Heteroclinic Ratchets for a System of Identical Coupled Oscillators

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Abstract

We study an unusual but robust phenomenon that appears in an example system of four coupled phase oscillators. The coupling is preserved under only one symmetry, but there are a number of invariant subspaces and degenerate bifurcations forced by the coupling structure, and we investigate these. We show that the system can have a robust attractor that responds to a specific detuning Δ between a certain pair of the oscillators by a breaking of phase locking for arbitrary $\Delta > 0$ but not for $\Delta \leq 0$. As the dynamical mechanism behind this is a particular type of heteroclinic cycle, we call this a 'heteroclinic ratchet' because of its dynamical resemblance to a mechanical ratchet.

1 Introduction

Coupled oscillators arise as simplified models for coupled limit cycle oscillators in case of weak coupling [20]. They have been receiving an increasing interest not only because of their various application areas such as electrochemical oscillators[21, 8] and neural systems[14] but also because they present analytically tractable models to understand various kinds of dynamical phenomena[1, 5, 7]. These include complete phase synchronization, partial synchronization due to the existence of stable synchronized clusters and slow switching between unstable clusters. The last phenomenon takes place if there is an attractor composed of unstable cluster states which are connected to each other by heteroclinic connections and thus form a heteroclinic network in state space.

Heteroclinic networks (or heteroclinic cycles in particular) are used to explain slow switching behaviour of physical systems where an observable of the system stays at some equilibrium or periodic orbit for a long period, then changes its state to another stationary state relatively fast and repeats this process for another or same stationary state. This type of behaviour is first observed in fluid dynamics and explained by the existence of robust heteroclinic cycle in [6] and [9]. Heteroclinic networks are used to explain slow switching phenomenon in population dynamics[2], electrochemical oscillators[8] and neural systems[15]. They also may have some applications in computational engineering as some recent works[16, 17, 18] suggest. Especially in neural systems, the use of heteroclinic networks are quite promising since this opens a new sight to complex neural systems that is analysing the transient behaviours rather than stationary ones which can be relatively easily handled by classical dynamical system theory[14].

Despite the fact that heteroclinic networks are not structurally stable they can exist robustly if the system considered is already restricted by some conditions, such as symmetry. This is due to the existence of invariant subspaces on which heteroclinic connections between saddle equilibria can exist robustly. In case of fully permutation symmetry (all-to-all coupling), a system of N coupled oscillators is shown to admit robust heteroclinic networks for $N = 4$ or greater[19]. It is important to note that due to the symmetry these heteroclinic networks can not have arbitrary forms. On the other hand, symmetry is not necessary for robust heteroclinic networks to exist. It is recently shown in [12] that robust heteroclinic cycles can exist for coupled cell systems with nonsymmetric coupling structure.

In this work, we study a coupled phase oscillator system for which robust heteroclinic networks appear in the phase difference space as a result of the coupling structure rather than the symmetry of the coupling. This also gives rise to the presence of some heteroclinic networks which are not seen for symmetric system. We emphasize one of them which we call heteroclinic ratchet as it resembles to mechanical ratchet. A heteroclinic ratchet on an N-torus contains heteroclinic cycles winding in some directions but no other heteroclinic cycles winding in the reverse directions. We show that this type of heteroclinic networks exist as attractors in the phase difference space resulting in noise induced desynchronization of certains couple of oscillators in such a way that a certain oscillator has a larger frequency than the other. This phenomenon is new in coupled phase oscillator systems and can not take place in all-to-all coupled systems, since the permutation symmetry enforce the system to have desynchronization of a couple, if there is any, in both ways. We will show that the existence of heteroclinic ratchets for a coupled phase oscillator system is mainly related to the coupling structure. Moreover, heteroclinic ratchets have important dynamical consequences such as sensitivity to detuning and noise.

The main model for coupled phase oscillators is the Kuramoto model of N oscillators where each oscillator is coupled to all the others by a specific 2π-periodic coupling function [1]. We consider the same model with a specific connection structure and using a more general coupling function $g(x)$. Each oscillator has dynamics given by

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} c_{ij} g(\theta_i - \theta_j). \tag{1}$$

Here $\dot{\theta}_i \in \mathbb{T} = [0, 2\pi)$ and w_i is the natural frequency of the oscillator i. The connection matrix $\{c_{ij}\}$ represents the coupling between oscillators. $c_{ij} = 1$ if the oscillator i receives an input from the oscillator j and $c_{ij} = 0$ otherwise. The coupling function g is a 2π-periodic function. For weakly coupled oscillators it is well know that (1) will have an S^1 phase shift symmetry, that is the dynamics of (1) are invariant under the phase shift

$$(\theta_1, \theta_2, \dots, \theta_N) \mapsto (\theta_1 + \epsilon, \theta_2 + \epsilon, \dots, \theta_N + \epsilon)$$

for any $\epsilon \in \mathbb{T}$. We will initially consider identical oscillators, that is,

$$\omega_i = \omega \ , \quad i = 1, \dots, N \tag{2}$$

before discussing at a latter stage the effect of detuning where the oscillators can have different natural frequencies. Because the coupling function g is 2π-periodic it is natural to consider a Fourier series expansion

$$g(x) = \sum_{k=1}^{\infty} r_k \sin(kx + \alpha_k) \tag{3}$$

where r_k must converge to zero fast enough and α_k's are arbitrary. Several truncated cases of the general case (3) have been considered in the literature:

- Setting $r_k = 0$ for $k = 2, 3, \ldots$ and $\alpha_1 = 0$ then coupling function (3) gives the Kuramoto model, which exhibits frequency synchronization and clustering phenomena [1].

- Setting $r_k = 0$ for $k = 2, 3, \ldots$ but leaving arbitrary α_1 then coupling function (3) gives the Kuramoto-Sakaguchi model with essentially the same dynamics as the Kuramoto model.

- Setting $r_k = 0$ for $k = 3, 4, \ldots$ and $\alpha_2 = 0$ gives the model of Hansel et al. [5]. They showed that one can observe new phenomena not present in the above cases. For example, taking

$$r_1 = -1, r_2 = 0.25, a_1 = 1.25$$

  and setting all other parameters to zero, they show that one can observe slow switching phenomenon as a result of the presence of an asymptotically stable robust heteroclinic cycle connecting a pair of saddles.

We investigate a particular four-coupled cell system that admits a robust heteroclinic ratchet as an attractor only in presence of a third harmonic in the coupling function, i.e. we will require $r_3 \neq 0$. Note that, without loss of generality, we also set $K = N$ and $r_1 = -1$ by a scaling of time.

The coupling structure considered in this work (see Figure 1) arises as an inflation of the all-to-all coupled 3-cell network[12]. As the network admits an S_3-symmetric quotient network there may exist symmetry broken branches of solutions for the coupled systems associated to this network[13]. This is a direct result of the Equivariant Branching Lemma[4]. We will show that for the associated coupled oscillator systems such a synchrony breaking bifurcation includes two extra pitchfork branches as a result of the S^1-phase shift symmetry. These correspond to the saddle cluster states which may form heteroclinic ratchets for some parameter region.

This work consist of three parts. In Section 2, we will apply both the equivariant dynamics theory and the groupoid formalism approach to a coupled cell system of four identical phase oscillators and find the invariant subspaces where heteroclinic networks can exist. Theorem 1 characterizes a syncrony breaking bifurcation in such systems with the types of occuring branches. In section 3, we consider a particular coupling function and explain the emergence of heteroclinic ratchet connecting two pitchfork branches given in Theorem 1. Finally in Section 4, we discuss dynamical consequences

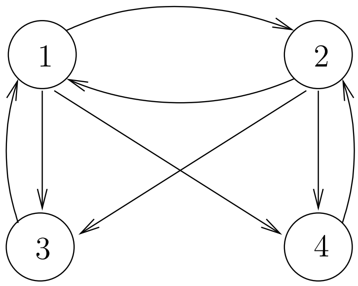


Figure 1: A 4-cell network: this gives coupled systems of the form (13). Observe that the system has a single symmetry given by the permutation (12)(34).

of the heteroclinic ratchet considering nonideal conditions such as noise and detuning of natural frequencies.

2 An example of four asymmetrically coupled oscillators

In this section, we consider a specific case of four identical oscillators coupled by a connection structure shown in Figure 1. More specifically, the system we consider is

$$\begin{aligned} \theta_1 &= f(\theta_1; \theta_2, \theta_3) \\ \theta_2 &= f(\theta_2; \theta_1, \theta_4) \\ \theta_3 &= f(\theta_3; \theta_1, \theta_2) \\ \theta_4 &= f(\theta_4; \theta_1, \theta_2) \end{aligned} \tag{4}$$

Although there is the permutation symmetry

$$f(x; y, z) = f(x; z, y) \quad \forall x, y, z \in \mathbb{T} \tag{5}$$

the system has far fewer symmetries than for example present in an all-to-all coupled system of four oscillators. We will also assume the presence of the phase shift symmetry

$$f(x + \epsilon; y + \epsilon, z + \epsilon) = f(x; y, z)\ , \quad \forall x, y, z, \epsilon \in \mathbb{T}. \tag{6}$$

For the present section, the form of coupling we assume will be more general than (1).

In the following we discuss the invariant subspaces of (4) using the balanced colouring method and will give a result about the solution branches on the invariant subspaces that emanates from the fully synchronized solutions.

2.1 Invariant subspaces

The network in Figure 1 has an S_2-symmetry, that is changing the first and third cells with the second an forth, respectively. Let Γ be an S_2-action on $\mathbb{T}^4$ generated by

$$\sigma\colon (\theta_1, \theta_2, \theta_3, \theta_4) \to (\theta_2, \theta_1, \theta_4, \theta_3).$$

The symmetry of the network implies that the system (4) is Γ-equivariant and the fixed point subspace of Γ, that is,

$$\mathrm{Fix}(\Gamma) = \{x \in \mathbb{T}^4 \mid \sigma x = x, \forall \sigma \in \Gamma\}$$

is invariant under the dynamics of (4). Note that $\mathrm{Fix}(\Gamma)$ consists of two disjoint subsets each of which is invariant: $V_2^{s3} = \{\theta \in \mathbb{T}^4 \mid \theta_1 = \theta_2, \theta_3 = \theta_4\}$ and $\bar{V}_2^{s3} = \{\theta \in T^4 \mid \theta_1 = \theta_2 + \pi, \theta_3 = \theta_4 + \pi\}$. On the other hand, there are many other invariant subspaces which do not base merely on the symmetry group of the network but on the groupoid structure of the input sets of cells (see [10] for groupoid formalism). These invariant subspaces can be obtained using the balanced coloring method. A coloring of cells, that is, a partition of the set of all cells is called balanced if each pair of cells with same color receive same number of inputs from the cells with any given color. Each balanced coloring gives rise to an invariant subspace obtained by equalizing the state of cells with same color. Moreover, each balanced coloring admits to some quotient network which comprises the dynamics reduced to the corresponding invariant subspace. For the system (4) the invariant subspaces obtained by the balanced coloring method are listed in Table 1.

Note that the subscripts indicate dimensions of subspaces and there exist a partial ordering for the set of these subspaces that represents the containment relation between them, that is,

$$V_x \prec V_y \Leftrightarrow V_x \subset V_y.$$

This ordering of invariant subspaces is illustrated in Figure 2. Consider the balanced coloring $\{3, 4\}$, where only third and forth cells have same color.

Dimensions	Invariant Subspaces
4	$V_4 = \mathbb{T}^4$
3	$V_3^s = \{\theta \in \mathbb{T}^4 \mid \theta_3 = \theta_4\}$
3	$V_3^1 = \{\theta \in \mathbb{T}^4 \mid \theta_1 = \theta_3\}$
3	$V_3^2 = \{\theta \in \mathbb{T}^4 \mid \theta_2 = \theta_4\}$
2	$V_2 = \{\theta \in \mathbb{T}^4 \mid \theta_1 = \theta_3, \theta_2 = \theta_4\}$
2	$V_2^{s1} = \{\theta \in \mathbb{T}^4 \mid \theta_1 = \theta_3 = \theta_4\}$
2	$V_2^{s2} = \{\theta \in \mathbb{T}^4 \mid \theta_2 = \theta_3 = \theta_4\}$
2	$V_2^{s3} = \{\theta \in \mathbb{T}^4 \mid \theta_1 = \theta_2, \theta_3 = \theta_4\}$
1	$V_1 = \{\theta \in \mathbb{T}^4 \mid \theta_1 = \theta_2 = \theta_3 = \theta_4\}$

Table 1: Invariant subspaces forced by the coupling structure in Figure 1 for the system (4)

The corresponding invariant subspace is V_3^s and the quotient network is the S_3-symmetric all-to-all coupled 3-cell network. Necessarily all the fixed point subspaces of this 3-cell quotient lift to some invariant subspaces of the 4-cell system and these are labelled by the superscript s. Note that V_2^{s3} is the only one of these that is contained in $\mathrm{Fix}(\Gamma)$, but there are some pairs of subspaces for which one subspace is related to the other by the symmetry of the system. These pairs are (V_2^{s1}, V_2^{s2}) and (V_3^1, V_3^2), as $\sigma(V_2^{s1}) = V_2^{s2}$ and $\sigma(V_3^1) = V_3^2$.

Exploiting the phase shift symmetry (6), the 4-dimensional system (4) can be reduced to a 3-dimensional one by defining new variables

$$(\phi_1, \phi_2, \phi_3) \hat{=} (\theta_1 - \theta_3, \theta_2 - \theta_4, \theta_3 - \theta_4)$$

so that

$$\begin{aligned}
\dot{\phi}_1 &= f(\phi_1; \phi_2 - \phi_3, 0) - f(0; \phi_1, \phi_2 - \phi_3) \\
\dot{\phi}_2 &= f(\phi_2; \phi_1 + \phi_3, 0) - f(0; \phi_1 + \phi_3, \phi_2) \\
\dot{\phi}_3 &= f(\phi_3; \phi_1 + \phi_3, \phi_2) - f(0; \phi_1 + \phi_3, \phi_2)
\end{aligned} \tag{7}$$

The symmetry of the system (4) has implications for this system. Let $\tilde{\Gamma}$ be an S_2-action on $\mathbb{T}^3$ generated by $\rho : (\phi_1, \phi_2, \phi_3) \to (\phi_2, \phi_1, -\phi_3)$. Then the system (7) is $\tilde{\Gamma}$ equivariant. In this case the fixed point subspaces are the lines $\{\phi \in \mathbb{T}^3 \mid \phi_1 = \phi_2,\ \phi_3 = 0\}$ and $\{\phi \in \mathbb{T}^3 \mid \phi_1 = \phi_2,\ \phi_3 = \pi\}$. Other invariant subspaces can be obtained projecting the previously found invariant subspaces onto $\mathbb{T}^3$. These are illustrated in Figure 3.

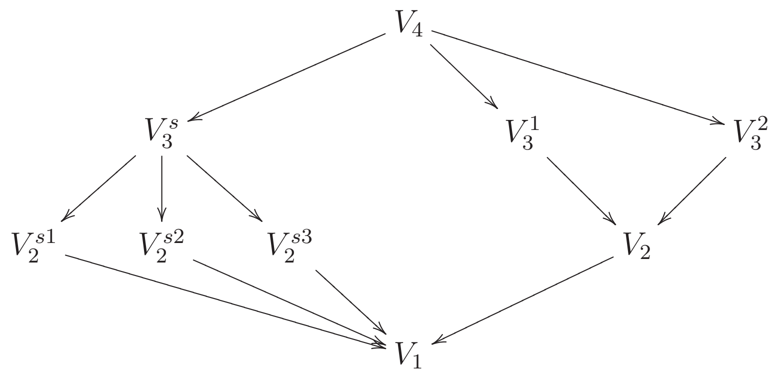


Figure 2: Containment of invariant subspaces given in Table 1. The subscripts indicates the dimensions of the invariant subspaces and the superscript s labels the fixed point subspaces related to the S_3-symmetry of the quotient network for $\theta_3 = \theta_4$.

2.2 Synchrony breaking bifurcations

In the work [13] it is shown that any coupled cell system that has a connection structure as in Figure 1 admits an S_3-transcritical bifurcation on V_3^s at the origin. More concretely, there exist three transcritical branches of unstable solutions on V_2^{s1}, V_2^{s2}, and V_2^{s3} simultaneously emanating from the origin if $f_x(0) - f_y(0) = 0$ and some transversality inequalities are satisfied. However, for the coupled phase oscillators of type (4), apart from the connection structure, dynamical properties affect the bifurcation scheme. Now we will show in Theorem 1 how the S^1-symmetry of f gives rise to a pitchfork bifurcation on V_2 that takes place simultaneously with the transcritical bifurcations mentioned above. The occurence of simultaneous branches on invariant lines is not only a consequence of the Equivariant Branching Lemma [4] but also a result of the connection structure and the property of the individual dynamics, that is the S^1-symmetry of f.

Theorem 1 *Assume that α is a parameter of the system (7). Then, there exist a pitchfork bifurcation of the origin of (7) on V_2 at $\alpha = \alpha^*$ appearing simultaneously with the transcritical bifurcations on V_2^{s1}, V_2^{s2} and V_2^{s3} if f satisfies $f_x(0, \alpha^*) = 0$, $f_{x\alpha}(0, \alpha^*) \neq 0$, and $f_{xxx}(0, \alpha^*) \neq 0$.*

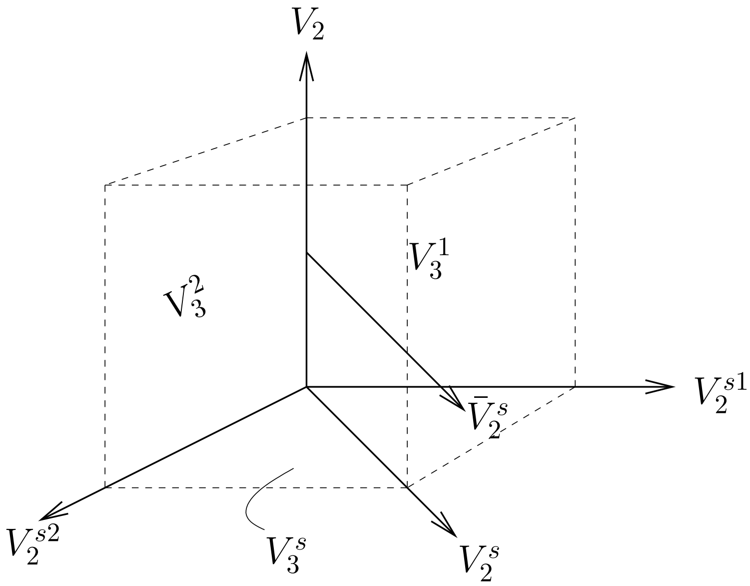


Figure 3: INVARIANT SUBSPACES GIVEN IN TABLE 1 PROJECTED ON $\mathbb{T}^3$ (SUBSCRIPTS INDICATES THE SUBSPACE DIMENSIONS ON $\mathbb{T}^4$)

Adjacency matrix	eigenvalues and eigenvectors
$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$	$\mu_1 = -1, \quad \nu_1 = (1,-1,0,0)^{\mathrm{T}}$ $\mu_2 = -1, \quad \nu_2 = (0,-1,1,1)^{\mathrm{T}}$ $\mu_3 = 0, \quad \nu_3 = (1,-1,1,-1)^{\mathrm{T}}$ $\mu_4 = 2, \quad \nu_4 = (1,1,1,1)^{\mathrm{T}}$

Table 2: ADJACENCY MATRIX WITH ITS EIGENVALUES AND EIGENVECTORS

Remark 1 *A direct consequence of this Theorem is that a generic bifurcation of the fully synchronized periodic solution (x,x,x,x) of (4) will give rise to three branches of periodic solutions of the form*

$$\begin{gathered}(x,y,x,x)\\(y,x,x,x)\\(x,x,y,y)\end{gathered}$$

and two other branches of the form (x,y,x,y), where the first three appear by transcritical bifurcations and the final two via a pitchfork bifurcation.

Proof. Consider the adjacency matrix A of the network (see Table 2). The eigenvalues of A and partial derivatives of f_x and f_y ($f_z = f_y$) at the origin determine the stability of the origin (see Proposition 2 in [13]). The eigenvalues of (7) at the origin are

$$\lambda_i = f_x(0, \alpha) + \mu_i f_y(0, \alpha) \tag{8}$$

where μ_i is an eigenvalue of A. The eigenvectors of (7) are the same as the A's. It is important to note that the S^1 phase shift symmetry of (7) induce a relation between partial derivatives:

$$f_x + f_y + f_z \equiv 0 \tag{9}$$

This can be obtained taking the derivative of (6) with respect to ϵ. Thus, there exist a linear relationship between partial derivatives

$$f_x \equiv -2f_y \tag{10}$$

Derivatives of (9) with respect to x and y give

$$f_{xx} \equiv -2f_{yx} \equiv 4f_{yy} \tag{11}$$

$$f_{xxx} \equiv -2f_{yxx} \tag{12}$$

The Eq. 8 and 10 imply that the eigenvalues λ_i become zero simultaneously for all invariant lines passing through the origin when $f_x(0, \alpha) = 0$. To see that there exist a pitchfork branch on V_2 we consider the solutions of type $(x, x + u/2, x, x + u/2)$. Substituting this into (7) and using the Eq. 6, one gets $\dot{u} = F(u) := f(0, 0, -u, \alpha) - f(0, u, 0, \alpha)$. Thus the assumptions $f_{x\lambda}(0, \alpha^*) \neq 0$, $f_{xxx}(0, \alpha^*) \neq 0$ and the Eq.'s 10, 11 and 12 imply the pitchfork bifurcation conditions $(\partial^2 F/\partial u^2)(0, \alpha^*) = 0$, $(\partial^2 F/\partial\lambda\partial u)(0, \alpha^*) \neq 0$ and $(\partial^3 F/\partial u^3)(0, \alpha^*) \neq 0$. Since these also imply the assumptions of Theorem 1 in [13], there exist simultaneous transcritical bifurcations on V_2^{s1}, V_2^{s2} and V_2^{s3}. ■

3 Robust Heteroclinic Ratchets for the System of Four Coupled Oscillators

In the previous section, it is shown that the connection structure of the system (4) induce the existence of invariant subspaces. These subspaces persist under the perturbations that preserve the connection structure. For this reason, as in symmetric systems, one can find robust heteroclinic networks laying on the invariant subspaces of the system (4). By "robust" we mean the persistence under the connection structure preserving small perturbations. We will see that for the phase difference system (7) some unusual heteroclinic networks exist, which are not seen for symmetric systems. We distinguish one type of these heteroclinic networks, which we call as heteroclinic ratchet because it includes connections that wind around the torus in one direction only.

Definition 1 *For a system on $\mathbb{T}^N$, an invariant set is a heteroclinic ratchet if it includes a heteroclinic cycle with nontrivial winding in one direction but no heteroclinic cycles winding in the opposite direction. More precisely, we say a heteroclinic cycle C parametrized by $x(s), s \in [0,1)$ has nontrivial winding in some direction if there is an angular variable $P \colon \mathbb{T}^N \to \mathbb{R}$ with $P(\mathbb{T}^N) = [0, 2\pi)$ and $P(x(0)) = 0$ such that $\lim_{s\to 1} P(x(s)) - \lim_{s\to 0} P(x(s)) = 2\pi$. A heteroclinic cycle winding in the opposite direction would similarly have $\lim_{s\to 1} P(x(s)) - \lim_{s\to 0} P(x(s)) = -2\pi$.*

Remark 2 *If a system has a heteroclinic ratchet, then there exist pseudo orbits winding in one direction on the torus.*

In this section, we will first explain how a heteroclinic ratchet emerges for the system (7) after a synchrony breaking bifurcation. Then, we will discuss the stability of the heteroclinic ratchet and show that a third harmonic in the coupling function g is necessary for the heteroclinic ratchet to be an attractor. Finally, different heteroclinic networks will be shown to exist for different parameter values.

3.1 Heteroclinic ratchets for the four coupled oscillators

In order to give examples to heteroclinic networks, we consider the particular case of

$$f(x; y, z) = \omega + g(x - y) + g(x - z) \tag{13}$$

Using this, we can write (7) in the form

$$\begin{aligned}\dot{\phi}_1 &= g(\phi_1+\phi_3-\phi_2)+g(\phi_1)-g(-\phi_1)-g(\phi_3-\phi_2)\\ \dot{\phi}_2 &= g(\phi_2-\phi_3-\phi_1)+g(\phi_2)-g(-\phi_3-\phi_1)-g(-\phi_2)\\ \dot{\phi}_3 &= g(-\phi_1)+g(\phi_3-\phi_2)-g(-\phi_3-\phi_1)-g(-\phi_2).\end{aligned} \tag{14}$$

Here the new variables are the phase differences $\phi_1 \hat{=} \theta_1-\theta_3$, $\phi_2 \hat{=} \theta_2-\theta_4$, and $\phi_3 \hat{=} \theta_3-\theta_4$. We consider the coupling function g up to three harmonics as

$$g(x) = -\sin(x+\alpha_1)+r_2\sin(2x)+r_3\sin(3x). \tag{15}$$

For this coupling function, there may exist different types of robust heteroclinic networks for different parameter values. We will first demonstrate the heteroclinic ratchet that exist for an open set of parameters.

Robust heteroclinic connections must lay on some invariant subspaces, which persist under the connection structure preserving perturbations. These are visualized in Figure 3. A robust connection on an invariant subspace has to be from saddle to sink when the dynamics on an invariant subspace is considered. Consider the 2-dimensional (for the phase difference space $\mathbb{T}^3$) invariant subspaces V_3^s, V_3^1, and V_3^2. The quotient networks corresponding to these subspaces are shown in Table 3. Recall that the subspaces V_3^1 and V_3^2 are related by the symmetry. The heteroclinic networks will be shown to lay on these subspaces where the dynamics are conjugate and governed by N_2 and N_3. Note that, according to the Theorem 1, there exist a syncrony breaking bifurcation at the origin for the system (14). This bifurcation results in two equilibria on $V_2 = V_3^1 \cup V_3^2$, say p and q, which are related to each other by the symmetry of (14). That is $q = \rho(p) = 2\pi - p$. Now, assume that there are robust connections from p to q on V_3^1. Due to the symmetry, this implies that there are robust connections from q to p on V_3^2. Since these connections form a robust heteroclinic network, it is enough to show the existence of saddle to sink connections on V_3^1. This is done by identifying robust connections from p to q using the simulation tool XPPAUT [3] (see Figure 4a).

In Figure 4b, a heteroclinic network is seen for the parameter values $\xi_1 = (\alpha_1, r_2, r_3) = (1.4, 0.3, -0.1)$. Note that this is a heteroclinic ratchet since it includes phase slips in one direction only. Since the dynamics of (14) is trapped inside the 2π-cube, phase slips are impossible. However, addition of arbitrary small noise give rise to phase slips in one direction. A solution

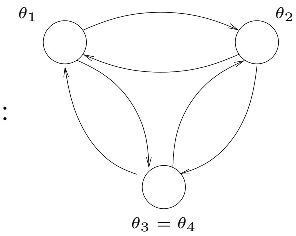
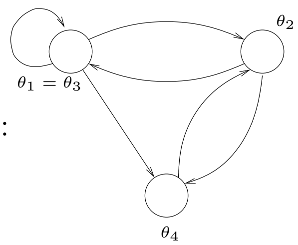
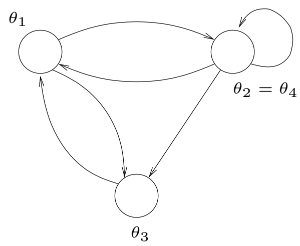
Balanced Colourings	Invariant Subspaces	Quotient Networks
$(3,4)$	V_3^s	N_1: 
$(1,3)$	V_3^1	N_2: 
$(2,4)$	V_3^2	N_3: 

Table 3: Quotient Networks on Three Dimensional Invariant Subspaces

of the system (14) with a white Gaussian noise is given in Figure 5 . Thus the presence of a heteroclinic ratchet for the phase difference dynamics of an oscillator system gives rise to behaviours where some oscillators oscillate faster than the others. We will turn to this point in the next section and discuss the relation between the presence of heteroclinic ratchets and the synchronization of oscillators.

The heteroclinic ratchet found for the parameter set ξ_1 is an asymptotically stable attractor. The asymptotic stability of robust heteroclinic networks can be observed considering the transversal, contracting and expanding eigenvalues at the equlibria p, that is, $\lambda_t(p)$, $\lambda_c(p)$, and $\lambda_e(p)$ respectively [11]. Since in our example there is no transversal direction and the eigenvalues at q are the same as the eigenvalues at p, a heteroclinic network connecting the equilibria p and q is asymptotically stable if $|\lambda_e(p)/\lambda_c(p)| < 1$ and not stable if $|\lambda_e(p)/\lambda_c(p)| > 1$. For the parameter set ξ_1, the equlibrium is found at $p = 1.4432$ using the Monte Carlo method in XPPAUT. Then, $\lambda_e(p)$ and $\lambda_c(p)$ can be found as 0.74 and -1.2 by linearizing (14) at p (see Eq. 17). This implies the asymptotic stability of the heteroclinic ratchet.

Note that, since the conditions for the asymptotic stability is open and the heteroclinic connections are robust, one can find an open set of the parameter space $\{(\alpha_1, r_2, r_3) \mid 0 \leq r_2, r_3, 0 \leq \alpha_1 < 2\pi\}$, for which the system (14) admits an asymptotically stable robust heteroclinic ratchet. On the other hand, for the system (14), the robust heteroclinic ratchets connecting a pair of saddles p and q on V_3^s can not be asymptotically stable if $r_3 = 0$ (see Appendix). Therefore, the heteroclinic ratchets for the system (14) can not be asymptotically stable unless the third or larger harmonics of the coupling function g are taken into account.

3.2 Bifurcation from one heteroclinic network to another

Although the subspace V_3^s does not include any part of the heteroclinic networks, the dynamics restricted on this subspace, that is, the dynamics of the network N_1 give rise to a bifurcation from a heteroclinic cycle to a heteroclinic ratchet as seen in Figure 6. The detailed bifurcation analysis of the 3-cell all-to-all coupled oscillators with a coupling function having the first two harmonics is given in [19]. In this work it is stated that apart from the transcritical bifurcation of the origin there exist a saddle node bifurcation on the invariant lines. On the reverse direction at this saddle-node bifurcation, for the system (14) a heteroclinic cycle bifurcates to a heteroclinic ratchet as seen in Figure 6b). This bifurcation should also exist for nonzero r_3 values and thus one can obtain the heteroclinic cycle in Figure 6d as attracting. In fact, for the parameter set $\xi_2 = (1.2, 0.3, -0.05)$, the same cycle exists and it turns out to be stable since $\lambda_e(p) = 0.68$ and $\lambda_c(p) = -0.70$ Thus the heteroclinic cycle for ξ_2 is attracting the nearby points with ϕ_1 and ϕ_2 less than 2π and repelling the other nearby points with ϕ_1 or ϕ_2 greater than 2π to the sink s. That is, this heteroclinic cycle has a basin with positive measure, so it is a Milnor attractor, though not stable. Note that this type of a heteroclinic cycle is also unusual for symmetric systems.

4 Dynamical Consequences of the Heteroclinic Ratchet

Heteroclinic ratchets induce important results in the dynamics of coupled oscillators, which are not obvious in view of the special coupling or symmetry of the system. In this section, the relation between the presence the heteroclinic ratchet for the phase difference system (14) and the onset of different synchronization types for the oscillator system will be explained. The imperfectness of the system, such as the presence of small noise and detuning of natural frequencies of oscillators will also be shown to play a crucial role in the dynamical consequences of heteroclinic ratchets.

Let us consider the phase variables of 4 lifted to $\mathbb{R}^4$, that is $\theta_i \in \mathbb{R},\ i = 1, \ldots 4$. We define the average frequency difference of oscillators i and j as

$$f_{ij} = \lim_{t\to\infty} \|\theta_i - \theta_j\|/t = 0.$$

Different synchronization types of oscillators can be defined as follows:

Definition 2 *(Synchronization of oscillators)*

- *Oscillators i and j are* completely synchronized *if their phase difference $\|\theta_i - \theta_j\|$ is constant.*
- *Oscillators i and j are* phase synchronized *if their phase difference is bounded uniformly for $t > 0$.*

- *Oscillators i and j are* frequency synchronized *if* $\lim_{t\to\infty} \|\theta_i - \theta_j\|/t = 0$.

Let us consider the phase variables lifted to $\mathbb{R}^4$. Then the reduced dynamics of phase differences can be considered on $\mathbb{R}^3$. Obviously, oscillators of the system (14) are phase synchronized since the reduced phase difference dynamics are trapped inside the 2π-cube in $\mathbb{R}_3$. However, addition of small noise or frequency detuning may change the type of synchronization. These switchings of systems between different behaviours are intimately related to the presence of robust heteroclinic ratchets. Note that without any perturbation an initial state in the vicinity of the heteroclinic ratchet results in a slow switching behaviour close to the nonwinding heteroclinic cycle, which is a part of the heteroclinic ratchet, and phase synchronization persist. However, when a white Gaussian noise is added, using the Wiener variables in XPP, phase slips occur either in $+\phi_1$ or $+\phi_2$ direction. These are due to the winding connections of the heteroclinic ratchet. Since the variable ϕ_3, that is the phase difference between the oscillators 3 and 4, is still trapped in the interval $[0, 2\pi)$, only these two oscillators are phase synchronized, whereas the other couples are completely desynchronized. On the other hand, when the added small noise is uniformly distributed on $\phi_1 - \phi_2$ plane, then the oscillators 1 and 2 turn out to maintain frequency synchronization although they are not phase synchronized. In order to see this, consider the illustration of the heteroclinic ratchet given in Figure 4c. Since a solution converging to a heteroclinic network stays longer and longer near equlibria we can consider the effect of the noise only for the states near equilibria. Then, considering the lower (upper) equilibrium p (q), it is obvious that the possibility of the phase slips in $+\phi_1$ ($+\phi_2$) direction and the possibility of switching to the upper (lower) equilibrium without any phase slips are equal. As a result, the number of phase slips in $+\phi_1$ and $+\phi_2$ directions tend to be equal as long as the system evolves with a uniform small noise. Thus, the oscillators 1 and 2 and the oscillators 3 and 4 are frequency synchronized although the phases of the former oscillator pair advance faster than the phases of the latter pair, which means both phase and frequency synchronizations are lost for the oscillator pairs $(1, 3)$ and $(2, 4)$ (see Figure 5). Note that in this case none of the pairs except the $(3, 4)$ are phase synchronized since due to the noise any bound can be excessed.

Another imperfectness that should be taken into account to be more realistic is the frequency detuning, that is considering oscillators with different natural frequencies. Considering the heteroclinic ratchet in Figure 4, it is reasonable to think that frequency detuning of the oscillator pairs $(1,3)$ and $(2,4)$ may destroy their synchrony. Let $\Delta_{13} = \omega_1 - \omega_3$ and $\Delta_{24} = \omega_2 - \omega_4$ be the frequency detunings of the oscillator pairs $(1,3)$ and $(2,4)$, respectively. The difference between the observed average frequencies of the oscillators 1 and 3 for the different values of Δ_{13} is given in Figure 7. Since the heteroclinic ratchet includes winding connections in the $+\Phi_1$ direction but no connections winding in the $-\Phi_1$ direction, the oscillator system responds to the positive Δ_{13} by breaking the synchrony of the oscillator pair $(1,3)$, whereas negative Δ_{13}'s have no qualitative effect on the dynamics of the oscillator system. Note that, because of the symmetry, a similar result can be obtained for the oscillator pair $(2,4)$ considering the detuning Δ_{24}.

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Appendix: Unstability of the heteroclinic ratchet for $r_3 = 0$

Note that for $r_3 = 0$ the new equilibrium after the pitchfork bifurcation on V_3^s is

$$p = \cos^{-1}\left(\frac{\cos\alpha_1}{2r_2\cos\alpha_2}\right) \tag{16}$$

This can be obtained from (14) by setting $\phi_1 = \phi_2 = \dot{\phi}_3 = 0$. Let us calculate the eigenvalues at p. Linearising (14) at p gives

$$\lambda_{1,2} = g'(\mp p) + 2g'(0) \tag{17}$$

, where λ_1 and λ_2 corresponds to the eigenvectors laying on V_3^1 and V_3^2 respectively. For a heteroclinic ratchet to be stable it is necessary to have $\lambda_1 + \lambda_2 \leq 0$. However, from (17),

$$\begin{aligned}
\lambda_1 + \lambda_2 &= g'(p) + g'(-p) + 4g'(0) \\
&= -2\cos p\cos\alpha_1 + 8r_2\cos^2 p\cos\alpha_2 - 4r_2\cos\alpha_2 - 2\cos\alpha_1 + 4r_2\cos\alpha_2
\end{aligned}$$

and substituting (16) one gets

$$\begin{aligned}
\lambda_1 + \lambda_2 &= -\frac{\cos p\cos\alpha_1}{r_2\cos\alpha_2} + \frac{2\cos p\cos\alpha_1}{r_2\cos\alpha_2} + 4r_2\cos\alpha_2 - 4\cos\alpha_1 \\
&= \left(\frac{\cos\alpha_1}{\sqrt{r_2\cos\alpha_2}} - 2\sqrt{r_2\cos\alpha_2}\right)^2 \geq 0.
\end{aligned}$$

Thus, the expanding eigenvalue is greater in absolute value than the contracting eigenvalue.

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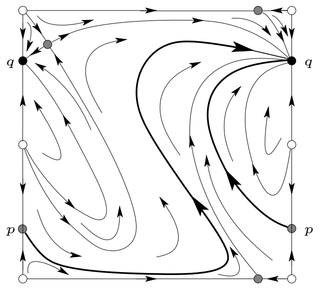
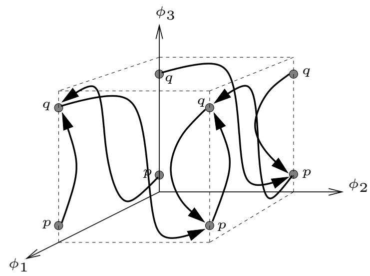
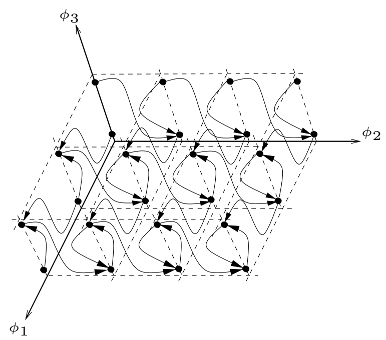


(a) Phase portrait on V_3^1. The horizontal axis is V_2^{s1} (ϕ_1-axis) and the vertical axis is V_2 (ϕ_3-axis).



(b) The heteroclinic ratchet on the boundary of $\mathbb{T}^3$



(c) The heteroclinic ratchet seen on the lift of $\mathbb{T}^3$ to $\mathbb{R}^3$

Figure 4: Heteroclinic Ratchet for the system (14) with the parameter set ξ_1. Sources, saddles and sinks are indicated by small disks filled with white, grey or black color.

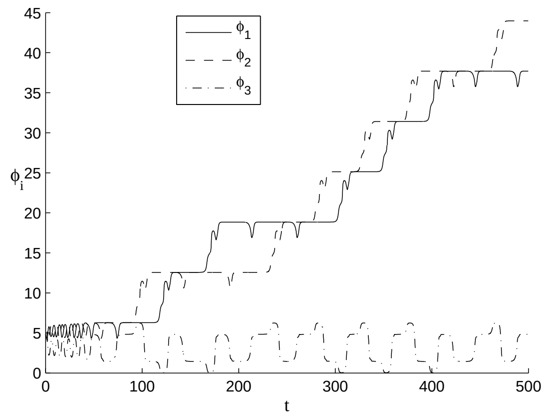


Figure 5: A solution of the system (14) with the additive white Gaussian noise ($\mu = 0$, $\sigma = 10^{-6}$) on the right hand side and for the parameter set ξ_1. The solution converges to the heteroclinic ratchet until the noise draws it to the other side of the invariant surfaces V_3^1 and V_3^2 resulting in phase slips in the ϕ_1 and ϕ_2 directions.

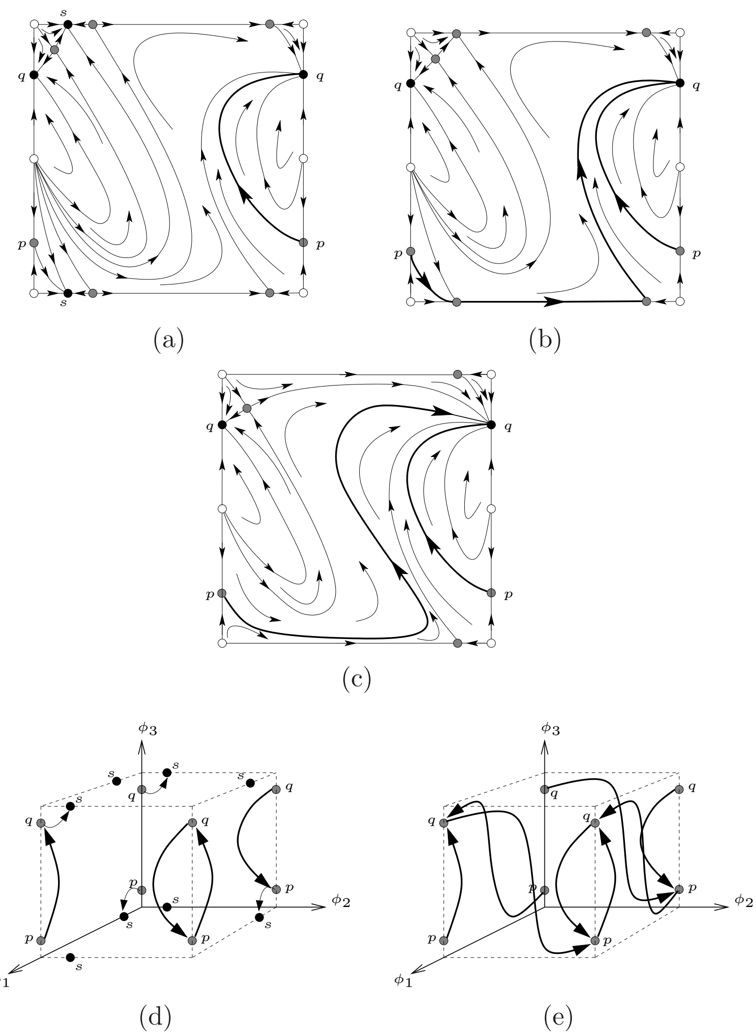


Figure 6: Phase Portraits on V_3^1 for $r = 0.3$ and $\alpha = 1.2$(a), $\alpha \cong 1.315$(b), and $\alpha = 1.4$(c) demonstrating a bifurcation from a heteroclinic cycle to a heteroclinic ratchet. (For each graph the horizontal axis is V_2^{s1} (ϕ_1-axis) and the vertical axis is V_2 (ϕ_3-axis). Sources, saddles and sinks are indicated by small disks filled with white, grey or black color. The parts of the heteroclinic networks are shown by thick lines.)

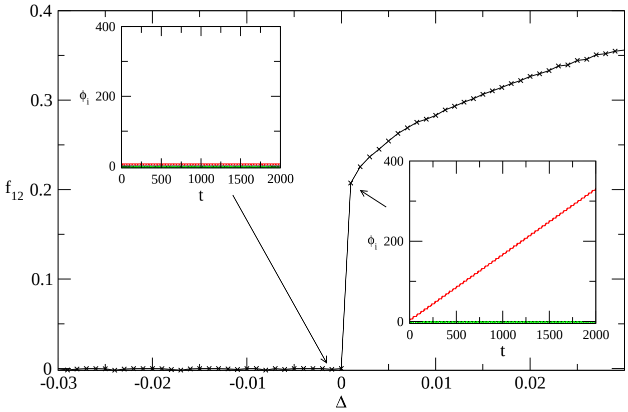


Figure 7: Detuning of ....