Abstract

We consider the dynamics of semiflows of patterns on unbounded domains that are equivariant under a noncompact group action. We exploit the unbounded nature of the domain in a setting where there is a strong ‘global/uniform’ norm and a weak ‘local/weighted’ norm. Relative equilibria whose group orbits are closed manifolds for a compact group action need not be closed in a noncompact setting; the closure of a group orbit of a solution can contain ‘co-solutions’.

The main result of the paper is to show that co-solutions inherit stability in the sense that co-solutions of a Lyapunov stable pattern are also stable. This means that the existence of a single group orbit of stable relative equilibria may force the existence of quite distinct group orbits of relative equilibria, and these are also stable. This is in contrast to the case for finite dimensional dynamical systems where group orbits of relative equilibria are typically isolated.

1 Introduction

Dynamical systems on unbounded domains has progressed mostly by restricting to specific classes of pattern-forming systems, such as parabolic partial differential equations (notably the Ginzburg-Landau and Swift-Hohenberg equations and their generalizations; see e.g. [1, 7, 14]) to model generic instabilities with nontrivial spatial dependence. This has
been very successful in characterising solutions of specific types in specific systems of equations; for example wave-like solutions, fronts between them [5, 18], defects [19], spirals etc. Complementary to this approach there have been attempts to find a ‘qualitative theory’ of partial differential equations (see for example [6]) where reductions to ODE models can explain many universal features of patterns in unbounded systems.

In attempting any analysis of dynamics on unbounded domains, there is always the problem of deciding which function space/topology/norm is appropriate. Choice of norm can lead to qualitatively different behaviour, and unbounded domains naturally lead to consideration of at least two norms [4, 14, 15]. For example, consider a solution $u(x, t)$ to the heat equation $u_t = u_{xx}$ with initial condition $u_0(x) \geq 0$ satisfying $\int |u_0(x)| \, dx < \infty$. Then $u(x, t)$ decays to zero in sufficiently ‘local’ norms; for example a weighted $L^1$ norm $\|u\| = \int \rho(x) |u(x, t)| \, dx$ with weight $\rho > 0$ such that $\rho(x) \rightarrow 0$ as $|x| \rightarrow \infty$. By contrast, in a more global or uniform norm such as the $L^1$ norm, the solution remains bounded away from zero.

If there is a stable propagating front between states $A$ and $B$ it seems reasonable to ask whether $A$ and $B$ inherit the stability of the front. In this paper, we consider such a problem for the stability of ‘far field’ patterns. Our results give a rigorous version of statements such as “for spirals to be stable the [...] asymptotic plane wave has to be absolutely stable” in reference to the stability of spirals in the 2D Complex Ginzburg-Landau equation [1, p117]. More precisely, we consider relative equilibria $u_0$ for a semiflow satisfying some assumptions on local and uniform topologies (see Section 2). We focus on systems where the semiflow commutes with the action of a noncompact group $\Gamma$ such that we can characterise the unboundedness by group symmetries. Section 3 shows that under quite general assumptions on the semiflow, solutions often force co-solutions. These are patterns that are in the closure of the group orbit in the weak/local topology. For example, a rigidly rotating spiral wave will have co-solutions that are roll solutions given by the far-field behaviour of the spiral. Proposition 3.4 gives a precise statement of the following:

**Proposition (Co-solutions)** Suppose that $u_0$ is a relative equilibrium and $v_0$ is a relative equilibrium that is a co-solution of $u_0$ (i.e. $v_0$ is in the closure of the group orbit of $u_0$, relative to the weak topology), then $v_0$ is also a relative equilibrium.

The main results in Section 4 show that stability of a relative equilibrium implies stability of its co-solutions. Theorem 4.2 gives a precise statement of the following:

**Theorem (Inheritance of stability)** Suppose that $u_0$ is a relative equilibrium and $v_0$ is a relative equilibrium that is a co-solution of $u_0$. If $u_0$ is stable then $v_0$ is also stable.

The stability is qualitative and nonlinear in the sense of Lyapunov stability with respect to certain norms; it is not a statement about spectral stability and in this sense it is a strong result. On the other hand the perturbations we consider do not allow perturbations to the
wavenumber that can be considered in spectral stability and in this sense it is weaker. We emphasise that we are not proposing this result as a way of practically proving stability of roll solutions from stability of spirals. The latter stability will always be much harder to determine than stability of roll solutions. The power of our results is rather the converse; for example a stable spiral must have a stable far-field pattern, so that proving instability (or nonexistence) of the far-field solution enables us to deduce instability (or nonexistence) of the spiral solution with that far-field. There are general statements to this effect as noted above, but we are not aware of any proofs except for spectral stability results such as [17, 19].

Another important problem is to understand convective and absolute instabilities not just in terms of spectral linearised stability but also in terms of nonlinear stability. Section 5 discusses this and other questions.

2 Semiflows with noncompact domain symmetries

We consider the behaviour of patterns that evolve on an infinite domain for a semiflow equivariant under a group $\Gamma$ acting on this domain.

For definiteness and motivation, we first consider the nonlinear heat equation on $\mathbb{R}^n$

$$\frac{\partial u}{\partial t} = \Delta u + N(u), \tag{1}$$

where $N$ is a polynomial in $u : \mathbb{R}^n \rightarrow \mathbb{R}$ (eg. $N(u) = au + bu^3$). The symmetry group for this equation is the Euclidean group $\Gamma = E(n)$ consisting of isometries $\gamma = (A, a)$, $A \in O(n)$, $a \in \mathbb{R}^n$, acting on $x \in \mathbb{R}^n$ as $\gamma x = Ax + a$. Equation (1) is $E(n)$-equivariant under the action $(\gamma u)(x) = u(\gamma^{-1}x) = u(A^{-1}(x - a))$.

Let $\mathcal{B}$ denote the vector space of bounded uniformly continuous functions $u : \mathbb{R}^n \rightarrow \mathbb{R}$ which is a Banach space for the sup-norm $\|u\|_s = \|u\|_\infty = \sup_{x \in \mathbb{R}^n} |u(x)|$. From now on, we refer to this as the strong norm. It is well-known [11] that the evolution equation (1) defines a semiflow $(u, t) \mapsto \Phi_t(u)$ on $(\mathcal{B}, \|\cdot\|_s)$ for $t \geq 0$, satisfying

$$\Phi_t \circ \Phi_s = \Phi_{s+t}, \quad \Phi_0 = \text{Id}.$$ 

The $E(n)$-equivariance of (1) implies $E(n)$-equivariance of the semiflow:

$$\Phi_t(\gamma u) = \gamma \Phi_t(u) \quad \text{for all } \gamma \in E(n), \ u \in \mathcal{B}, \ t \geq 0.$$

**Proposition 2.1** Let $u \in \mathcal{B}$, $\gamma \in E(n)$. Then $\|\gamma u\|_s = \|u\|_s$.

**Proof** This is immediate. 

In Subsection 2.1, we introduce a weak norm $\|\cdot\|_w$ and a family of weights which interpolate between the strong and weak norms. The basic properties of these norms that we require in the remaining sections are also derived in Subsection 2.1.

In Subsection 2.2, we describe axioms that ensure the main results of this paper go through, clarifying the necessary properties obtained for the specific equation (1). In Subsection 2.3, we indicate how the framework applies to more general classes of PDEs.
2.1 The weak topology

We continue to let \((B, \|\|_s)\) denote the Banach space of bounded uniformly continuous functions \(u : \mathbb{R}^n \to \mathbb{R}\) with the sup-norm. Define the weak norm \(\|u\|_w = \|\rho u\|_s\) where \(\rho(x) = (1 + |x|^2)^{-1}\) and \(|x|^2 = x_1^2 + \cdots + x_n^2\). Note that \((B, \|\|_w)\) is a normed vector space but not a Banach space. Nevertheless, the semiflow defined by (1) on \((B, \|\|_s)\) is also continuous in the weak norm \(\|\|_w\); namely \(\Phi_t : B \to B\) is continuous in the weak norm for each fixed \(t\) [15, Example 2.2, Theorem 2.6(A2)].

**Proposition 2.2**  
(a) For each fixed \(\gamma \in \Gamma\), the linear map \(u \mapsto \gamma u\) is bounded in the weak norm. (So \(\|\gamma u\|_w \leq K\|u\|_w\) for all \(u \in B\), where \(K = \|\gamma\|_w < \infty\) is the operator norm of \(\gamma\).)

(b) The map \(\gamma \mapsto \|\gamma\|_w\) is a continuous map from \(E(n)\) to \(\mathbb{R}\).

(c) \(\|u\|_s = \sup_{x \in \Gamma} \|\gamma u\|_w\).

**Proof**  
For any \(a \in \mathbb{R}^n\), \(\lim_{|x| \to \infty} \rho(x + a)/\rho(x) = 1\) so we can define \(C(a) = \sup_x \rho(x + a)/\rho(x) < \infty\). Writing \(\gamma x = Ax + a = A(x + A^{-1}a)\), we claim that \(\|\gamma\|_w \leq C(A^{-1}a)\). It suffices to show that \(\|\gamma\|_w = 1\) for \(\gamma x = Ax\) and \(\|\gamma\|_w \leq C(a)\) for \(\gamma x = x + a\).

Note that \(\|\gamma u\|_w = \sup_x \rho(x)|u(\gamma^{-1}x)| = \sup_x \rho(x)|u(x)|\). If \(\gamma x = Ax\), then \(\rho(\gamma x) = \rho(x)\) and \(\|\gamma\|_w = 1\). If \(\gamma x = x + a\) is a translation, then it follows from the definition of \(C(a)\) that \(\|\gamma\|_w \leq C(a)\). This completes the proof of part (a).

Let \(\text{Id} = (I, 0)\) denote the identity element of \(\Gamma\). Note that \(C(a) \to 1\) as \(|a| \to 0\). Hence \(\limsup_{\gamma \to \text{Id}} \|\gamma\|_w \leq 1\). For general \(\gamma, \gamma_0 \in \Gamma\),

\[
\|\gamma \gamma_0^{-1}\|_w \leq \|\gamma\|_w \leq \|\gamma\|_w \leq \|\gamma\|_w \leq \|\gamma\|_w,
\]

so that \(\lim_{\gamma \to \gamma_0} \|\gamma\|_w = \|\gamma_0\|_w\) proving (b). Part (c) is clear.

Next, consider the family of weights \(\lambda_\alpha(x) = \rho(\alpha x), 0 \leq \alpha \leq 1\) that can be thought of as interpolating between unweighted \(\alpha = 0\) and weighted \(\alpha = 1\) norm \(\sup_x |\lambda_\alpha(x)u(x)|\).

**Proposition 2.3**  
(a) \(\|u\|_s \leq \|\lambda_\alpha u\|_s \leq \|u\|_s\) for all \(u \in B\), \(\alpha \in (0,1]\).

(b) For all \(M > 0\) and \(\epsilon > 0\), there exists \(\alpha \in (0,1]\) such that

\[
\|u\|_s \leq M \implies \|(1 - \lambda_\alpha)u\|_w < \epsilon.
\]

**Proof**  
Let \(\alpha > 0\) and note that for any \(x\)

\[
|\rho(x)u(x)| \leq |\lambda_\alpha u(x)| \leq |u(x)|
\]

proving (a). To verify (b), suppose that \(\|u\|_s < M\). Then

\[
\|(1 - \lambda_\alpha)u\|_w = \sup_x |\rho(x)(1 - \lambda_\alpha(x))u(x)| \leq P_\alpha M
\]

\(\|\|_w = \sup_{u \in B, u \neq 0} \|\gamma u\|_w/\|u\|_w\).
where

\[ P_\alpha = \sup_r \frac{1}{1 + r^2} \frac{\alpha^2 r^2}{1 + \alpha^2 r^2} \leq \alpha^2 \to 0 \]

as \( \alpha \to 0 \).

We associate to each \( u \in B \) the group orbit \( \Gamma u \in B \). Let \( \text{clos}_w(\Gamma u) \) denote the closure of \( \Gamma u \) with respect to the weak norm \( \| \cdot \|_w \).

**Proposition 2.4** For all \( u \in B \), \( v \in \text{clos}_w(\Gamma u) \), \( \alpha \in (0, 1] \) and \( \epsilon > 0 \), there exists \( \gamma \in \Gamma \) such that

\[ \| \lambda_\alpha(\gamma u - v) \|_s < \epsilon. \]

**Proof** Let \( M = \| u \|_s + \| v \|_s \). Since \( v \in \text{clos}_w(\Gamma u) \), there is a sequence \( \gamma_n \) such that \( \gamma_n u - v \to_w 0 \). In particular, \( \gamma_n u - v \to 0 \) pointwise. Moreover, \( \sup_x |\gamma_n u(x) - v(x)| \leq M \) and so \( \lambda_\alpha(\gamma_n u - v) \to 0 \) uniformly for each fixed \( \alpha \).

2.2 Hypotheses on the weak and strong topologies

The results in this paper go through in the following more general context. Let \( B \) be a vector space with norms \( \| \cdot \|_s \) and \( \| \cdot \|_w \). Suppose that \((u, t) \mapsto \Phi_t \) defines a semiflow on \( B \) (in particular \( B \) is forward invariant for \( \Phi_t \)) and \( B \) is invariant under action of the noncompact group \( \Gamma \). Suppose also that \( \Phi_t : B \to B \) is continuous with respect to the weak norm \( \| \cdot \|_w \) for each fixed \( t \). (Note that we do not require that \( B \) is a Banach space in either of these norms.)

Let \( \Lambda = \{ \lambda \} \) be a family of bounded linear operators on \((B, \| \cdot \|_s)\). (These correspond to pointwise multiplication by the weights \( \lambda_{\alpha} \) in Subsection 2.1.)

We make the following hypotheses:

(H1) \( (a) \) \( \| \gamma u \|_s = \| u \|_s \) for all \( u \in B \) and \( \gamma \in \Gamma \).

\( (b) \) For each fixed \( \gamma \in \Gamma \), the linear map \( u \mapsto \gamma u \) is bounded in the weak norm.

(H2) \( (a) \) \( \| u \|_w \leq \| \lambda u \|_s \leq \| u \|_s \), and

\( (b) \) For all \( M > 0 \) and \( \epsilon > 0 \), there exists \( \lambda \in \Lambda \) such that

\[ \| u \|_s \leq M \text{ implies that } \| (1 - \lambda)u \|_w < \epsilon. \]

\( (c) \) For all \( u \in B \), \( v \in \text{clos}_w(\Gamma u) \), \( \lambda \in \Lambda \) and \( \epsilon > 0 \), there exists \( \gamma \in \Gamma \) such that

\[ \| \lambda(\gamma u - v) \|_s < \epsilon. \]

(H3) \( (a) \) The map \( \gamma \to \| \gamma \|_w \) is continuous.

\( (b) \) For any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( u \in B \)

\[ \sup_{\gamma \in \Gamma} \| \gamma u \|_w < \delta \text{ implies that } \| u \|_w < \epsilon. \]
It follows from Propositions 2.1, 2.2, 2.3 and 2.4 that hypotheses (H1-H3) are satisfied for the specific equation (1).

2.3 Generalisations

The set up in Section 2.2 extends to systems of reaction-diffusion equations in the obvious way, and can be adapted equally to apply to equations with higher order derivatives such as the Swift-Hohenberg equation $u_t = -(\Delta + 1)^2 u + N(u)$. In these examples, the action of $E(n)$ is still given by the “scalar” action $(\gamma u)(x) = u(\gamma^{-1}x)$.

More generally, let $\chi : O(n) \to O(s)$ be an orthogonal action of $O(n)$ on $\mathbb{R}^s$. Then we allow actions of $\gamma = (A, a)$ of the form $(\gamma u)(x) = (\chi A u)(\gamma^{-1}x)$. Such actions are called “physical” in [13] and include the actions that arise say for Navier-Stokes. Since the weights $\lambda_\alpha$ are invariant under the action of $\chi A$, the results in Subsection 2.1 go through unchanged.

Remark 2.5 Babin & Vishik [4] work with $B = L^2$, $\|u\|_s = \|u\|_2 = (\int_{\mathbb{R}^n} |u(x)|^2 \, dx)^{1/2}$, and $\|u\|_w = \|\rho^{1/2}u\|_s$, where $\rho$ is as in Subsection 2.1. In this situation, hypotheses (H1) and (H2) are valid, and so our main result, Theorem 4.2 holds, but the proof of the refinement in Theorem 4.4 breaks down.

3 Co-solutions and relative equilibria

In this section, we continue to assume that $B$ is a normed vector space with norms $\| \|_s$ and $\| \|_w$ and that $\Phi_t$ is a locally defined semiflow on $B$ such that $\Phi_t : B \to B$ is continuous in the weak norm $\| \|_w$ for each fixed $t$. We suppose also that $\Gamma$ is a Lie group acting on $B$ and that $\Phi_t$ is $\Gamma$-equivariant.

We say $u \in B$ has symmetry or isotropy $\Sigma_u = \{ \gamma \in \Gamma : \gamma u = u \}$ (see for example [10]). A subgroup $\Sigma$ of $\Gamma$ is cocompact if the coset space $\Gamma/\Sigma$ is compact.

We say $u_0 \in B$ is a relative equilibrium if $\Phi_t(u_0) = e^{\eta t}u_0$ for some $\eta \in L\Gamma$ which is called the drift of $u_0$. Note that for any $\gamma \in \Gamma$, $\gamma u_0$ is also a relative equilibrium with drift $\eta_\gamma$, where $\eta_\gamma$ is given by the adjoint action of $\gamma$ on $\eta$ (i.e. $e^{\eta_\gamma t} = \gamma e^{\eta t} \gamma^{-1}$ for all $t > 0$.)

Remark 3.1 Associated to the relative equilibrium $u(t) = e^{\eta t}u_0$ is the closed Lie subgroup $K(\eta) = \{ \exp \eta t : t \in \mathbb{R} \}$.

Recall that for $\Gamma$ compact and any $\eta$ the group $K(\eta)$ is a torus and for generic $\eta$ this torus is maximal [8, 12]. For $\Gamma$ non-compact, $K(\eta)$ is either isomorphic to $\mathbb{R}$ or to a torus; generically $K(\eta)$ is either a maximal torus or $\mathbb{R}$, see [3]. For $E(n)$, generically $K(\eta)$ is a torus for $n$ even, and generically $K(\eta) \cong \mathbb{R}$ for $n$ odd.
For continuous action of compact groups, relative equilibria are compact and hence closed. As noted in [3], this is not true for noncompact groups unless one makes further assumptions. Generally speaking, in the situations of interest in this paper, the relative equilibria are closed in the strong topology but not in the weak topology.

**Definition 3.2** Let \( u_0, v_0 \in \mathcal{B} \). We say that \( v_0 \) is a co-solution of \( u_0 \) if \( v_0 \in \text{clos}_w(\Gamma u_0) \).

If \( u_0, v_0 \in \mathcal{B} \) and \( u(t) = \Phi_t(u_0), \ v(t) = \Phi_t(v_0) \) are the corresponding solutions, then we say that \( v(t) \) is a co-solution of \( u(t) \) if \( v_0 \) is a co-solution of \( u_0 \). It follows from weak-continuity and \( \Gamma \) -equivariance of \( \Phi_t \) that the property \( v(t) \in \text{clos}_w(\Gamma u(t)) \) holds for one value of \( t \) if and only if it holds for all \( t \). Hence the set of co-solutions of any given solution is also an invariant set.

**Remark 3.3** (a) Note that \( v_0 \) being a co-solution for \( u_0 \) means that one can find arbitrarily large patches of \( u_0 \) that resemble \( v_0 \) arbitrarily closely, up to transformation by elements of \( \Gamma \).

(b) Our definition of co-solution is in terms of the weak topology. We can also define a strong notion of co-solution. However, in many situations of interest, the notion is vacuous. Indeed, suppose that \( u_0 \) is a relative equilibrium with isotropy \( \Sigma \). Following [3, Definition 5.2], we say that a sequence \( \{\gamma_n\} \subset \Gamma / \Sigma \) is an approximate symmetry of \( u_0 \) if \( \gamma_n \) has no convergent subsequences and \( \|\gamma_n u_0 - u_0\|_w \to 0 \). If no such approximate symmetries exist, then in the strong topology \( \Gamma u_0 \) is a closed submanifold diffeomorphic to \( \Gamma / \Sigma \) (see [3, Proposition 5.3]).

(c) Arguing as in (b), we note that relative equilibria with cocompact isotropy subgroup are compact and hence closed in both the strong and weak topologies. In particular, such relative equilibria cannot have nontrivial co-solutions.

It is clear that co-solutions of equilibria are themselves equilibria. In certain cases, co-solutions of relative equilibria are also relative equilibria.

**Proposition 3.4** Assume hypothesis (H1)(b). Suppose \( u_0 \) is a relative equilibrium with drift \( \xi \). If there is a sequence \( \gamma_n \in \Gamma \), an \( \eta \in L\Gamma \) and a \( v_0 \in \mathcal{B} \) such that \( \|\gamma_n u_0 - v_0\|_w \to 0 \) and

\[
\|(e^{e^{\xi_n t}} - e^{e^{\eta t}})\gamma_n u_0\|_w \to 0
\]

as \( n \to \infty \) (where \( \xi_n = \xi_{\gamma_n} \) is the drift of \( \gamma_n u_0 \)) for all \( t > 0 \), then \( v_0 \) is a relative equilibrium with drift \( \eta \).

**Proof** We set \( u_n = \gamma_n u_0 \) and calculate

\[
\|e^{e^{\eta t}} v_0 - \Phi_t(v_0)\|_w = \|e^{e^{\eta t}} v_0 - e^{e^{\eta t}} u_n + e^{e^{\eta t}} u_n - e^{e^{\xi_n t}} u_n + e^{e^{\xi_n t}} u_n - \Phi_t(v_0)\|_w \\
\leq \|e^{e^{\eta t}} v_0 - e^{e^{\eta t}} u_n\|_w + \|e^{e^{\eta t}} u_n - e^{e^{\xi_n t}} u_n\|_w + \|\Phi_t(u_n) - \Phi_t(v_0)\|_w.
\]

We fix \( t \) and take the limit as \( n \to \infty \). The first term goes to zero by (H1)(b). The second term goes to zero by the hypothesis. The third term goes to zero by continuity of the semiflow. Hence \( \Phi_t(v_0) = e^{e^{\eta t}} v_0 \) meaning that \( v_0 \) is a relative equilibrium with drift \( \eta \).
Figure 1: (a) A relative equilibrium for a Euclidean-equivariant system with two co-solutions that are relative equilibria; a uniform state (by taking translations to the right) and a family of spatially periodic states (by taking translations to the left). (b) A relative equilibrium that is a defect of this form has stripe co-solutions that are relative equilibria on taking translations in any direction. In both cases the co-solutions have additional symmetries.

Remark 3.5 (a) A special case where Proposition 3.4 applies is when $v_0 \in \text{clos}_w(\Gamma u_0)$ and $v$ has drift $\xi$ in the centre of $\Gamma$. In such a case $\xi_n = \xi$ and we can choose $\eta = \xi$ to satisfy the hypotheses.

(b) Another special case satisfying these hypotheses is where $v$ has full symmetry, in which case $\eta = 0$.

3.1 Examples of co-solutions

To motivate the results we give a few examples of patterns that have a nontrivial set of co-solutions, building on ideas in [2]. Figure 1 shows two examples. Figure 1(a) shows a front between a spatially periodic pattern for $x > 0$ and a uniform state for $x < 0$. If this pattern is a relative equilibrium for a semiflow that fits our setting then there are two families of co-solutions; the uniform pattern for $x \to -\infty$ and the periodic pattern for $x \to +\infty$. Figure 1(b) shows a defect solution that implies the existence of stripe solutions as well. Note that in both cases all co-solutions have cocompact symmetry, so the co-solutions have no further co-solutions.

Another example is illustrated in Figure 2; this shows one component of reaction diffusion system with a spiral relative equilibrium that rotate anticlockwise. By taking limits of large translations in any direction we obtain weak co-solutions that are propagating spatially
Figure 2: A spiral relative equilibrium for a Euclidean-equivariant system with co-solutions given by rolls that drift if the spiral rotates; these are be found by considering translates of the spiral in any direction.

periodic stripe patterns.

3.2 Symmetries of co-solutions

In spite of the fact that co-solutions are generated by symmetries of the system there is not a simple relationship between the symmetries of a relative equilibrium $v$ and the symmetries of a co-solution. Figures 1 and 2 show cases where the co-solutions have more symmetry than the original pattern. By contrast Figure 3 has a reflection symmetry in the vertical axis that is missing on the cosolutions obtained by translating to the left or right. Hence symmetry may be gained or lost in passing from a relative equilibrium to a co-solution.

4 Inheritance of stability

Let $u_0 \in \mathcal{B}$. We say that $u_0$ is (Lyapunov) \textit{sw-stable} if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$
\|u - u_0\|_s < \delta \text{ implies that } \|\Phi_t(u) - \Phi_t(u_0)\|_w < \epsilon \text{ for all } t > 0.
$$

(2)

Similarly, we say $u_0$ is \textit{ss-stable} if (2) holds with $\|.,\|_w$ replaced by $\|.,\|_s$. This corresponds to the usual notion of stability. Observe that ss-stability clearly implies sw-stability.

In a similar way one could define $ww$-stability but this is probably too weak to be of use and so we do not discuss it further here.
Proposition 4.1 Assume hypothesis (H1). If $u_0 \in B$ is sw-stable (resp. ss-stable) then so is $\gamma u_0$ for all $\gamma \in \Gamma$.

Proof Suppose that $u_0$ is sw-stable and $\gamma \in \Gamma$. By (H1)(b), $K = \|\gamma\|_w < \infty$. For any $\epsilon > 0$, there exists $\delta > 0$ such that $\|u - u_0\|_s < \delta$ implies that $\|\Phi_t(u) - \Phi_t(u_0)\|_w < \epsilon/K$ for all $t > 0$.

We show that $v_0 = \gamma u_0$ is sw-stable. Suppose that $\|v - v_0\|_s < \delta$. By (H1)(a), $\|\gamma^{-1}v - u_0\|_s < \delta$. Hence $\|\Phi_t(\gamma^{-1}v) - \Phi_t(u_0)\|_w < \epsilon/K$. By (H1)(b) and equivariance of the semiflow,

$$\|\Phi_t(v) - \Phi_t(v_0)\|_w \leq K\|\gamma^{-1}(\Phi_t(v) - \Phi_t(v_0))\|_w = K\|\Phi_t(\gamma^{-1}v) - \Phi_t(u_0)\| < K\epsilon/K = \epsilon,$$

proving that $v_0$ is sw-stable.

The proof that ss-stability of $u_0$ is inherited by $v_0$ is simpler (with $K = 1$) by (H1)(a).

Theorem 4.2 Assume hypotheses (H1) and (H2). Suppose that $u_0, v_0 \in B$ and $v_0 \in \text{clos}_w(\Gamma u_0)$. If $u_0$ is ss-stable then $v_0$ is sw-stable.

Proof We prove the statement by contradiction, assuming that $v_0$ is sw-unstable and arguing that $u_0$ must be ss-unstable.

Since $v_0$ is sw-unstable, there is an $\epsilon > 0$ such that for all $\delta > 0$ we can find a $T > 0$ and $v_1$ (both depending on $\delta$) such that

$$\|v_1 - v_0\|_s < \delta,$$  \hspace{1cm} (3)
but
\[ \| \Phi_T(v_1) - \Phi_T(v_0) \|_w \geq \epsilon. \]  
(4)

By weak continuity of \( \Phi_t \), there exists \( \eta \in (0, \delta) \) such that
\[ \| \Phi_T(z) - \Phi_T(v_0) \|_w < \frac{\epsilon}{3} \quad \text{for all } z \in B \text{ with } \| z - v_0 \|_w < \eta. \]  
(5)
\[ \| \Phi_T(z) - \Phi_T(v_1) \|_w < \frac{\epsilon}{3} \quad \text{for all } z \in B \text{ with } \| z - v_1 \|_w < \eta. \]  
(6)

Set \( M = \| u_0 \|_s + \| v_1 \|_s \). Then \( \| \gamma u_0 - v_1 \|_s \leq M \) for all \( \gamma \in \Gamma \) by (H1)(a), and hence by (H2)(b) there exists \( \lambda_0 \in \Lambda \) such that
\[ \| (1 - \lambda_0)(\gamma u_0 - v_1) \|_w < \eta \quad \text{for all } \gamma \in \Gamma. \]  
(7)

Since \( v_0 \in \text{clos}_w(u_0) \), it follows from (H2)(c) that there exists \( \gamma_0 \in \Gamma \) such that
\[ \| \lambda_0(\gamma_0 u_0 - v_0) \|_s < \eta < \delta. \]  
(8)

By hypothesis (H2)(a) and estimate (8),
\[ \| \gamma_0 u_0 - v_0 \|_w \leq \| \lambda_0(\gamma_0 u_0 - v_0) \|_s < \eta, \]
so it follows from (5) that
\[ \| \Phi_T(\gamma_0 u_0) - \Phi_T(v_0) \|_w < \frac{\epsilon}{3}. \]  
(9)

Now define
\[ u_1 = (1 - \lambda_0)\gamma_0 u_0 + \lambda_0 v_1. \]
Then \( u_1 - \gamma_0 u_0 = \lambda_0(v_1 - \gamma_0 u_0) \) and we compute that
\[ \| \gamma_0^{-1} u_1 - u_0 \|_s = \| u_1 - \gamma_0 u_0 \|_s = \| \lambda_0(v_1 - \gamma_0 u_0) \|_s \leq \| \lambda_0(v_1 - v_0) \|_s + \| \lambda_0(\gamma_0 u_0 - v_0) \|_s \]
\[ \leq \| v_1 - v_0 \|_s + \| \lambda_0(\gamma_0 u_0 - v_0) \|_s < \delta + \delta = 2\delta, \]
where we have used hypotheses (H1)(a) and (H2)(a), and estimates (3) and (8).
Moreover, \( u_1 - v_1 = (1 - \lambda_0)(\gamma_0 u_0 - v_1) \) so \( \| u_1 - v_1 \|_w < \eta \) by (7). It follows from (6) that
\[ \| \Phi_T(u_1) - \Phi_T(v_1) \|_w < \frac{\epsilon}{3}. \]  
(10)

Writing
\[ \Phi_T(u_1) - \Phi_T(\gamma_0 u_0) = [\Phi_T(u_1) - \Phi_T(v_1)] + [\Phi_T(v_1) - \Phi_T(v_0)] + [\Phi_T(v_0) - \Phi_T(\gamma_0 u_0)] \]
we have
\[ \| \Phi_T(\gamma_0^{-1} u_1) - \Phi_T(u_0) \|_s = \| \Phi_T(u_1) - \Phi_T(\gamma_0 u_0) \|_s \]
\[ \geq \| \Phi_T(v_1) - \Phi_T(v_0) \|_w - \| \Phi_T(u_1) - \Phi_T(v_1) \|_w - \| \Phi_T(\gamma_0 u_0) - \Phi_T(v_0) \|_w \]
\[ \geq \epsilon - \frac{\epsilon}{3} - \frac{\epsilon}{3} = \frac{\epsilon}{3}. \]
where we have used hypothesis (H1)(a), $\Gamma$-equivariance of $\Phi_T$, and estimates (6), (9) and (10).

Summarizing, we have shown that there is an $\epsilon > 0$ such that for all $\delta > 0$ there is a $T > 0$ and a $w = \gamma_0^{-1}u_1$ such that

$$
\|w - u_0\|_s < 2\delta \quad \text{and} \quad \|\Phi_T(w) - \Phi_T(u_0)\|_s \geq \frac{\epsilon}{3}
$$
giving ss-instability of $u_0$ and the proof is complete.

In certain situations we obtain a more powerful result:

**Lemma 4.3** Assume hypotheses (H1) and (H3) and suppose that $u_0 \in \mathcal{B}$ has cocompact isotropy $\Sigma$. Then $u_0$ is ss-stable if and only if $u_0$ is sw-stable.

**Proof** We prove the nontrivial direction, namely that sw-stability implies ss-stability. Let $\epsilon > 0$ and choose $\delta > 0$ as in (H3)(b). By Proposition 4.1, $\gamma u_0$ is sw-stable for all $\gamma \in \Gamma$. Hence, for each $\gamma$, there exists $\eta = \eta(\gamma) > 0$ such that $\|u - \gamma u_0\|_s < \eta$ implies that $\|\Phi_t(u) - \Phi_t(\gamma u_0)\|_w < \delta$ for all $t > 0$. By the proof of Proposition 4.1, we can take $\eta(\gamma) = \eta(1_\Gamma)/\|\gamma\|_w$ where $1_\Gamma$ is the identity element in $\Gamma$. By (H3)(a), $\eta(\gamma)$ depends continuously on $\gamma$. Clearly $\eta(\gamma)$ can be chosen to be constant on $\Sigma$-cosets. Since $\Gamma/\Sigma$ is compact, it follows that $\eta > 0$ can be chosen independent of $\gamma$.

Suppose that $\|v - u_0\|_s < \eta$ and let $\gamma \in \Gamma$. By hypothesis (H1)(a), $\|\gamma v - \gamma u_0\|_s < \eta$, so by the above argument with $u = \gamma v$ we have $\|\Phi_t(\gamma v) - \Phi_t(\gamma u_0)\|_w < \delta$ for all $t > 0$. By equivariance, $\|\gamma(\Phi_t(v) - \Phi_t(u_0))\|_w < \delta$ for all $t > 0$ and all $\gamma \in \Gamma$. By (H3)(b), we deduce that $\|\Phi_t(v) - \Phi_t(u_0)\|_s < \epsilon$ for all $t > 0$ and so $u_0$ is ss-stable.

Combining Theorem 4.2 and Lemma 4.3 we have:

**Theorem 4.4** Assume hypotheses (H1-H3). Suppose that $u_0, v_0 \in \mathcal{B}$ and $v_0 \in \text{clos}_w(\Gamma u_0)$. Suppose further that $v_0$ has cocompact isotropy. If $u_0$ is ss-stable then $v_0$ is also ss-stable.

This means, for example, that the existence of an ss-stable relative equilibrium of the form in Figure 1(a) implies that both the uniform and the stripe co-solutions are ss-stable. Similarly if the spiral solution in Figure 2 is ss-stable then the ‘far field’ roll solutions are ss-stable.

Note that this result does not contradict results of Mielke and co-workers [9, 16] on diffusive mixing of roll solutions; they show in a number of contexts that rolls are only diffusively stable, meaning that the wavenumber of rolls diffuses within the unbounded domain. The perturbations we consider in both ss- and sw-stability are more stringent and will not vary wavenumber. Moreover we only consider Lyapunov stability rather than asymptotic convergence meaning that we permit small perturbations along the group orbit.
5 Discussion

We have focussed on some aspects of the qualitative behaviour of dynamics on unbounded domains. There is clearly a great deal more that might be investigated by making use of assumptions such as (H1-H3); one direction that seems worth pursuing is the generalisation to transients. In particular, an initial condition may converge to relative equilibria in a weak but not in a strong sense, and this may give further predictions for the existence of co-solutions (see for example the spiral wind-up discussed in [2]); we simply note here that the results in Section 4 apply equally for solutions and co-solutions that are not relative equilibria.

The results above are purely ‘topological’ in nature and do not attempt to explain or predict the smooth dynamics. However, the setting may give a clue as to how to obtain results that relate smooth dynamical properties such as spectral stability to topological properties such as ss- and sw-stability and indeed what ingredients a bifurcation theory for such systems should have. In particular one would assume that an sw-stable state can show convective but not absolute instability whereas an ss-stable state can have neither convective nor absolute instability, for sensible definitions of these notions.

It would be interesting to understand asymptotic stability as well as Lyapunov stability in our setting; we observe that, similar to Lyapunov stability, there are several inequivalent notions of asymptotic stability depending on choice of norm (see e.g. [16]).

Finally, we remark that there are situations where change of a parameter means that a semiflow that is continuous in the weak and strong topology becomes continuous in only the strong topology due to the appearance of mean flow effects [20]. At this point, certain of our hypotheses are violated, and it would be interesting to understand how this impacts on the relationship between local and global dynamics. In such cases we expect there not necessarily to be a simple relationship between the stability of a solution and stability of its co-solutions.

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References


