_2}{Y_1} + \frac{Q_2}{Y_1^2}} \]
where $Q_k$ is a constant and

$$p'_2 = \frac{2Av_R - 3Av_L + 2u_L\alpha - 3u_R\alpha}{A}.$$ 

For large negative $Y_1$, we have $Y_0 > Y_1 - \sigma$, hence if $\sigma < 0$ then $\sigma > Y_1$ for all large enough $Y_1$. Hence $\sigma < 0$ is a sufficient condition for a global attractor of the map to exist. The case for (b2) is argued in a similar way to (b1).

Note that we do not have a converse for this proposition. Namely, in case (b) if $A > 0$ and $\sigma$ is positive but not too large, there may still be a global attractor. We have numerical evidence that necessary conditions may be quite difficult to obtain.

Figure 1 shows some example trajectories for this map on the plane. Note that for the parameters used in (a) we have

$$p_L = 0.0186695, \quad p_R = -0.0137981, \quad A = 0.01, \quad \sigma = -22.0176.$$ 

Hence in the case we are guaranteed global attraction by Proposition 2(b1). By contrast, for (c) we have the same signs of $p_{1,R}$ and $A$ but $\sigma = 2.9074$ and no global attractor. Figure 1(b) gives a detailed image of the attractor (plotted for one long orbit) for the parameters in case (a); observe that it has a nontrivial structure and is clearly a nonconvex set. It appears to have a piecewise smooth boundary composed of a finite number of parabolic pieces. We have not studied the detailed dynamics on the attractor but it appears to typically have dense orbits.

Some degenerate cases are not treated in the above proposition; for example if $u_L/u_R = \alpha$ and $v_R/v_R = -\alpha$ (i.e. $s_L = s_R = 0$) then every trajectory remains on a line of slope $\alpha$ and may or may not go to infinity.

4. General Properties of Planar Piecewise Parabolic Maps

It is a routine but nontrivial exercise to determine whether a PWP map $f : P \rightarrow P$ is invertible; this can be done by determining, for instance that $f(P_i) \cap f(P_j) = \emptyset$ for all $i \neq j$, up to a set of zero measure. In cases where $f$ is noninvertible then there may or may not be a global attractor $X \subset P$ such that $f$ is invertible on $X$.

Note that for proper parabolic matrices $M_1$ and $M_2$, their product $M_1M_2$ is a parabolic matrix if and only if $M_1$ and $M_2$ have a common eigenvector, and in this case they are commutative. We say a piecewise parabolic map $f$ is a consistent PWP map if $f^k$ is a PWP for $k = 1, 2, \ldots$. Clearly, if $f$ is a simple PWP map then so are all its iterates; hence simple PWP maps are consistent. More generally, a sufficient condition for a PWP map to be consistent is that all $M_i$ have a common eigenvector, but this is by no means necessary; for example, the map (12) on the upper half-plane maps the first and second quadrants into themselves; so the map is consistent even though the two parabolic matrices have different eigenvectors.
The paper [Wu & Chua, 1994] discusses the measure of cells for general piecewise linear maps, and planar piecewise parabolic maps are a special case of piecewise linear maps.

**Definition 1.** The partition \( \{P_0, P_1, \ldots, P_{N-1}\} \) of \( P \) associated with a PWP map \( f \) gives rise to a coding map \( S : P \rightarrow \Omega = \{0, 1, \ldots, N-1\}^N \) for \( f \) with \( S(x) = s_0s_1 \cdots \), where \( f^k(x) \in P_k \).

**Definition 2.** The set of all points following the same (finite or infinite) coding \( s \in \Omega \) under \( f \), i.e. \( S^{-1}(s) \) is called a cell (or a k-cell if we consider \( s \) finite and of length \( k \)).

For a consistent PWP map \( f_i(z) = M_i(z) + b_i \) for \( i \in P_k \) if for each \( k \)-cell with periodic admissible coding with its periodic segment \( s_0s_1 \cdots s_k \), i.e. there exist \( x \in P \) such that \( S(x) = s_0s_1 \cdots s_k \), the corresponding linear partition

\[
M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} & \text{if } 0 \leq y < \frac{1}{2} \\
\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} & \text{if } \frac{1}{2} \leq y < \frac{3}{2} \\
\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} & \text{if } \frac{3}{2} \leq y < \frac{5}{2} \\
\begin{pmatrix} u_4 \\ v_4 \end{pmatrix} & \text{if } \frac{5}{2} \leq y < \frac{7}{2} \\
\begin{pmatrix} u_5 \\ v_5 \end{pmatrix} & \text{if } \frac{7}{2} \leq y
\end{cases}
\]

This corresponds to the inverse of \( f_{1-3} \) in the special case of

\[
u_1 = \frac{1}{3}, \quad u_2 = 0, \quad u_3 = \frac{2}{3}, \quad u_4 = \frac{1}{3}, \quad u_5 = 0, \quad v_1 = -1, \quad v_2 = 0, \quad v_3 = 1, \quad v_4 = 2, \quad v_5 = 3.
\]

4.1. **Example: Extending the inverse of a two-torus parabolic map**

We now consider a generalization of the inverse of the map \( f_{1-3} = g \circ M_{1-3} \). This inverse can be written as a PWP map but not as a two-torus map. One can extend this to a map of the plane as follows: Let \( M = M_{1-3} \) and write

\[
f(x, y) = \begin{cases}
M \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} & \text{if } \frac{3}{2} \leq y < \frac{1}{2} \\
M \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} & \text{if } \frac{1}{2} \leq y < \frac{3}{2} \\
M \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} & \text{if } \frac{3}{2} \leq y < \frac{5}{2} \\
M \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_4 \\ v_4 \end{pmatrix} & \text{if } \frac{5}{2} \leq y < \frac{7}{2} \\
M \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_5 \\ v_5 \end{pmatrix} & \text{if } \frac{7}{2} \leq y
\end{cases}
\]

For these parameters, if a trajectory enters the unit square, then it will remain within the unit square and move along some invariant lines [Ashwin et al., 2000]. It is not clear whether the unit square is a global attractor for all initial conditions on the plane though numerical experiments (see Fig. 2) suggest that it is.

4.2. **Example: A planar PWP map with a simple global attractor**

The following example is a PWP on the whole plane with a simply computable global attractor. Consider the planar PWP map

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \quad (x, y) \in P_i
\]
where

\[ P_1 = \{(x, y) | x \geq 0, y \geq 0\}, \]
\[ P_2 = \{(x, y) | x < 0, y > 0\}, \]
\[ P_3 = \{(x, y) | x \leq 0, y \leq 0\} \setminus \{(0, 0)\}, \]
\[ P_4 = \{(x, y) | x > 0, y < 0\} \]

and

\[ (u_1, v_1) = (-1, -1), \quad (u_2, v_2) = (1, 0), \]
\[ (u_3, v_3) = (1, 1), \quad (u_4, v_4) = (-1, 0). \]

This map has a bounded global attractor [Deane, 2002; Fu & Duan, 2007; Goetz, 2000a] that will be a compact set \( X \) such that for all sufficiently large bounded sets \( A, X = \bigcup_{n=0}^{\infty} f^n(A) \).

We note that after a finite number of iterations, every point in the plane will enter the square \([-1, 1]^2\), and once a point enters that square it will stay in it. The following argument supports this claim. Consider any point \((x, y)\) with \( x > 1 \), \( y' = x - 1 \) until \( x \leq 1 \), and when \( x < -1 \), \( x' = x + 1 \) until \( x \geq -1 \), so the first coordinate of an arbitrary point will enter \([-1, 1]\). Next we suppose a point is in the strip \([-1, 1] \times \mathbb{R} \), and we discuss the second coordinate in every partition. In the first partition \( P_1 \), \( y' = y \) for \( y > -1 \), and when \( x = 1 \), \( y' = y + 1 \), so \( y' < y \) when \( x \neq 1 \), and when \( x = 1 \), \( y > 1 \), \( (1, y) \to (0, y) \to (-1, y - 1) \), the second coordinate will also decrease after two iterations. In the second partition, \( y' = x + y, -1 \leq x < 0, y > 0 \), so \( y' < y \). In the third and fourth partitions, \( y \) will increase towards 1 after some iterates. Hence, every point will move into \([-1, 1]^2\). This shows that the maximal invariant set of \([-1, 1]^2\) under \( f \) (the dark region in Fig. 3, which is the image of \([-1, 1]^2\) under \( f \)) is the global attractor for \( f \).

4.3. Discussion and further problems

Piecewise parabolic maps are an interesting transitional case between elliptic and hyperbolic maps; however, understanding their dynamics requires different techniques to either of these. Under certain circumstances one can reduce orbits of piecewise parabolic maps to an interval exchange on a number of intervals [Ashwin et al., 2000].

For planar piecewise parabolic maps on the plane, a natural question is to find necessary and sufficient conditions ensuring the existence of a global attractor. This has been discussed for piecewise isometries [Goetz, 2000a, 2000b; Goetz et al., 2007; Deane, 2002; Fu & Duan, 2007], but the equivalent for PWP maps is likely to be difficult to frame except for subclasses such as simple PWP maps.

It is known by a general result of Buzzi [2003] that piecewise isometric maps in any dimension have zero topological entropy. On the other hand, hyperbolic area-preserving maps will typically have positive topological entropy; this begs the question

Fig. 2. The trajectory for the initial point \((-0.66, -0.66)\) under the map (13) with parameters (14). After 499 iterates the trajectory enters the unit square and remains therein.

Fig. 3. The shaded region shows the location of the global attractor for the piecewise parabolic map (15) on the plane.
of whether the topological entropy of PWP maps can be nonzero. We do not have an answer to this, but at least for certain limited cases the topological entropy can be easily shown to be zero. In general, we believe that at least all consistent PWP maps will have zero topological entropy, while their dynamical complexity can be positive.

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