Robust Heteroclinic Behaviour, Synchronization, and Ratcheting of Coupled Oscillators

Peter Ashwin and Özkan Karabacak

Abstract This review examines some recent work on robust heteroclinic networks that can appear as attractors for coupled dynamical systems. We focus on coupled phase oscillators and discuss a number of nonlinear dynamical phenomena that are atypical in systems without some coupling structure. The phenomena we discuss include heteroclinic cycles and networks between partially synchronized states. These networks can be attracting and robust to perturbations in parameters and system structure as long as the coupling structure is preserved. We discuss two related effects; extreme sensitivity to detuning (strongly coupled oscillators may lose their frequency synchrony for very small detunings) and heteroclinic ratchet where the sensitivity may only appear for detunings of one sign.

1 Introduction

Coupled dynamical systems are a very important source of examples of nonlinear systems that are of interest because of many applications. Additionally, they are of intrinsic interest as structured examples of high dimensional dynamical systems. The applications of coupled dynamical systems are very wide and include in particular solid state physics [2], neuroscience [21] and biological systems generally [37], rather than discuss applications here we refer to these articles. A fundamental concept of use for describing coupled dynamical systems (whether chaotic or not) is that of synchronization in its various forms, and this has been the topic of many papers over the last decade [30, 31, 10, 1].

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The topic that we focus on in the review is the robust appearance of dynamics that is neither chaotic nor periodic, but that is intermittent in the sense that repeated switchings are apparent between different saddle states. These "robust heteroclinic cycles" appear naturally in systems ranging from Lotka-Volterra dynamics to symmetric systems [27, 20, 16, 14]. Indeed the cycles may be between chaotic saddles in general [16, 29, 8].

The prototype of these cycles is the so-called *Guckenheimer-Holmes cycle* [18] though the same cycle has been studied in a variety of contexts by dos Reis [34], Busse and Clever [12] and others. This can be understood from studying the dynamics of the vector field

$$\begin{aligned} \dot{x} &= \mu x + (ax^2 + by^2 + cz^2)x \\ \dot{y} &= \mu y + (ay^2 + bz^2 + cx^2)y \\ \dot{z} &= \mu z + (az^2 + bx^2 + cy^2)z \end{aligned} \tag{1}$$

for the open set of parameters where $\mu > 0$, $a < 0$ and $b < -c < 0$. For this set of parameters (see e.g. [13, p61]) one can verify that the dynamics possesses an attracting heteroclinic cycle whose structure is illustrated in Figure 1. This cycle is robust because the system is preserved under a number of reflection symmetries $(x,y,z) \to (\mp x, \mp y, \mp z)$ and the permutation symmetry $(x,y,z) \to (y,z,x)$ meaning that the axis planes are invariant for the dynamics. Therefore, saddle-to-sink type heteroclinic connections between equilibria on these planes are robust under symmetry-preserving perturbations. Observe that the equation (1) can be viewed as a system of three coupled one-dimensional dynamical systems with a particular form of cyclic coupling.

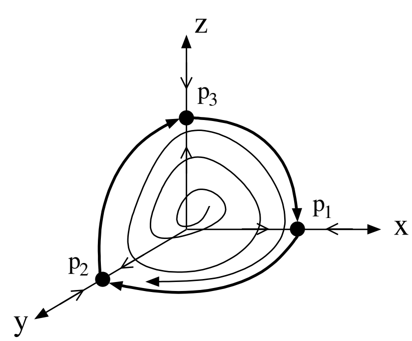


Fig. 1 The Guckenheimer-Holmes cycle: (1) has an attracting heteroclinic cycle between equilibria p_1, p_2 and p_3 for an open set of parameter values, that is robust to all perturbations that preserve a finite group of symmetries of the vector field.

Other families of dynamical systems for which heteroclinic cycles may appear robustly are coupled dynamical systems where the coupling between dynamical units respects to a directed graph (see [17, 14] and the references therein). It is because such families also admit dynamically invariant subspaces, this time forced by the coupling structure rather than the symmetry of the system. The heteroclinic cycles found in such systems are much richer in dynamics due to the lack of symmetry and give rise to a new phenomenon in case of coupled oscillators which we summarize in Section 4.

Much research has been done on the behaviour of robust heteroclinic cycles in symmetric systems; we will focus only on work that has linked this to coupled oscillators. The paper is organized as follows; in Section 2 we give an introduction to coupled oscillator dynamics and the reduction to phase oscillators. Section 3 discusses examples of robust heteroclinic networks and extreme sensitivity to detuning in such systems. Section 4 discusses some recent work on "heteroclinic ratchets" where attractors of the nonlinear system wind in a nontrivial manner around the torus. The final section 5 summarizes some open questions in this area and relevance to applications.

2 Synchronization properties of coupled oscillators

Many physical processes that are time-periodic in nature can be modelled by nonlinear oscillators, by which we mean dissipative dynamical systems with hyperbolic, attracting limit cycles. When several of these systems are coupled, various phenomena related to the synchronization of oscillators can arise. In this paper we will focus on some synchronization properties of oscillators that are well-modelled by coupled equations for the phases of each oscillator.

2.1 From limit cycle oscillators to phase oscillators

By a limit cycle oscillator, we mean a dynamical system $\dot{x} = f(x)$ on a manifold M that has an attracting hyperbolic periodic solution $\gamma(t)$. Coupled limit cycle oscillator systems are dynamical systems of the form

$$\dot{x} = F(x, \kappa)\ , \ \ x \in M^N \tag{2}$$

which reduce to N uncoupled limit cycle oscillators when the coupling strength $\kappa = 0$. In the uncoupled case ($\kappa = 0$), the N-torus defined as the direct product of the limit cycles of each oscillator

$$\tau^N = \{x_i = \gamma_i(t+\theta_i) \colon (\theta_1, \dots, \theta_N) \in \mathbb{T}^N\}$$

is obviously invariant, attracting and normally hyperbolic. Therefore, one can predict that this attracting N-torus persists in the weak coupling case $\kappa \ll 1$. As a result, the asymptotic dynamics of (2) can be reduced to the dynamics reduced on this N-torus in the weak coupling case. Note that for a point on this torus, each oscillator can be represented by a phase variable $t + \theta_i \in \mathbb{T}$. Using an averaging technique [9], one can obtain a coupled phase oscillator system of the form

$$\dot{\theta} = \bar{F}(\theta, \kappa)\ , \ \ \theta \in \mathbb{T}^N, \tag{3}$$

where $\theta_i \in \mathbb{T}$ represents the phase of the oscillator i and F is invariant under the action of $\mathbb{S}^1$ given by $\theta \mapsto \theta + \varepsilon(1, \ldots, 1)$, $\varepsilon \in [0, 2\pi)$ (see [9] for details). This symmetry gives rise to a further reduction of the system on N-torus to a system on the quotient space $\mathbb{T}^N/\mathbb{S}^1$, which is an $(N-1)$-torus,

$$\dot{\phi} = \tilde{F}(\phi, \kappa)\ , \ \ \phi \in \mathbb{T}^{N-1}, \tag{4}$$

where ϕ_i's can be chosen as independent phase difference variables $\theta_{m_i} - \theta_{n_i}$. In the sequel, we refer to the space of phase difference $\mathbb{T}^{N-1}$ as *phase difference space* of the coupled oscillator system (3).

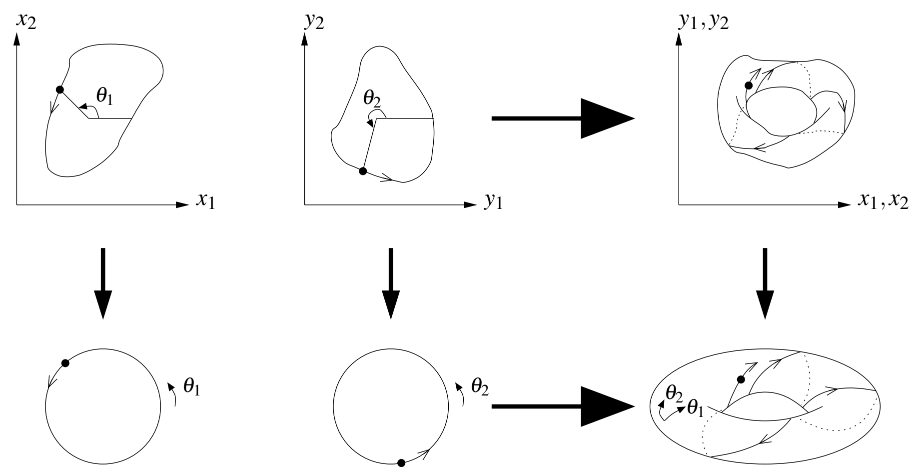


Fig. 2 Schematic diagram representing the reduction from limit cycle oscillators (upper figures) to phase oscillators (lower figures). In the uncoupled case $\kappa = 0$, the direct product of limit cycles (upper-right) is invariant and corresponds to the torus (lower-right) which is the phase space for the reduced system of coupled phase oscillators.

The idea of reducing the coupled phase oscillator systems to limit cycle oscillators was first proposed by Winfree in 1967. However, coupled phase oscillator systems began to be studied widely after Kuramoto's works in 1984 (See [37] and the references therein).

Kuramoto's model consists of N phase oscillators that are coupled globally with a sinusoidal coupling function. That is, the governing equation for each oscillator is

$$\dot{\theta}_i = \omega_i + \frac{\kappa}{N}\sum_{j=1}^{N} \sin(\theta_i - \theta_j), \tag{5}$$

where $\theta_i \in \mathbb{T} = [0, 2\pi)$ is the phase and ω_i is the natural frequency of the oscillator i

Considering an arbitrary coupling structure and a more general coupling function, the coupled phase oscillator dynamics can be written, more generally, as follows:

$$\dot{\theta}_i = \omega_i + \frac{\kappa}{N}\sum_{j=1}^{N} c_{ij} g(\theta_i - \theta_j). \tag{6}$$

Here, the connection matrix $\{c_{ij}\}$ represents the coupling between oscillators. $c_{ij} = 1$ if the oscillator i receives an input from the oscillator j and $c_{ij} = 0$ otherwise. The coupling function g is a 2π-periodic function. Therefore, it is natural to consider a Fourier series expansion of g

$$g(x) = \sum_{k=1}^{\infty} r_k \sin(kx + \alpha_k) \tag{7}$$

Note that, by scaling the time, we can set $\kappa = N$ and $r_1 = 1$. In this case, the coupling is modulated by the parameters $\alpha_1, \alpha_2, \ldots$ and $r_2, r_3, \ldots$.

Several truncated cases of the general case (7) was considered in the literature. Considering the first Fourier term only (as in the Kuramoto model, (5)), frequency synchronization and clustering phenomena were analyzed [28, 35]. Hansel et al. used first two Fourier terms and observed a new phenomenon, called slow switching, as a result of the presence of an asymptotically stable robust heteroclinic cycle [19, 25]. Recently, using the first three harmonics an attracting heteroclinic ratchet was found for a nonsymmetric connection structure [23], while four harmonics seem to be necessary to observe chaotic dynamics in four all-to-all coupled oscillators [38].

In the literature, there are different definitions for the phase or frequency synchronization of oscillators. Moreover, one can define other concepts related to the synchronization, such as sensitivity to detuning [6]. For an ordered pair of oscillators, we call such properties *synchronization properties* of the oscillator pair and these may include: Phase locking, Phase synchronization, Frequency synchronization, Sensitivity to detuning and Ratcheting. The former three are discussed for example in [31] while the latter two are discussed in [6, 23] and we outline some of the discussion and results from these papers.

2.1.1 Phase and Frequency Synchronization

For a solution $\theta(t) = (\theta_1(t), \ldots, \theta_N(t))$ of (6), let $\theta^L(t) = (\theta_1^L(t), \ldots, \theta_n^L(t))$ denote the lifted phase variables. We say the oscillator pair (i, j) is *phase synchronized* on the solution $\theta(t)$ if $\theta_i^L(t) - \theta_j^L(t)$ is bounded for all t and *phase locked* if $\lim_{t\to\infty}(\theta_i^L(t) - \theta_j^L(t))$ exists. We say oscillators are *frequency synchronized* if $\lim_{t\to\infty} \frac{\theta_i(t)-\theta_j(t)}{t} = 0$. Note that phase locking implies phase synchronization and phase synchronization implies frequency synchronization. However, the converses are not true in general. For example, on a typical solution approaching to a heteroclinic network, particular pairs of oscillators are never phase locked, but they can be phase synchronized if the heteroclinic network is contractible to the diagonal in $\mathbb{T}^n$. More interesting is the effect of heteroclinic ratchets on the synchronization properties of oscillators. On a solution approaching to a heteroclinic ratchet, some oscillator pairs can be frequency synchronized but not phase synchronized as we will see in Section 4.

2.1.2 Sensitivity to Detuning and Ratcheting

It is known that when the oscillators are synchronized, a mismatch in natural frequencies, that is, detuning may result in loss of synchronization depending on how large the detuning is. Let $\omega_{ij} = \omega_i - \omega_j$ denote the detuning and $\Omega_{ij} = \Omega_i - \Omega_j$ the difference in observed average frequencies. Here $\Omega_i = \lim_{t\to\infty} \frac{\theta_i^L}{t}$. The typical $(\omega_{ij}, \Omega_{ij})$ characteristic of coupled oscillators is as in Fig. 3a.

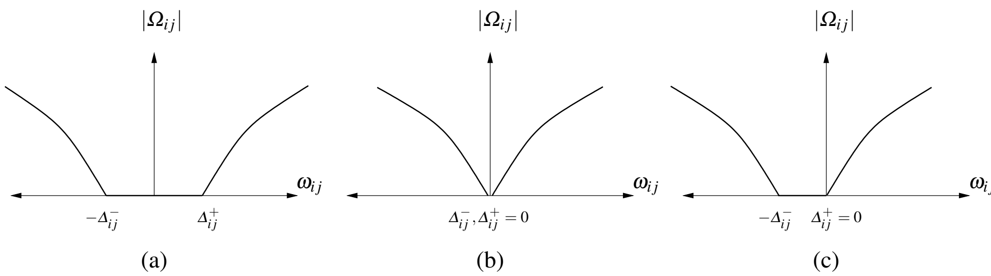


Fig. 3 Different $(\omega_{ij}, |\Omega_{ij}|)$-characteristics of coupled oscillators. (a) Usual case: Frequency synchronization of the oscillators persist in a certain tolerance range of detuning. (b) Extreme sensitivity to detuning: Although there is a dynamically stable frequency synchronized behaviour at $\omega_{ij} = 0$, synchronization is broken by arbitrarily small detuning. This can happen if there is an attracting heteroclinic cycle in state space (see Section 3). (c) Unidirectional extreme sensitivity to detuning (or ratcheting): Under small detuning synchronization is broken only if the detuning is positive.

For an ordered oscillator pair (i, j), we generalize notions in [6] to define the *tolerance to positive detuning* and *tolerance to negative detuning* as

$$\Delta_{ij}^+ := \sup\{\Delta : 0 \le \omega_{ij} < \Delta \implies \text{(i,j) is phase synchronized on all attractors of (6)}\}$$

$$\Delta_{ij}^- := \sup\{\Delta : -\Delta < \omega_{ij} \le 0 \implies \text{(i,j) is phase synchronized on all attractors of (6)}\},$$

We call $\Delta_{ij} := \min(\Delta_{ij}^-, \Delta_{ij}^+)$ the *tolerance to detuning* of (i, j). If $\Delta_{ij} = 0$ then the oscillator pair (i, j) is said to have extreme sensitivity to detuning. If $\Delta_{ij}^+ = 0$ but $\Delta_{ij}^- > 0$, we say that the oscillator pair (i, j) is *ratcheting* (see Fig. 3) (for the details see [22]). Note that ratcheting is an asymmetric relation on the set of oscillators, that is, if (i, j) is ratcheting then (j, i) is not ratcheting. In the following, we will show that heteroclinic networks may result in extreme sensitivity to detuning (Section 3) and the heteroclinic ratchets give rise to ratcheting of some oscillator pairs (Section 4).

3 Heteroclinic cycles and extreme sensitivity

An attracting heteroclinic cycle in the phase difference space of a coupled oscillator system has a strong effect on the synchronization properties of oscillators. For instance, a solution approaching to a heteroclinic cycle implies the absence of phase locking of certain oscillator pairs. Moreover, heteroclinic cycles are related to the extreme sensitivity phenomenon [6].

Heteroclinic cycles induce an intermittent behaviour called *slow switching* where the dynamics stays long time near one cluster and then passes to another cluster. Slow switching behaviour of coupled oscillator systems was first studied by Hansel et al. in [19]. They found heteroclinic cycles for four globally coupled phase oscillator system with a coupling function up to second order Fourier terms ($\alpha_1 = 1.25$, $r_2 = 0.5$). After this work, heteroclinic cycles associated with slow switching were also studied for different oscillator types, such as delayed pulse-coupled integrate-and-fire oscillators [26, 11], limit cycle oscillators [25]. In the following, we will describe the heteroclinic behaviour observed in coupled phase oscillators [5, 6, 4, 7], and explain its effect on synchronization properties. This effect has been investigated for fully symmetric (all-to-all coupled) systems but not in many other configurations.

3.1 *Symmetric heteroclinic cycles for all-to-all coupled phase oscillators*

All-to-all coupling gives rise to S_N-permutation symmetry. This imposes many dynamically invariant subspaces arising as fixed point subspaces of subgroups of S_N. Therefore, the dynamics is trapped in invariant regions bounded by these fixed point subspaces. Let us choose the phase difference variables as $\phi_i = \theta_1 - \theta_i + 1$, $i = 1, \ldots, N-1$. Then, the invariant regions are $\{\phi \in \mathbb{T}^{N-1} \colon \phi_{\sigma(1)} \leq \phi_{\sigma(2)} \leq \cdots \leq \phi_{\sigma(N-1)}\}$ where σ is a permutation of oscillators. When σ is identity, this region is called *canonical invariant region* [9]. Since all these regions are symmetric images of each other, it suffices to study the dynamics on the canonical invariant region. Note that, since the dynamics is trapped in these invariant regions in the phase difference space, oscillators are always phase synchronized and therefore frequency synchronized (the subspace $\theta_i = \theta_j$ being invariant implies phase synchronization of oscillators i and j [15]). We will be more interested in the extreme sensitivity properties of oscillators for which the existence of heteroclinic networks are crucial.

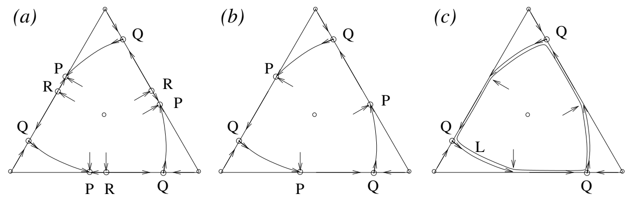


Fig. 4 Schematic diagrams illustrating a bifurcation of all-to-all coupled 3-oscillator system in the canonical invariant regions. The edges of the triangles represent the fixed point subspaces of the form $\{\theta_i = \theta_j\}$. On these lines two equilibria P and R (a) join together by a saddle-node bifurcation (b) and disappear giving birth to a periodic orbit in the interior of the canonical invariant region (c). At the bifurcation point (b), a heteroclinic cycle appears connecting the saddles P and Q on the invariant lines. (Adapted from [5]).

For N coupled phase oscillators, heteroclinic behaviour can arise if $N \geq 3$. The case $N = 3$ and $N = 4$ is analyzed in detail by Ashwin et al. in [5]. Considering second order Fourier truncation of the coupling function, they show that for $N = 3$ a heteroclinic cycle appears as a codimension one phenomenon in phase difference space (see Figure 4). This heteroclinic cycle connects the saddles labelled by P and Q on the invariant lines, which have $S_2 \times S_1$ isotropy [5]. Note that, the heteroclinic network on $\mathbb{T}^{N-1}$ formed by these heteroclinic cycles contains winding heteroclinic cycles in each $\theta_i - \theta_j$ direction. Therefore any detuning Δ_{ij} gives rise to a periodic orbit that breaks the synchronization of the oscillators i and j (see [6] for details). As a result, this heteroclinic network leads to extreme sensitivity to detuning (see [6]). However, this phenomenon is not robust for $N = 3$ as it occurs at a bifurcation point.

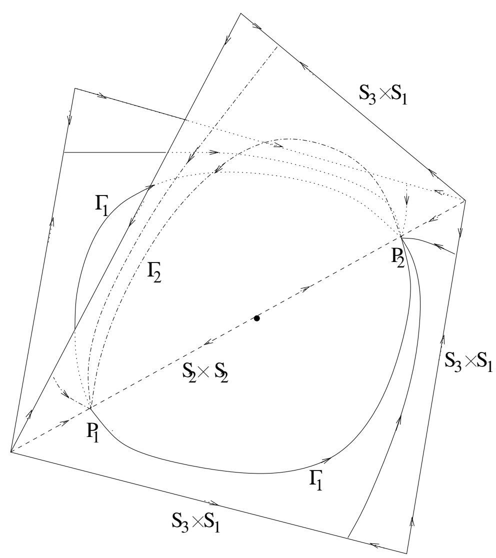


Fig. 5 A robust heteroclinic cycle for the all-to-all coupled 4-oscillator system. The heteroclinic cycle consists of two saddle equilibria P_1 and P_2 with $S_2 \times S_2$ isotropy and two connections Γ_1 and Γ_2 on the two dimensional invariant subspaces. The invariant subspaces are embedded in a cube that represents a unit cell for the torus of phase difference space- in this representation all vertices of the cube represent in-phase solutions where all oscillators are synchronized. (Adapted from [5]).

For the case $N = 4$, one can observe robust heteroclinic cycles (see Fig. 5). In this case the canonical invariant region is a tetrahedron whose lines have either $S_2 \times S_2$ or $S_3 \times S_1$ isotropy. The heteroclinic cycle shown in Figure 5 exists robustly for an open set in the parameter space (see [5] for details.) This time the heteroclinic network formed by these heteroclinic cycles in different invariant regions does not contain any winding heteroclinic cycle, except for the critical case when the heteroclinic cycles first appear and lie on the invariant lines. As a result, although the heteroclinic behaviour is robust when $N = 4$, the extreme sensitivity phenomenon is again not robust.

Robust extreme sensitivity behaviour arises when one considers an all-to-all coupled oscillator system with $N \geq 5$. It is numerically shown in [6] that for $N = 5$, the extreme sensitivity is robust. In [4], a heteroclinic network for the 5-oscillator all-to-all coupled system is shown to exist on the phase difference space $\mathbb{T}^4$. In this case, the heteroclinic network contains winding heteroclinic cycles in any direction breaking the frequency synchronization of oscillators, and this happens robustly under small parameter changes. This robust extreme sensitivity behaviour is bidirectional due to the presence of full permutation symmetry.

4 Heteroclinic ratchets for nonsymmetric coupling

The heteroclinic cycles described in Section 3 are robust for $N > 3$ because they are contained in invariant subspaces forced by the symmetries of the coupling structure in such a way that connections are saddle-to-sink type in each subspace. However, these symmetries impose some restrictions on the types of possible robust heteroclinic cycles. Namely, such a cycle necessarily has the symmetries that are related to the invariant subspaces which contain parts of the heteroclinic cycle. For instance, in the case of all-to-all coupled oscillators, the heteroclinic network found in [4] have S_5 permutation symmetry. Therefore, the dynamics near the heteroclinic network is the same for each oscillator. This means one expects the same synchronization properties for all pairs of oscillators.

On the other hand, as shown in Section 1, one can find nonsymmetric robust heteroclinic cycles. These are contained in invariant subspaces not forced by the symmetry but by the balanced equivalence relations of the underlying graph [17, 3]. The balanced equivalence relations result in invariant subspace without having much restrictions on the overall dynamics as symmetry. Therefore, it is possible to find richer dynamics in such systems.

For coupled oscillators an example of a nonsymmetric heteroclinic network is discussed in [23]. This robust heteroclinic network induces different effects on different oscillators, since it includes heteroclinic cycles winding in one direction around the torus and no other cycles winding in the opposite direction. This type of heteroclinic network is called heteroclinic ratchet as its dynamical consequences are similar to a mechanical ratchet, a device that allows rotary motion on applying a torque in one direction but not in the opposite direction:

Definition 1. [23] For a system on $\mathbb{T}^N$, a heteroclinic network is a *heteroclinic ratchet* if it includes a heteroclinic cycle with nontrivial winding in one direction but no heteroclinic cycles winding in the opposite direction. More precisely, we say a heteroclinic cycle $C \subset \mathbb{T}^N$ parametrized by $x(s)$ $(x\colon [0,1) \to \mathbb{T}^N)$ has *nontrivial winding in some direction* if there is a projection map $P\colon \mathbb{R}^N \to \mathbb{R}$ such that the parametrization $\bar{x}(s)$ $(\bar{x}\colon [0,1) \to \mathbb{R}^N)$ of the lifted heteroclinic cycle $\bar{C} \subset \mathbb{R}^N$ satisfies $\lim_{s\to 1} P(\bar{x}(s)) - P(\bar{x}(0)) = 2k\pi$ for some positive integer k. A heteroclinic cycle winding in the opposite direction would satisfy the same condition for a negative integer k.

In Fig. 6, three heteroclinic networks on a 2-torus are shown. The first network is not a heteroclinic ratchet because it does not contain a winding heteroclinic cycle. The second network contains a heteroclinic cycle winding around $+x$ direction, but it also contain a cycle winding in the opposite direction $-x$. Therefore, the only heteroclinic ratchet in the figure is the third one, which has a winding cycle in $+x$ direction.

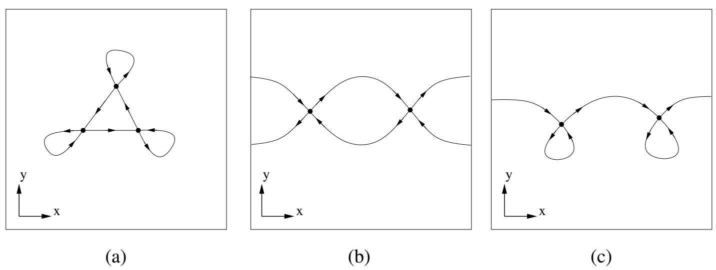


Fig. 6 Three different heteroclinic networks on $\mathbb{T}^2$ containing (a) no winding heteroclinic cycle (b) winding heteroclinic cycles in opposite directions $+x$ and $-x$ (c) one winding heteroclinic cycle in $+x$ direction. Therefore, only the network in (c) is a heteroclinic ratchet.

4.1 A simple example of ratcheting

Heteroclinic ratchets have strong effects on the synchronization properties of oscillators. An example of a heteroclinic ratchet in coupled oscillator systems is first introduced and analyzed in [23].

The coupled oscillator system considered in [23] is given by

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 + f(\theta_1;\theta_2,\theta_3)\\
\dot{\theta}_2 &= \omega_2 + f(\theta_2;\theta_1,\theta_4)\\
\dot{\theta}_3 &= \omega_3 + f(\theta_3;\theta_1,\theta_2)\\
\dot{\theta}_4 &= \omega_4 + f(\theta_4;\theta_1,\theta_2),
\end{aligned} \tag{8}$$

which has a coupling structure as in Fig 7a. Here, the coupling function f is chosen as in 7. We first assume identical oscillators, that is

$$\omega = \omega_1 = \cdots = \omega_4. \tag{9}$$

Since then the oscillators are identical, one can use the balanced colouring method to find the invariant subspaces imposed by the coupling structure. A coloring of cells, that is, a partition of the set of all cells into a number of groups or colors is called *balanced* if each pair of cells with the same color receive same number of inputs from the cells with any given color. Three balanced colorings (beside the others) of

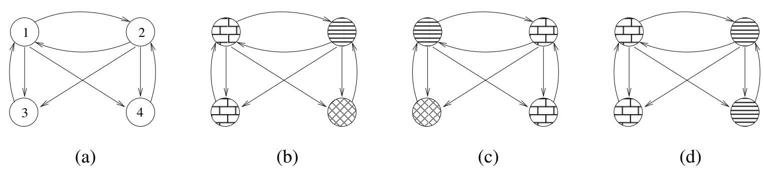


Fig. 7 The graph showing the connection structure of the system (8) (a) and three of its balanced colourings (b-d). Colors are represented by different filling patterns.

the graph in Fig. 7a are shown in Fig. 7b-d. These give three invariant subspaces:

$$\begin{aligned} V_1 &= \{\theta \in \mathbb{T}^4 : \theta_1 = \theta_3\} \\ V_2 &= \{\theta \in \mathbb{T}^4 : \theta_2 = \theta_4\} \\ \bar{V} &= V_1 \cap V_2 \end{aligned}$$

Using the phase-shift symmetry ($\{\theta_1, \ldots, \theta_4\} \to \{\theta_1 + \varepsilon, \ldots, \theta_4 + \varepsilon\}_{\text{ mod } 2\pi}$) of (8), one can reduce the dynamics to a phase difference system on 3-torus. Defining the new variables as $(\phi_1, \phi_2, \phi_3) := (\theta_1 - \theta_3, \theta_2 - \theta_4, \theta_3 - \theta_4)$, the phase difference dynamics can be written as

$$\begin{aligned} \dot{\phi_1} &= f(\phi_1; \phi_2 - \phi_3, 0) - f(0; \phi_1, \phi_2 - \phi_3) \\ \dot{\phi_2} &= f(\phi_2; \phi_1 + \phi_3, 0) - f(0; \phi_1 + \phi_3, \phi_2) \\ \dot{\phi_3} &= f(\phi_3; \phi_1 + \phi_3, \phi_2) - f(0; \phi_1 + \phi_3, \phi_2). \end{aligned} \tag{10}$$

Note that the invariant subspaces V_1, V_2 and $\bar{V}$ correspond to the planes $\phi_1 = 0$, $\phi_2 = 0$ and the line $\phi_1 = \phi_2 = 0$, respectively. Therefore, it is possible that a robust heteroclinic network exists on these invariant subspaces in the phase difference space. In fact, there exist a robust heteroclinic ratchet for the parameter values $(\alpha_1, r_2, r_3) = (1.4, 0.3, -0.1)$ (see Fig. 8).

Heteroclinic ratchets have two main effects on the synchronization properties of oscillators: ratcheting via noise and ratcheting via detuning as described in Section 2.1.2. Both arbitrary small noise and arbitrary small detuning in a certain direction

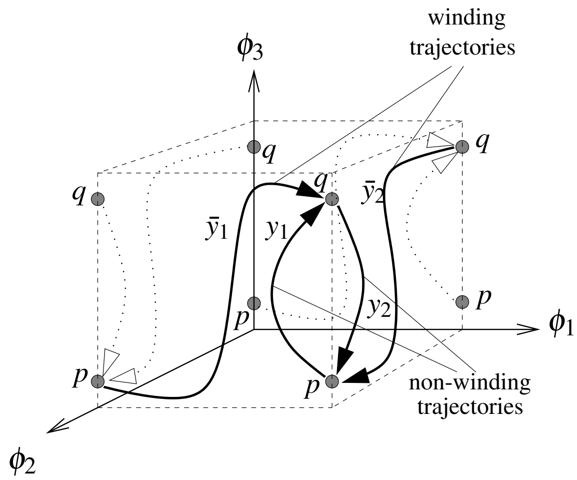


Fig. 8 A heteroclinic ratchet for the system (10). The heteroclinic network consist of two equilibria (p and q) and four heteroclinic trajectories $(y_1, y_2, \bar{y}_1, \bar{y}_2)$. It contains three winding heteroclinic cycles: $(p, \bar{y}_1, q, y_2)$, $(p, \bar{y}_2, q, y_1)$ and $(p, \bar{y}_1, q, \bar{y}_2)$. (Adapted from [23]).

give rise to perpetual one-directional phase slips, and therefore, loss of frequency synchronization.

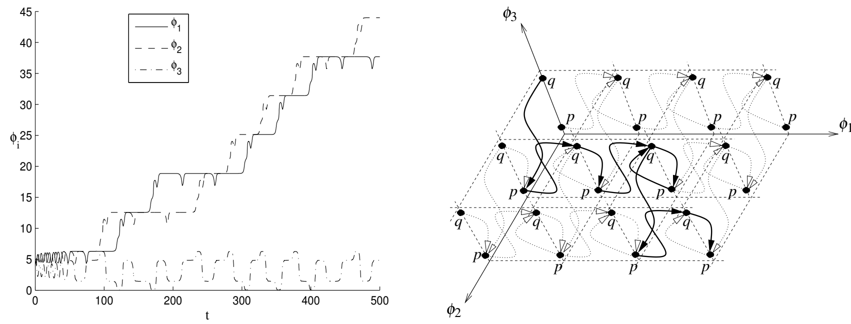
4.2 Ratcheting forced by noise

We consider as in [23] the dynamics of (10) near the heteroclinic ratchet shown in Figure 8. On applying small noise to the system, phase differences between oscillators grow in certain directions such that for some pairs one oscillator always has a larger average frequency than the other. Therefore, the effect of noise is not homogeneous for oscillators even though the added noise is homogeneous.

Figure 9 depicts a solution of (10) under small noise with amplitude 10^{-6}. Although the noise is homogeneous with a zero mean, phase slips occur only in $+\phi_1$ and $+\phi_2$ directions. Recall that $\phi_1 = \theta_1 - \theta_3$ and $\phi_2 = \theta_2 - \theta_4$. Therefore, the oscillator pairs $(1,3)$ and $(2,4)$ lose frequency synchronization such that the first oscillator has as greater average frequency then the second oscillator.

4.3 Ratcheting forced by detuning

The effect of detuning, setting $\Delta_{ij} = \omega_i - \omega_j$ nonzero, on a heteroclinic ratchet is similar to the effect of noise. A system with a heteroclinic ratchet winding in some direction on the phase difference space $\mathbb{T}^{N-1}$, say $+\phi_i = \theta_{m_i} - \theta_{m_j}$, responds to a positive detuning $\Delta_{m_i n_i} > 0$ by breaking frequency synchronization, whereas a small enough negative detuning, $\Delta_{m_i n_i} < 0$, leaves the frequency synchronization unchanged. We call this phenomenon *unidirectional extreme sensitivity to detuning*.



(a) (b)

Fig. 9 The figure in (a) shows a time series solution of the system (10) under white noise with amplitude 10^{-6}. For the first half of the solution in (a) the switchings between saddle states are shown in (b). (Adapted from [23]).

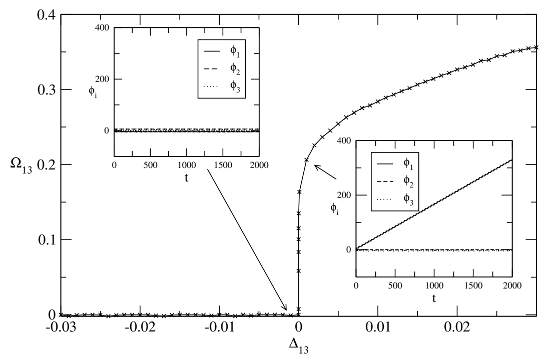
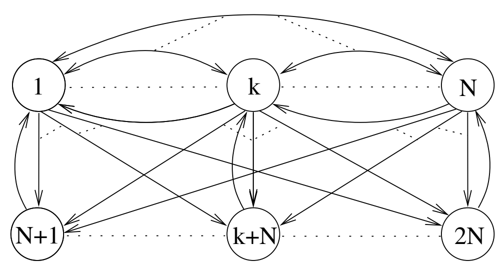


Fig. 10 Difference in observed average frequencies of the oscillators 1 and 3 are plotted for different detuning values $\Delta_{13} = \omega_1 - \omega_3$. Insets show time series solutions of (10) for small negative and positive detuning. (Adapted from [23]).

Fig. 11 Schematic diagram of coupling for a $2N$-cell network that admits heteroclinic ratchets. Upper cells are all-to-all coupled between themselves and each upper cell receives an extra input from the cell below itself. A lower cell receives one input from each upper cell.



As shown in [23] for the system (10), oscillators 1 and 3 (2 and 4) loose frequency synchronization when $\Delta_{13} > 0$ ($\Delta_{24} > 0$) (see Figure 10). It is also noted in [23] that a $2N$-cell coupled oscillator system as in Figure 11 can admit heteroclinic ratchets ratcheting in $\theta_k - \theta_{k+N}$, $k = 1, \dots, N$ directions. A solution of such coupled systems for $2N = 6$ is illustrated in Figure 12.

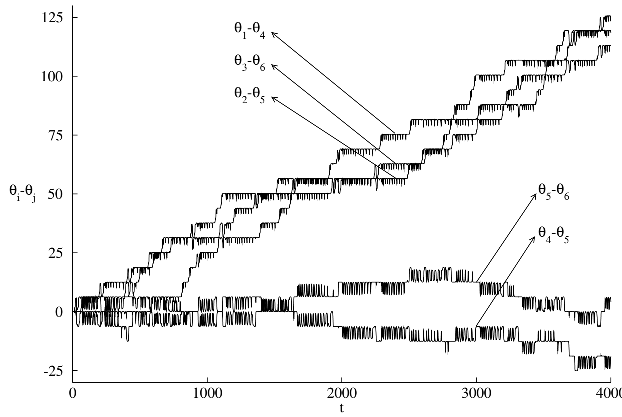


Fig. 12 A solution of coupled 6-oscillator system coupled as in Figure 11 (with $2N = 6$) under small noise with amplitude 10^{-6}. Coupling function is chosen as $f(x) = \sin(x+1.15) + 0.3\sin 2x - 0.1\sin 3x$. One directional phase slips in $\theta_k - \theta_{k+3}$, $k = 1,2,3$ directions suggest the existence of an attracting heteroclinic ratchet on $\mathbb{T}^6$. (Adapted from [23]).

5 Discussion

In summary, we have reviewed some of the basic properties of robust heteroclinic networks that arise in coupled systems of nonlinear oscillators; these can manifest themselves as intermittent switching between various different partially synchronized states. Such dynamics have been observed in various systems including coupled chemical reactors [39, 24] and models of neural activity [32].

These systems provide a rich set of examples of nontrivial dynamical behaviours where the dynamics of individual systems, the topology of the network and the nature of the coupling can give networks within phase space of surprising richness. These heteroclinic networks may wind around the torus which is the natural phase space for coupled oscillator systems to give rise to topologically nontrivial networks (leading to robust extreme sensitivity to detuning) and to with nontrivial unidirectional winding in such networks (leading to heteroclinic ratcheting).

Robust extreme sensitivity is of interest in that it can only appear in globally coupled systems of five or more coupled identical oscillators and in that sense it is a truly high dimensional phenomenon. It remains to be seen to what extent it can be found systems that are not globally coupled. The heteroclinic ratcheting can be understood by a careful analysis of the nonlinear dynamics in phase space. We conjecture they may be of interest as analogues of brownian ratcheting systems in molecular dynamics with a significant difference that they are based on detuning or noise perturbed dissipative systems rather than diffusing systems in a modulated periodic potential [33]. They may also be of use as a possible "circuit elements" or "dynamical motifs" in neural computational systems [40, 36] where we suggest that the presence of such a network can lead to a robust clamping of one oscillator frequency to be above another.

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References

1. D. M. Abrams and S. H. Strogatz. Chimera states for coupled oscillators. *Phys. Rev. Lett.*, 93(17):174102, Oct 2004.
2. J. A. Acebron, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler. The Kuramoto model: A simple paradigm for synchronization phenomena. 77(1):137–185, 2005.
3. M. A. D. Aguiar, P. Ashwin, A. P. S. Dias, and M. Field. Robust heteroclinic cycles in coupled cell systems: identical cells with asymmetric inputs. *preprint*, 2009.
4. P. Ashwin and J. Borresen. Encoding via conjugate symmetries of slow oscillations for globally coupled oscillators. *Phys. Rev. E*, 70(2):026203, 2004.
5. P. Ashwin, O. Burylko, and Y. Maistrenko. Bifurcation to heteroclinic cycles and sensitivity in three and four coupled phase oscillators. *Physica D*, 237:454–466, 2008.
6. P. Ashwin, O. Burylko, Y. Maistrenko, and O. Popovych. Extreme sensitivity to detuning for globally coupled phase oscillators. *Phys. Rev. Lett.*, 96(5):054102, 2006.
7. P. Ashwin, G. Orosz, J. Wordsworth, and S. Townley. Dynamics on networks of clustered states for globally coupled phase oscillators. *SIAM J. Appl. Dyn. Sys.*, 6(4):728–758, 2007.
8. P. Ashwin, A. M. Rucklidge, and R. Sturman. Cycling chaotic attractors in two models for dynamics with invariant subspaces. *Chaos*, 14(3):571–582, 2004.
9. P. Ashwin and J. W. Swift. The dynamics of n weakly coupled identical oscillators. *J. Nonlinear Sci.*, 2(1):69–108, 1992.
10. S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou. The synchronization of chaotic systems. *Phys. Rep.*, 366:1–101, 2002.
11. H. Broer, K. Efstathiou, and E. Subramanian. Heteroclinic cycles between unstable attractors. *Nonlinearity*, 21(6):1385–1410, 2008.
12. F. H. Busse and R. M. Clever. Nonstationary convection in a rotating system. In U. Müller, K.G. Roesner, and B. Schmidt, editors, *Recent Developments in Theoretical and Experimental Fluid Dynamics*, pages 376–385. Springer Verlag, Berlin, 1979.
13. M. J. Field. *Lectures on bifurcations, dynamics and symmetry*, volume 356 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, 1996.
14. M. J. Field. *Dynamics and symmetry*, volume 3 of *ICP Advanced Texts in Mathematics*. Imperial College Press, London, 2007.
15. M Golubitsky, K Josic, and E Shea-Brown. Winding numbers and average frequencies in phase oscillator networks. *J. Nonlinear Sci.*, 16(3):201–231, 2006.
16. M. Golubitsky and I. Stewart. *The Symmetry Perspective*. Birkhäuser Verlag, Basel, 2002.

17. M. Golubitsky and I. Stewart. Nonlinear dynamics of networks: the groupoid formalism. *Bull. Amer. Math. Soc. (N.S.)*, 43(3):305–364 (electronic), 2006.
18. J. Guckenheimer and P. Holmes. Structurally stable heteroclinic cycles. *Math. Proc. Camb. Phil. Soc.*, 103:189–192, 1988.
19. D. Hansel, G. Mato, and C. Meunier. Clustering and slow switching in globally coupled phase oscillators. *Phys. Rev. E*, 48(5):3470–3477, 1993.
20. J. Hofbauer and K. Sigmund. *Evolutionary Games and Population Dynamics*. Cambridge University Press, Cambridge, 1998.
21. F. C. Hoppensteadt and E. M. Izhikevich. *Weakly connected neural networks*, volume 126 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1997.
22. O. Karabacak. *(in preparation)*. PhD thesis, University of Exeter, 2010.
23. O. Karabacak and P. Ashwin. Heteroclinic ratchets in networks of coupled oscillators. *J. Nonlinear Sci.*, pages DOI 10.1007/s00332–009–9053–2, 2009.
24. I. Z. Kiss, C. G. Rusin, H. Kori, and J. L. Hudson. Engineering complex dynamical structures: sequential patterns and desynchronization. *Science*, 316:1886–1889, 2007.
25. H. Kori and Y. Kuramoto. Slow switching in globally coupled oscillators: robustness and occurence through delayed coupling. *Phys. Rev. E*, 63(046214), 2001.
26. H. Kori and Y. Kuramoto. Slow switching in a population of delayed pulse-coupled oscillators. *Phys. Rev. E*, 68(021919), 2003.
27. M. Krupa. Robust heteroclinic cycles. *J. Nonlinear Sci.*, 7(2):129–176, 1997.
28. Y. Kuramoto. *Chemical Oscillations, Waves and Turbulence*. Springer-Verlag, Berlin, 1984.
29. A. S. Kuznetsov and J. Kurths. Stable heteroclinic cycles for ensembles of chaotic oscillators. *Phys. Rev. E (3)*, 66(2):026201, 4, 2002.
30. E. Mosekilde, Y. Maistrenko, and D. Postnov. *Chaotic synchronization: applications to living systems*. World Scientific Publishing Co. Inc., River Edge, NJ, 2002.
31. A. Pikovsky, M. Rosenblum, and J. Kurths. *Synchronization, a universal concept in nonlinear sciences*. Cambridge University Press, Cambridge, 2001.
32. M. I. Rabinovich, R. Huerta, P. Varona, and V. S. Afraimovich. Generation and reshaping of sequences in neural systems. *Bio. Cyb.*, 95:519–536, 2006.
33. P. Reimann. Brownian motors: noisy transport far from equilibrium. *Phys. Rep.-Rev. Sec. Phys. Lett.*, 361(2-4):57–265, 2002.
34. G. L. dos Reis. Structural Stability of Equivariant Vector-Fields on 2-Manifolds. *Trans. Am. Math. Soc.*, 283(2):633–643, 1984.
35. H. Sakaguchi and Y. Kuramoto. A soluble active rotator model showing phase transitions via mutual entrainment. *Prog. Theor. Phys.*, 76(3):576–581, 1986.
36. O. Sporns and R. Kötter. Motifs in brain networks. *PLoS Biol*, 2(11):1910–1918, 2004.
37. S. H. Strogatz. From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators. *Physica D*, 143:1–20, 2000.
38. M. Timme. (pers. comm.). 2009.
39. Y. M. Zhai, I. Z. Kiss, H. Daido, and J. L. Hudson. Extracting order parameters from global measurements with application to coupled electrochemical oscillators. *Physica D*, 205:57–69, 2005.
40. P. Z. Zhigulin. Dynamical motifs: building blocks of complex dynamics in sparsely connected random networks. *Phys. Rev. Lett.*, 92(23):238701, 2004.