1 Noise-induced switching near a depth two heteroclinic network
and an application to Boussinesq convection

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We investigate the robust heteroclinic dynamics arising in a system of ordinary differential equations
in R4 with symmetry D4 × (Z2)2. This system arises from the normal form reduction of a 1:√2
mode interaction for Boussinesq convection. We investigate the structure of a particular robust
heteroclinic attractor with “depth two connections” from equilibria to subcycles as well as connect-
ions between equilibria. The “subcycle” is not asymptotically stable, due to nearby trajectories
undertaking an “excursion,” but it is a Milnor attractor, meaning that a positive measure set of
nearby initial conditions converges to the subcycle. We investigate the dynamics in the presence of
noise and find a number of interesting properties. We confirm that typical trajectories wind around
the subcycle with very occasional excursions near a depth two connection. The frequency of
excursions depends on noise intensity in a subtle manner; in particular, for anisotropic noise, the
depth two connection may be visited much more often than for isotropic noise, and more generally
the long term statistics of the system depends not only on the noise strength but also on the
anisotropy of the noise. Similar properties are confirmed in simulations of Boussinesq convection
for parameters giving an attractor with depth two connections. © 2010 American Institute of Phys-

It is well known that symmetries can cause rather strange
and uncharacteristic behavior in smooth dynamical sys-
tems. A particularly counterintuitive phenomenon is the
appearance of robust heteroclinic attractors,14 where tra-
jectories of symmetric systems visit a sequence of saddle
points, such that the sequence of residence times in a
given neighborhood of a saddle tends to infinity. In the
presence of noise and/or symmetry breaking perturba-
tions the sequence of residence times attains a finite mean
value after a transient.25 This paper explores a particular
example of a heteroclinic attractor in R4 with a depth two
connection;3 this means that some connections in the attrac-
tor are not between equilibria, but between an equi-
lbrium and a heteroclinic subcycle. For the example, we
find that the “subcycle” is an attractor in its own right; it
attracts most, but not all, nearby trajectories. Some tra-
jectories starting arbitrarily close to the subcycle un-
dergo an “excursion” before returning. The approach to
the attractor can therefore show periods of approach to-
ward to subcycle interrupted by the excursions. The fre-
quency of the excursions per cycle is found to depend on
the details of the added noise.

47 I. INTRODUCTION

Robust heteroclinic cycles can be attractors in the sense
of Milnor, while failing to be asymptotically stable. They
may attract trajectories starting at a positive measure set
nearby in phase space while repelling another, complemen-
tary, set—this phenomenon has been called essential
asymptotic stability.15,17 In addition, robust heteroclinic
cycles may share a common connection to form a network
with competition between different cycles. In Ref. 13 such a
system is studied, where both cycles can be simultaneously
Milnor attractors. One may find heteroclinic networks where
no subcycle is an attractor, but the whole network is attract-
ing and asymptotic behavior involves chaotic switching near
the network.1,10,11,21,22

Numerical and analytical studies in Ref. 20 found an
open region in parameter space for an eight-dimensional
(8D) symmetric system of ordinary differential equations
(ODEs), where the only attractor is a robust depth two het-
eroclinic network involving a variety of saddle equilibria
classified according to their symmetries. In this paper we
consider a considerable simplification of the 8D system of
Ref. 20 to a four-dimensional (4D) subspace, such that the
simplified system retains the essential dynamical properties
of the depth two attractor.

The addition of noise to such attractors can have a dra-
matic effect: typical ω-limit sets of trajectories become much
larger and a simple argument says that typical trajectories
approaching a heteroclinic network will recurrently explore
all unstable manifolds of points on the network. The depen-
dence of trajectory properties on noise amplitude is of inter-
est and has been explored in Refs. 2, 21, and 24, but so far
not for any networks with depth two connections.

In this paper we make three observations concerning the

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FIG. 1. Chawanya’s example of a network with a “child cycle”—a heteroclinic network that is not of depth one. Observe that (a) the unstable manifold of the equilibrium $p_1$ is a connection that accumulates on a simple heteroclinic cycle connecting $(p_1, p_2, p_1, p_2)$, and (b) the unstable manifolds of $p_1$ and $p_2$ are two-dimensional.

switching between competing cycles within a heteroclinic network may become more, rather than less, regular on addition of noise. Second, anisotropic noise may have a significant influence on how often excursions are made from the subcycle. Third, we verify that this behavior appears in simulations of problem in Boussinesq convection.

The paper is organized as follows: we give some definitions and introduce in Sec. I A the particular set of four coupled ODEs that we study, briefly motivating where they arise in a normal form problem. We discuss some dynamics and attractors of this system. Section II studies the dynamics and attractors of this system. Section II discusses implications for more general systems in Sec. V as well as some open questions concerning depth two networks.

A. Heteroclinic networks and depth

Suppose we have an ODE $\dot{x} = f(x)$ for $x \in \mathbb{R}^n$ with $f$ smooth that generates a smooth flow $\Phi_t(x)$. We say a compact invariant chain recurrent set $\Sigma$ is a heteroclinic network of depth one (between equilibria) if the following hold:

1. $\Sigma$ contains a finite set $\{x_i\}$ of hyperbolic equilibria.
2. If $y \in \Sigma \setminus \{x_i\}$ then $y \in W^u(x_i) \cap W^s(x_j)$ for some $i, j \neq i, j$, i.e.,
   \[ \lim_{t \to +\infty} \Phi_t(y) = x_i \quad \text{and} \quad \lim_{t \to -\infty} \Phi_t(y) = x_j. \]

We say a heteroclinic network is a simple heteroclinic cycle if it is topologically a circle. This means that all equilibria have one-dimensional unstable manifolds within $\Sigma$, and $\Sigma$ can, with a suitable reordering, be seen as a union of $n$ equilibria and $n$ connections from one equilibrium to the next. As a result, a simple heteroclinic cycle has depth one. However, this notion of heteroclinic network does not include all possible situations that can appear even if we restrict to robust cases. Figure 1 schematically shows an example of Chawanya’s from a game dynamical system—note the presence of a connection that is not from equilibrium to equilibrium, but rather from equilibrium to subcycle.

As a way to understand the dynamics of more general heteroclinic networks, the concept of depth of a heteroclinic network was introduced by Ref. 3. We say a heteroclinic network of depth two if it is a compact invariant chain recurrent set such that the following hold:

1. $\Sigma$ contains a finite set $\{x_i\}$ of hyperbolic equilibria.
2. $\Sigma = \bigcup_{x \in \Sigma} o(x) \cup o(x)$ is a union of heteroclinic networks of depth one and equilibria.

We assume that $\Sigma \neq \Sigma \neq \Sigma$.

Almost all previously documented heteroclinic networks are of depth one. Nonetheless, in symmetric systems of dimension higher than three, previous work suggests that depth two networks can be robust, and that paper gives an example in nine dimensions; the example of Ref. 7 is in five dimensions, meaning that the example in this paper (in four dimensions) is of lower dimension than either. Finally, we mention that Ref. 8 gives an example flow on a manifold with boundary in three dimensions that has a robust depth two heteroclinic network; however, the latter is presumably not robust for any compact symmetry group action.

B. A model with a depth two heteroclinic network

We consider a system that is a restriction to an $\mathbb{R}^4$ subspace of a $C^4$ normal form for a mode interaction present in a Boussinesq convection problem. We consider the dynamics of the following system of four coupled ODEs on $(x_1, x_2, y_1, y_2) \in \mathbb{R}^4$ with added noise:

\[
\begin{align*}
\dot{x}_1 &= (\lambda_1 + B_{1x} x_1^2 + B_{2x} x_2^2 + B_{3x} y_1 y_2 + B_{4x} y_2^2) x_1 + \eta_1, \\
\dot{x}_2 &= (\lambda_1 + B_{1x} x_1^2 + B_{2x} x_2^2 + B_{3x} y_1 y_2 + B_{4x} y_2^2) x_2 + \eta_2, \\
\dot{y}_1 &= (\lambda_2 + B_{5y} y_1^2 + B_{6y} y_2 y_1 + B_{7y} x_1^2 + B_{8y} x_2^2) y_1 + \eta_3, \\
\dot{y}_2 &= (\lambda_2 + B_{5y} y_1^2 + B_{6y} y_2 y_1 + B_{7y} x_1^2 + B_{8y} x_2^2) y_2 + \eta_4.
\end{align*}
\]

The $\eta_i$ represent independent additive white noise processes. The real quantities $\lambda_i$ and $B_i$ are system parameters and we assume the noise is independent, $\langle \eta_i, \eta_j \rangle = 0$ for $i \neq j$, and has amplitude $\langle \eta_i^2 \rangle = \epsilon_i \delta(t-t')$. We say the noise is isotropic if $\epsilon_i = \epsilon$ independent of $i$, otherwise it is anisotropic. The system (1) is symmetric (equivariant) under the action of the transformations

\[
\begin{align*}
\rho(x_1, x_2, y_1, y_2) &= (x_2, x_1, y_2, y_1), \\
\kappa(x_1, x_2, y_1, y_2) &= (-x_1, x_2, -y_1, y_2), \\
\kappa_1(x_1, x_2, y_1, y_2) &= (x_1, x_2, -y_1, y_2), \\
\kappa_2(x_1, x_2, y_1, y_2) &= (x_1, x_2, y_1, -y_2)
\end{align*}
\]

that generate the group $D_4 \ltimes (\mathbb{Z}_2)^2$, where $D_4 = \langle (\rho, k) \rangle$ and $(\mathbb{Z}_2)^2 = \langle (k_1, k_2) \rangle$ (note that this is not a minimal set of generators as $k_1^2 = \rho^3 k_1 \rho$).

We will concentrate on a depth two network that appears robustly in the dynamics of the system (1). First, we list in Table I the various equilibria that exist in invariant subspaces forced by symmetries of the system; in the last two columns of the table we present eigenvalues determining the stability...
TABLE I. Invariant subspaces and nontrivial equilibria involved in the heteroclinic attractor for Eq. (1), shown with eigenvectors and eigenvalues determining their stability. The same labels as in Ref. 20 are employed except that AR corresponds to AR1 and WR to WR2 in that reference. Typical points are given in terms of the coordinates $(x_1, x_2, y_1, y_2)$. The last column gives the eigenvalues (or just their signs) for the particular parameter values given in Table II.

<table>
<thead>
<tr>
<th>Subspace</th>
<th>Invariant subspace</th>
<th>Eigenvalues, formulae</th>
<th>Eigenvalues, particular values</th>
</tr>
</thead>
<tbody>
<tr>
<td>LR (Large rolls)</td>
<td>$x_1^2 = -\lambda_1 B_1$</td>
<td>$(q, 0, 0, 0)$</td>
<td>$-2\lambda_1$</td>
</tr>
<tr>
<td>SS (Small squares)</td>
<td>$x_1^2 = -\lambda_1 B_2$</td>
<td>$(0, q, 0, 0)$</td>
<td>$\lambda_1 (B_2 - B_3) / B_1$</td>
</tr>
<tr>
<td>LS (Large squares)</td>
<td>$x_2^2 = -\lambda_2 / (B_1 + B_2)$</td>
<td>$(0, q, 0, 0)$</td>
<td>$\lambda_2 + B_2 x_2^2$</td>
</tr>
<tr>
<td>SR (Small rolls)</td>
<td>$x_2^2 = -\lambda_2 / (B_3 + B_2)$</td>
<td>$(0, q, 0, 0)$</td>
<td>$\lambda_2 + B_2 x_2^2$</td>
</tr>
<tr>
<td>AR (Asymmetric rolls)</td>
<td>$x_3^2 = -\lambda_3 (x_1, x_2, x_3)$</td>
<td>$(0, q, 0, 0)$</td>
<td>$\lambda_3 (x_2 B_3 - B_3) / B_3$</td>
</tr>
<tr>
<td>WR (Wavy rolls)</td>
<td>$x_3^2 = -\lambda_3 (x_1, x_2, x_3)$</td>
<td>$(0, q, 0, 0)$</td>
<td>$\lambda_1 + B_2 x_2^2$</td>
</tr>
</tbody>
</table>

II. THE HETEROCLINIC NETWORK

The 4D system (1) with coefficients in Table II contains an attracting heteroclinic network whose structure is shown schematically in Fig. 3. We have verified this (a) by examination of the existence and stabilities of the equilibria from Table I, (b) by verification of robust connections, by examining eigenvalues, by direct (noise-free) simulation within invariant subspaces to ensure transverse intersections and by applying Theorem 1 from Ref. 20, and (c) by direct simulations using initial conditions away from invariant subspaces, both with and without added noise.

For direct simulation without noise the full network is not easily observable: apart from transients, the typical observed $\omega$-limit set is just the subcycle between $\xi_{LR}$ and $\xi_{SS}$. However, on addition of noise, one can observe much more of the network, including the excursions along the depth two cycle.

TABLE II. Parameter values that give the depth two heteroclinic attractor for Eq. (1); we note that $\lambda_1 = \cos \theta$, $\lambda_2 = \sin \theta$ where $\theta = 1.34$ is a parameter used in Ref. 20 to unfold a mode interaction.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>0.228 752</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>0.973 485</td>
</tr>
<tr>
<td>$B_1$</td>
<td>-0.062 000</td>
</tr>
<tr>
<td>$B_2$</td>
<td>-0.115 655</td>
</tr>
<tr>
<td>$B_3$</td>
<td>0.064 731</td>
</tr>
<tr>
<td>$B_4$</td>
<td>-0.072 599</td>
</tr>
</tbody>
</table>

TABLE III. Connections between equilibria listed in Table I. The third column indicates the equilibria contained within the unstable manifold of that equilibrium (i.e., for which there is a connecting trajectory), while the fourth column indicates the equilibria onto which the unstable manifold accumulates (i.e., lying in the closure of the unstable manifold).

<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>Dim of $W^u$</th>
<th>$W^u$ connects to</th>
<th>$W^u$ accumulates on</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_{LR}$</td>
<td>1</td>
<td>$\xi_{SS}$</td>
<td>$\xi_{LS}, \xi_{SR}, \xi_{AR}$</td>
</tr>
<tr>
<td>$\xi_{SS}$</td>
<td>2</td>
<td>$\xi_{LR}, \xi_{SR}, \xi_{AR}$</td>
<td>$\xi_{LS}, \xi_{SR}, \xi_{AR}$</td>
</tr>
<tr>
<td>$\xi_{LS}$</td>
<td>2</td>
<td>$\xi_{SR}, \xi_{AR}$</td>
<td>$\xi_{LR}, \xi_{SR}, \xi_{AR}$</td>
</tr>
<tr>
<td>$\xi_{SR}$</td>
<td>2</td>
<td>$\xi_{LR}, \xi_{AR}$</td>
<td>$\xi_{LR}, \xi_{SR}, \xi_{AR}$</td>
</tr>
<tr>
<td>$\xi_{AR}$</td>
<td>1</td>
<td>...</td>
<td>$\xi_{LR}, \xi_{SR}, \xi_{AR}$</td>
</tr>
<tr>
<td>$\xi_{WR}$</td>
<td>2</td>
<td>...</td>
<td>$\xi_{LR}, \xi_{SR}, \xi_{AR}$</td>
</tr>
</tbody>
</table>
The LR/SS-subcycle is shown at the base of Fig. 3 while example trajectories are shown in phase space in Fig. 2 and time series in Fig. 4. Predominantly, an almost periodic temporal behavior is displayed in the latter figure that involves jumps between states where one of the $x_i$ is nonzero ($\xi_{LR}$ states) and states where one of the $y_i$ is nonzero ($\xi_{SS}$ states). The approximate period decreases from Fig. (a) to (b) to (c) as the level of noise in the $y_i$ components is increased.

FIG. 2. Projections of a typical trajectory for the system (1) with parameters in Table II (a)–(d) for noise amplitudes $\epsilon_1 = \epsilon_2 = \epsilon_3 = 10^{-3}$ and (e)–(h) for $\epsilon_1 = \epsilon_2 = 10^{-9}$, $\epsilon_3 = \epsilon_4 = 10^{-3}$, after initial transients are allowed to decay. (a)–(d) show that the dynamics only explores symmetric copies of the subcycle between the four symmetric copies of $\xi_{LR}$, while (e)–(h) show repeated excursions around a number of connections that accumulate back on the subcycle. Note that in projection [(a) and (e)] $\xi_{SR}$ and $\xi_{SS}$ are both projected onto the origin, and only a selection of equilibria is labeled. Panels (d) and (h) show the dynamics in a group invariant projection ($x_1^2 + x_2^2, y_1^2 + y_2^2$) and distinguish all of the equilibria involved in the network. By contrast, projections of solutions where $\epsilon_1 = \epsilon_2 = 10^{-3}$, $\epsilon_3 = \epsilon_4 = 10^{-3}$ look very similar to (a)–(d) — the dynamics is much more sensitive to noise added to the last two components.
the time series also show a transient. In addition (b) and (c) show excursions characterized by a distinctive pattern of long visits to an equilibrium where all $x$'s and $y$'s are non-zero, followed by a rapid and growing oscillation leading back to the LR/SS subcycle. These features of the attractors are visible in Fig. 2 where in (a) only the LR/SS subcycle is present, while in (e)–(h) the excursions also take place.

In the following Sec. II A, we perform an analysis of the dynamics near this attractor; we construct a local return map and show that the LR/SS subcycle is a Milnor attractor and (b) the measure of its local basin of attraction includes all except a very small set in a neighborhood of the subcycle (i.e., it is essentially asymptotically stable)—recall that the local basin of attraction is the set of points whose trajectories remain near the cycle, and that are asymptotic to the cycle. We construct a return map for the excursions and use this to describe both deterministic and noise-induced excursions.

### A. Analysis of the heteroclinic attractor

The LR/SS subcycle is a heteroclinic cycle

\[ \cdots \rightarrow \xi_{LR} \rightarrow \xi_{SS} \rightarrow \xi_{LR} \rightarrow \cdots \]

where, in the coordinates $(x_1, x_2, y_1, y_2)$, the LR states are

\[ \xi_{LR}^{1,2} = (p, 0, 0, 0), \quad \xi_{LR}^{3,4} = (0, p, 0, 0) \]

and the SS states are

\[ \xi_{SS}^{1,2} = (0, 0, \pm q, 0), \quad \xi_{SS}^{3,4} = (0, 0, 0, \pm q). \]

There is switching between the $\pm$ images of these states in the presence of noise (see, for example, in Fig. 4(a), around 2500 time units) but we will identify all such points in the argument below.

We study the stability of this cycle by considering an approximate Poincaré map (similar to Refs. 1, 11, 13, 16, and 19) that is a composition of two local maps and two global maps between local sections as follows:

\[ \cdots H_{LR}^{(in)} \rightarrow H_{LR}^{(out)} \rightarrow H_{LS}^{(in)} \rightarrow H_{SS}^{(out)} \rightarrow H_{LR}^{(in)} \rightarrow \cdots \]

where $H_{LR}^{(in)}$ and $H_{LR}^{(out)}$ are local incoming/outgoing sections near $\xi_{LR}$ (see Fig. 5). Near $\xi_{LR} = (x, 0, 0, 0)$ we consider local coordinate $(u_1, z_1, v_1, w_1)$ aligned with the original coordinate axes $(x_1, x_2, y_1, y_2)$. Assume that $\xi = (u_1, |z_1|, |v_1|, |w_1| \leqslant 1)$ is a scaling of a small neighborhood of $\xi_{LR}^{1,2}$, where the behavior of the flow is almost linear, and define

\[ H_{LR}^{(in)} = (u_1, z_1, v_1, w_1): u_1^2 + v_1^2 = 1, |w_1|, |z_1| \leqslant 1. \]
For components of the compositions and hence the cycle. We identify symmetrically placed equilibria of leading order, only the coordinates \( w, z \), and \( w \) and \( z \) are significant for stability of the cycle. We identify symmetrically placed equilibria and illustrate these sections in Fig. 5.

Near \( \xi_{SS}=(0,0,0,y) \) we consider local coordinates \((x_3, y_3, z_3, u_3, v_3)\) aligned to the original coordinate axes \((x_1, x_2, y_1, y_2)\). The local map \( \phi_2: H_{SS}^{in} \rightarrow H_{SS}^{out} \) is defined for

\[
H_{SS}^{in} = \{(v_3, z_3, u_3): |v_3|, |z_3| \leq 1, w_3 = 1\}
\]

and

\[
H_{SS}^{out} = \{(v_3, z_3, u_3): |v_3|, |z_3| \leq 1, w_3 = 1\},
\]

and to leading order this can be expressed as

\[
\psi_2: H_{SS}^{in} \rightarrow H_{SS}^{in}
\]

and hence

\[
(w_3, z_3) = \psi_2^{in}(v_3, z_3, u_3, v_3) = (a_3 z_3, b_3 u_3),
\]

respectively, where \( a_3 \) and \( b_3 \) are constants. The \( (w, z) \) components of the compositions \( g_1 = \psi_1 \circ \phi_1: H_{LR}^{in} \rightarrow H_{SS}^{in} \) and \( g_2 = \psi_2 \circ \phi_2: H_{SS}^{in} \rightarrow H_{LR}^{in} \) to leading order are therefore

\[
(w_2, z_2) = g_1^{in}(w_1, z_1) = (\bar{a}_1 z_1 w_1^{1/2} e^{i \zeta_1}, \bar{b}_1 w_1^{1/2} e^{i \zeta_1}),
\]

\[
(w_1, z_1) = g_2^{in}(w_2, z_2) = (\bar{a}_2 z_2 w_2^{1/2} e^{i \zeta_2}, \bar{b}_2 w_2^{1/2} e^{i \zeta_2}),
\]

where \( \bar{a}_j \) and \( \bar{b}_j \) are bounded by constants. Defining logarithmic variables \( \zeta_j = \ln w_j \), \( j = 1,2 \), the mappings \( g_j^{in} \) are, to leading order, linear,

\[
\eta_j(t+1) = M_j \eta_j(t),
\]

with

\[
M_1 = \begin{pmatrix} c_{12}/e_{14} & 1 \\ c_{13}/e_{14} & 0 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} -c_{34}/e_{24} & 1 \\ c_{43}/e_{24} & 0 \end{pmatrix}.
\]

The product \( M = M_1 M_2 : H_{SS}^{in} \rightarrow H_{SS}^{in} \) (or equivalently \( M_2 M_1 : H_{LR}^{in} \rightarrow H_{LR}^{in} \)) approximates the dynamics in logarithmic coordinates for a small neighborhood of the cycle. The matrices \( M_j \) are sometimes called transition matrices, see, e.g., Ref. 16. To determine the local basin of attraction in a small neighborhood of a simple heteroclinic cycle [the subcycle (2) is simple], one needs to examine the eigenvalues \( \lambda_1, \lambda_2 \) and the associated eigenvectors \((v_1, v_2), (u_1, u_2)\) for the product \( M \) of transition matrices. The eigenvalues of \( M \) are 6.32 and 0.876, with associated eigenvectors \((1,0.46)\) and \((1,0.384)\) [see Eq. (3) and Table I]. If both eigenvalues of a transition matrix are real, \( \lambda_1 > \lambda_2 \), \( \lambda_1 > 1 \), \( v_1 v_2 > 0 \), and \( y_2 > 16.48 \), then it can be verified that

\[
\{(\xi, \eta): \xi \leq 0, \eta \leq 0, (v_1 v_2 - v_1 v_2)^{-1} (v_2 \xi - v_2 \eta) < 0\}
\]

is the set of points in logarithmic coordinates whose trajectories are attracted to the cycle (more general results of this type are discussed in Ref. 19). Applying this to the matrix \( M = M_1 M_2 \) we find that the starting points in \( H_{SS}^{in} \) for trajectories attracted by the cycle satisfy \(|w| < C_1 |z|^{0.384}\). This means that along the connection \( H_{LR}^{in} \rightarrow H_{SS}^{in} \) the trajectories starting at the points satisfying \(|x_2| < C_1 |y_2|^{0.384}\) are attracted to the cycle. Equivalently, points in \( H_{LR}^{in} \) that give rise to \( 320 \) trajectories attracted by the cycle will satisfy to leading order \(|w| > C_2 |z|^{16.48}\). Hence, along the connection \( H_{LR}^{in} \rightarrow H_{LR}^{in} \) the trajectories starting at the points satisfying

\[
|y_2| > C_2 |x_2|^{16.48}
\]

are attracted to the cycle. This implies that the LR/SS subcycle is essentially asymptotically stable,\(^6\) i.e., it attracts almost all trajectories starting in a sufficiently small neighborhood. Note that the high order of the power in Eq. (4) implies that the cusp of trajectories that escape from the LR/SS subcycle (see inset of Fig. 5) is extremely narrow.

III. DYNAMICS IN THE PRESENCE OF ANISOTROPIC NOISE

Note that isotropic noise is usually introduced as a model- ing assumption; it is not invariant under rescaling, but is justified by the fact that for many systems the response depends only on the amplitude of the noise. More generally, it is justifiable to explore the dynamics of Eq. (1) under vary...
TABLE IV. Frequency $F$ of excursions away from the LR/SS subcycle for Eq. (1) with parameters as in Table II. Note the monotonic increase of $F$ on increasing $\epsilon_{3,4}$, noise in the $y_i$ components. Note also the apparent cutoff below which the excursions become so infrequent that they are not apparent even for extended runs of 10^6 time units or more. Note that excursions were not found in the isotropic case, even for quite high noise amplitudes.

<table>
<thead>
<tr>
<th>$\epsilon_1 = \epsilon_2$</th>
<th>$10^{-3}$</th>
<th>$10^{-6}$</th>
<th>$10^{-9}$</th>
<th>$10^{-12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon_3 = \epsilon_4$</td>
<td>$10^{-3}$</td>
<td>$10^{-6}$</td>
<td>$10^{-9}$</td>
<td>$10^{-12}$</td>
</tr>
<tr>
<td>$F$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table IV shows some numerically measured values for $F$ as a function of noise for the 4D model (1) with parameters as in Table II. The frequencies were calculated by taking a Poincaré section at $x_1^2+x_2^2=3$ and then noting that returns close to the LR/SS subcycle came back at $y_1^2+y_2^2 \approx 9.83$, while the excursions correspond to cases where $y_1^2+y_2^2 > 9.83$. On taking a run of approximately 50 000 time units the proportion $F$ of excursions per cycle were then calculated (discounting repeated intersections during one excursion). The average was calculated over approximately 500–1000 cycles, meaning that the credible accuracy of $F$ in Table IV is only three or four decimal places at most. Note that one can count excursions more simply by examining, for example, intersections with $y_1^2+y_2^2 > 23$ [see Fig. 2(h)]; however, this requires taking two sections and does not remove the problem that one can include an arbitrary number of oscillations as one moves away from $\xi_{SR}$.

Note that the results in Table IV demonstrate the apparent lack of any excursions even for quite high amplitude of isotropic noise. A high level of anisotropy may be required to obtain excursions at all. Very long time series need to be computed as each cycle can take a long time, and in addition the excursions can occur very infrequently. Hence, we urge caution in interpreting the numerical values of $F$.

B. Dynamics of the return map with noise

Consider a small neighborhood of the equilibrium $\xi_{SS}=(0,0,0,y)$ in the local coordinates $(v_1, w_3, z_3, u_3)$. Without noise, the trajectories starting in the region

$$\{(1-\delta)|z_3|^2 + \epsilon_{34} > |z_3|^2 \cap H_{SS}^{(in)}\}$$

leave the neighborhood in the direction of $\xi_{LR}=(0,x,0,0)$, and the ones starting in

$$\{(1-\delta)|z_3|^2 + \epsilon_{34} > |w_3| \cap H_{SS}^{(in)}\}$$

leave in the direction of $\xi_{SR}=(0,0,y,y)$. ($\delta > 0$ is a small constant, $\epsilon_{34}$ and $\epsilon_{24}$ are given in Table I, $\epsilon_{34}/\epsilon_{24}=0.32$.)

Following Ref. 2, we assume that noise is significant only in neighborhoods of steady states (where the vector field is small) and that noise can be ignored outside these neighborhoods. The probability distribution of trajectories near a saddle equilibrium in a two-dimensional system was studied in the presence of noise in Ref. 24. It was shown that for an incoming normal distribution with a zero mean, the mean $\mu_{exit}$ and the standard deviation $\sigma_{exit}$ of the distribution of outgoing trajectories for asymptotically small noise will scale as

$$\mu_{exit} \sim \epsilon^{\lambda_h/|\lambda_0|}, \quad \sigma_{exit} \sim \epsilon^{1/2 + |\lambda_h|/|\lambda_0|}$$

where $\lambda_h < 0$ and $\lambda_0 > 0$ are the eigenvalues associated with the stable and the unstable directions of the saddle equilibrium, $\epsilon$ is the amplitude of the added noise, and $\epsilon_1, \epsilon_2$ are positive constants. For an equilibrium, where the saddle quantity $\lambda_h + |\lambda_0|$ is positive, $\mu_{exit} > \sigma_{exit}$ in the limit $\epsilon \rightarrow 0$, and hence outgoing trajectories are lifted away from the noise-free heteroclinic connection.

Consider a small neighborhood of the $\xi_{SS}$: since noise is important only near equilibria, the probability distribution of trajectories approaching $H_{SS}^{(in)}$ is essentially the same as in $H_{LR}^{(in)}$ (up to a linear transformation). Similarly, consider a
small neighborhood of $\xi_{LR}=(x,0,0,0)$. In the subspace $(0,x_3,0,y_3)$, $\lambda_x=-c_{12}\approx-0.105$, $\lambda_y=e_{14}\approx0.117$ (see Table 4), and lift-off occurs. In the subspace $(0,0,y_3)$, $\lambda_y=-c_{13}\approx-0.782$, $\lambda_y=e_{14}\approx0.117$, and so we expect no lift-off near the $\xi_{SS}$ states.

Assume that the added noise is isotropic ($\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4$) and so of equal amplitude in the $x_1$ and $y_1$ directions. Incom- mings ing trajectories in $H_{LS}\left(\xi_{LS}\right)$ have a distribution with the mean $\left(\mu_1, \mu_2\right)$ and standard deviations $(\sigma_1, \sigma_2)$, schematically shown in Fig. 6 as an ellipse (drawn by a thick line) centered at $\left(\mu_1, \mu_2\right)$ with axes $\sigma_1$ and $\sigma_2$ (note that the ellipse represents the covariance ellipse of the noisy trajectories on re- turn, not their support). The gray (white) area represents starting points for trajectories leaving the neighborhood in the direction of $\xi_{LR}$ (\xi_{LS}, respectively). Due to lift-off in the $(0,x_3,0,y_3)$ subspace [schematically shown by the shift of the noise ellipse in the $x_3$ direction in Fig. 6(a)], added isotropic noise makes the LR/SS subcycle even more stable.

Now consider anisotropic noise ($\epsilon_1=\epsilon_3=\epsilon_4=\epsilon_4$). On in- crease of $\epsilon_1$ relative to $\epsilon_2$, the vertical axis of the noise el- liptical is stretched [see Fig. 6(b)], implying that anisotropic noise may make the LR/SS subcycle unstable. Note that the techniques of Ref. 24 assume an incoming normal distribution with a zero mean and suggest that this is a good model for small enough noise amplitudes; a more sophisticated model has been recently developed by Bakhtin.

**IV. A DEPTH TWO HETEROCLINIC NETWORK**

**FOR CONVECTION**

Convection appears spontaneously when a horizontal layer of fluid, uniformly heated from below, commences mo- tion by the uniform conducting state becoming unstable. The onset of convection is determined by the nondimensional Rayleigh number $R$ (which measures the relative amplitude of the buoyancy and the viscous forces) and the Prandtl number $P$ (which measures the ratio of viscous to thermal diffu- sion). In a previous paper 20 we considered the nonlinear so- lutions near primary instability neighborhood of this mode interaction in $(R,k)$ space ($R$ being the Rayleigh number and $k$ the horizontal wavenumber) for a variety of Prandtl num- bers $P$.

In Ref. 20, we found parameter regimes where the only attractors are robust heteroclinic networks that can include connections from equilibria to subcycles by performing a reduction to a normal form that is an 8D system on $\mathbb{C}^4$. The latter system is detailed in the Appendix, Eq. (A11) with co- eficients (A5), $\theta=1.34$ and Table VII; the Appendix shows that this system reduces to Eq. (1) on restricting to a 4D invariant subspace. As a consequence of Secs. II and III, we see that the essential dynamics is reproduced by the system reduced to an $\mathbb{R}^4$ invariant subspace, namely, Eq. (1).

We turn now to the full equations for Boussinesq thermal convection as governed by the Navier–Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} = \nabla \times (\nabla \times \mathbf{v}) + P\Delta \mathbf{v} + PRT e_3 - \nabla p, \quad \text{(5)}$$

the incompressibility condition

$$\nabla \cdot \mathbf{v} = 0, \quad \text{(6)}$$

and the heat transfer equation

$$\frac{\partial T}{\partial t} = -(\mathbf{v} \cdot \nabla)T + v_3 + \Delta T, \quad \text{(7)}$$

where $\mathbf{v}$ is the flow velocity and $T$ is the deviation of tempera- ture of fluid from the linear profile in the trivial steady state. $R$ and $P$ are dimensionless parameters, the Rayleigh and Prandtl numbers, respectively. Stress-free horizontal boundaries held at fixed temperature are assumed,

$$\frac{\partial v_1}{\partial x_3} = \frac{\partial v_2}{\partial x_3} = v_3 = 0, \quad T = 0 \quad \text{at} \quad z = 0, 1. \quad \text{(8)}$$

The equations are solved numerically by applying stan- dard pseudospectral methods. The fields are represented as a Fourier series,

$$u_j = \sum_n \hat{u}_{j,n} \exp \left[ \frac{2 \pi i}{L} (n_1 x_1 + n_2 x_2) \right] q_j (\pi n_3 x_3), \quad \text{(9)}$$

where $q_j(x_3) = q_j^2(x_3) = \cos x_3$ and $q_j^3(x_3) = \sin x_3$ [here $\mathbf{v} = (u_1, u_2, v_3)$, $v_3 = T$]. Substitution of Eq. (9) into Eqs. (5) and (7) yields a system of equations for the Fourier coeffi- cients $\hat{w}_{j,n} = \hat{u}_{j,n}$, $j = 1, 2, 3, 4$ of the form

$$\hat{w}_{j,n} = F_{j,n}(\mathbf{w}). \quad \text{(10)}$$

With an added random force the system of Eq. (10) takes the form

$$\hat{w}_{j,n} = F_{j,n}(\mathbf{w}) + \eta_{j,n}, \quad \text{(11)}$$

where $\eta_{j,n}$ are mutually independent white noise forcing com- ponents, $\langle \eta_{i,n}^j(\eta_{j,m}^i) = 0 \quad \text{for} \quad i \neq j, \quad n \neq m$, and $\langle \eta_{j,n}^j(\eta_{j,n}^i) \rangle = \epsilon_{j,n}^i \delta_{i-j}$. The forcing affects the first four spherical shells in the Fourier space ($\eta_{j,0}^j = 0$ if $n_1^2+n_2^2+n_3^2 > 4$); its amplitude is

$$\epsilon = \sum_j \sum_n \epsilon_{j,n}, \quad \text{(12)}$$

FIG. 6. Schematic diagram showing a “noise ellipse” indicative of the distribution of incoming trajectories in $H_{LS}$ around the connection [here at $(0,0)$] for the (a) isotropic and (b) anisotropic noise. The gray region shows the local basin of attraction of the LR/SS subcycle for the system in the absence of noise; the white region indicates excursions via $\xi_{SS}$ take place. The region of the noise ellipse that lies outside the gray region represents the probability of an excursion between two visits to the Poincaré surface of section.
510 Consider the symmetry $s$, defined by spatial translation
511 by half a period with reference to both of the sides of the
512 enclosing box. For symmetric fields, $v_{j,n}=0$ with $n_1+n_2$ odd
513 ($j=1, \ldots, 4$); for antisymmetric fields, $v_{j,n}=0$ with $n_1+n_2$
514 even ($j=1, \ldots, 4$). Respectively, we introduce the amplitude
515 of symmetric and antisymmetric parts of the added noise,
516 $e_s = \sum_{j,n,1+n_2 \text{ even}} e_s^{jn}, \quad e_a = \sum_{j,n,1+n_2 \text{ odd}} e_a^{jn}$.
517
518 The symmetry $s$ action on the center manifold coordi-
519 nates is
519 $s(z_1,z_2,z_3,z_4) \rightarrow (-z_1,-z_2,z_3,z_4)$.
520 Hence, if random forcing $\eta$ in Eq. (11) is anisotropic with
521 $e_a^{jn} \gg 1$,
522 its projections $P_j(\eta)$ along axes $z_j, j=1, \ldots, 4$, satisfy
523 $\langle (P_{1j}(\eta))^2 \rangle = \langle (P_{2j}(\eta))^2 \rangle \leq \langle (P_{3j}(\eta))^2 \rangle = \langle (P_{4j}(\eta))^2 \rangle$,
524 and in this sense we can make a heuristic identification $e_{a}^{jn} = e_2$ and $e_a^{jn} = e_1$ in the 4D system (1). In the latter
525 system, the bifurcation parameter is $\theta$ which can be related to
526 the parameters of convection $R$ and $\delta = k - k_m$ as follows,
527 where the other constants are given in the Appendix:
528 $\tan \theta = \frac{\lambda_2}{\lambda_1} = \frac{R - R_m}{\delta}$.
529
530 Note that the system we consider is constrained to have exact
531 periodicity in both horizontal directions, and so will not be
532 sensitive to long-wavelength modes that will be important
533 for stability in an unconstrained setting.
534 We present numerical results for $\delta = -0.05k_m$ and $R = 713$, corresponding to $\theta = 1.34$. The convective attractor is
535 the LR/SS subcycle, similar to that shown in Fig. 8a of Ref.
536 20. With an added stochastic forcing we observe excursions
537 to $\xi_{SR}$ (e.g., as in Fig. 7), even when the amplitude of added
538 noise is as small as $10^{-14}$. During the excursion a sample
539 trajectory follows from the LR/SS subcycle to the LS/SS
540 cycle, makes a revolution around it, then from $\xi_{SS}$ it proceeds
541 to $\xi_{SR}$, followed by $\xi_{AR}$ and afterward by $\xi_{WR}$, from where it
542 returns to the LR/SS cycle. It can be briefly described by the
543 sequence of the steady states involved,
544 $\cdots \xi_{LR} \rightarrow \xi_{SS} \rightarrow \xi_{LS} \rightarrow \xi_{SS} \rightarrow \xi_{SR}$
545 $\rightarrow \xi_{AR} \rightarrow \xi_{WR} \rightarrow (\xi_{LR} \rightarrow \xi_{SS}) \cdots$.
547 Other trajectories make two revolutions around the LR/SS
548 cycle, or pass close to it. A different steady state $\xi_{LS}$ can be
549 visited instead of $\xi_{AR}$ and a transition from $\xi_{SR}$ to $\xi_{WR}$ can
550 occur, but the overall behavior is similar to that displayed in
551 Fig. 7.
552 Table V shows numerically measured values for the fre-
553 quency $F$ of excursions as a function of the noise amplitude
554 for Boussinesq convection. Computations with anisotropic
555 noise were done for two cases. If the added noise is isotropic,
556 or when relatively large amplitude noise is added in the $e_1$
557 direction (anisotropic case I), generally speaking, the fre-
558 quency of excursions decreases with the noise amplitude. By
559 contrast, if most of the noise is added in the $e_a$ direction
560 (anisotropic case II), an opposite effect is observed as the
561 amplitude of the added noise varies from $10^{-8}$ to $10^{-4}$; for
562 smaller amplitudes, from $10^{-14}$ to $10^{-8}$, the variation
563 with noise amplitude seems to be relatively weak. As discussed
564 above, $e_a$ corresponds to addition of noise in the $z_3,z_4$ direc-
565 tions of the normal form at the mode interaction, and this
566 corresponds to the $\xi_{SA}$ amplitudes for the 4D model (1). As
567 shown on the last panel of Fig. 7 (during an excursion the
568 value $|z_3|−|z_4|$ is large, while otherwise it is close to zero),
569 the distribution of excursions is nonuniform, and we presume
570 that the time span of $2 \times 10^4$ of simulations is not sufficiently
571 long to accumulate accurate statistics.
572 Although there are similarities as outlined above, there
573 are clearly differences compared to the 4D system (see Sec.
574 III); for example excursions in the 4D system are very rarely
575 observed if the added noise is isotropic. This may be related
576 to the fact that eigenvalues of linearizations (and the dimen-
577 sions of corresponding eigenspaces) near the steady state $\xi_{LR}$
578 in the two systems differ significantly. However, we have
579 calculated the eigenvalues for the convective system on im-
580 posing the relevant symmetries, and find $\varepsilon_{14}^{\text{conv}} =0.122$, $c_{12}^{\text{conv}}$
581 $=0.086$, and $c_{13}^{\text{conv}} =0.566$. The values are not very different to
582 the ones given in Table I, and the arguments in Sec. III B
583 about the lift-off within one of the invariant subspaces based
584 on the differences of eigenvalues remain true. In addition, for
585 convection the center manifold is properly 8D but we study
586 in detailed dynamics only in a 4D subspace. We have per-
587 formed simulations for the 8D normal form in the Appendix,
588 and find the excursions are even rarer than for the 4D system.
V. DISCUSSION

In the process of trying to understand the behavior of a two-dimensional unstable manifold of a robust example, the minimum number of dimensions needed to see such behavior robustly in a symmetric system of ODEs.

This behavior is not possible in three or fewer dimensions because, in order to connect robustly to a robust subcycle, it is necessary to have a three-dimensional proper invariant subspace for the dynamics.

Further study of the structure and dynamical properties of Eq. (1) at other parameter values is likely to bring other possible robust depth two network structures to light. In particular, for this example we find a subcycle that is essentially asymptotically stable, and hence excursions take place only as transients or in the presence of noise. This may no longer be the case if the coefficients $B$ are changed in a way that the subcycle ceases to be essentially asymptotically stable—although in this case it is possible that this will destroy attraction for the network as a whole. Depth two cycles have been observed in other settings, but it is still unclear whether there are any robust examples such that typical trajectories in the noise-free case explore the whole network in their $\omega$-limit.

We observe that addition of noise can stabilize switching via “lift-off” of trajectories, as observed in Ref. 2. Depending on the details of the noise one may find either apparently deterministic cycling around the LR/SS subcycle or irregular excursions alternating with a fairly small numbers of cycles.

This is dependent not just on the noise amplitude but also on the “shape” of the noise ellipse with respect to the local basin of attraction of the subcycle. Similar effects are presumably also observable for symmetry breaking perturbations of the network in the absence of noise, including the possibility of chaotic dynamics near the network.

In terms of the Boussinesq convection mode that originally motivated this study, we observe a depth two attractor with similar structure for numerical simulations in the presence of noise. The spatial-temporal structure of the noise is important in determining the frequency of excursions in that these depend on the noise strengths in the various spatial Fourier modes. We have not found a quantitative agreement of the frequency of excursions observed between the cycles from convection and that for the reduced model (1), but in fact there is no rigorous reason for the center manifold reduction to the truncated normal form (1) to be valid on addition of noise. For the 8D $(C^4)$ model in Ref. 20 we find apparently far fewer excursions than in the example discussed here. This may be linked to the fact that $\xi_{SR}$ has a three-dimensional unstable manifold for the 8D models, with a codimension two connection to $\xi_{SR}$. By contrast, for the 4D model (1) the corresponding $\xi_{SS}$ has an unstable manifold with a codimension one connection to $\xi_{SR}$. It is an interesting open problem to understand the relative roles of noise amplitudes, eigenvalue, and dimensionality in determining the attraction properties and relative frequency of excursions for such robust attractors.

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APPENDIX: REDUCTION FROM THE BOUSSINESQ NORMAL FORM

As detailed in Ref. 20 the normal form on center manifold $C^4$ with $D_4 \times T^2 \times Z_2$ symmetry truncated at cubic order can be written as

$$z_1 = \lambda_1 z_1 + z_1 (A_1 |z_1|^2 + A_2 |z_2|^2 + A_3 (|z_3|^2 + |z_4|^2)) + A_4 z_1 z_2 z_3 z_4,$$

$$z_2 = \lambda_2 z_2 + z_2 (A_1 |z_2|^2 + A_3 |z_1|^2 + A_4 (|z_3|^2 + |z_4|^2)) + A_5 z_1 z_2 z_3 z_4,$$

$$z_3 = \lambda_3 z_3 + z_3 (A_1 |z_3|^2 + A_5 |z_1|^2 + A_6 (|z_4|^2 + |z_2|^2)) + A_7 z_1 z_2 z_3 z_4,$$

$$z_4 = \lambda_4 z_4 + z_4 (A_1 |z_4|^2 + A_6 |z_3|^2 + A_7 (|z_1|^2 + |z_2|^2)) + A_8 z_1 z_2 z_3 z_4,$$

where $A_i$ are real numbers (the normal form coefficients). The linear growth rates $\lambda_1$ and $\lambda_2$ are close to zero for the truncation to be valid; otherwise, higher order terms become important.

For the problem of Boussinesq convection in a planar region, these coefficients can be calculated as functions of the Rayleigh number $R$, the Prandtl number $P$, and the critical wavenumber $k$ by introducing...
749 The parameter $\theta$ can be used to study the competition between the bifurcations to two different translation-invariant solutions (rolls parallel to the periodicity box or parallel to a diagonal) while the parameter $r>0$ gives the distance from the mode interaction. For $\theta=1.34$ we find robust heteroclinic cycles with depth two connections. This paper relies on note that the main features of this robust heteroclinic attractor are present for the cycle restricted to the invariant subspace

750 $$(z_1, z_2, z_3, z_4) = (x_1, x_2, ix_3, ix_4),$$

751 and hence we examine the system (A6) below, in the main part of the paper, which on addition of noise is simply Eq. (A1) within this subspace,

752 $${\dot{x}}_1 = (\lambda_1 + A_1 x_1^2 + A_2 x_2^2 + A_3 (x_3^2 + x_4^2) - A_4 x_3 x_4)x_1,$$

753 $${\dot{x}}_2 = (\lambda_1 + A_1 x_2^2 + A_2 x_3^2 + A_3 (x_3^2 + x_4^2) + A_4 x_3 x_4)x_2,$$

754 $${\dot{x}}_3 = (\lambda_2 + A_5 x_1^2 + A_6 x_2^2) x_3 + (A_7 (x_3^2 + x_4^2) + A_8 (x_2^2 - x_1^2)) x_3,$$

755 $${\dot{x}}_4 = (\lambda_2 + A_5 x_1^2 + A_6 x_2^2) x_4 + (A_7 (x_3^2 + x_4^2) + A_8 (x_2^2 - x_1^2)) x_4,$$

756 and hence we examine the system (A6) below, in the main part of the paper, which on addition of noise is simply Eq. (A1) within this subspace,

757 $${\dot{x}}_1 = (\lambda_1 + A_1 x_1^2 + A_2 x_2^2 + A_3 (x_3^2 + x_4^2) - A_4 x_3 x_4)x_1,$$

758 $${\dot{x}}_2 = (\lambda_1 + A_1 x_2^2 + A_2 x_3^2 + A_3 (x_3^2 + x_4^2) + A_4 x_3 x_4)x_2,$$

759 $${\dot{x}}_3 = (\lambda_2 + A_5 x_1^2 + A_6 x_2^2) x_3 + (A_7 (x_3^2 + x_4^2) + A_8 (x_2^2 - x_1^2)) x_3,$$

760 $${\dot{x}}_4 = (\lambda_2 + A_5 x_1^2 + A_6 x_2^2) x_4 + (A_7 (x_3^2 + x_4^2) + A_8 (x_2^2 - x_1^2)) x_4.$$


Recall that an invariant set \(A\) is chain recurrent if given any \(T > 0, x, y \in A\), and \(e > 0\) there is a finite set of points \(x=x_0, x_1, \ldots, x_k=y\) such that \(|\Phi^t(x_i)−x_{i+1}| < e\) for \(i=0,1,\ldots,k−1\).


The original normal form coordinates in Ref. 20 are related to ours by \((x_1, x_2, i(y_1+y_2)/2, i(y_1−y_2)/2)\). A brief summary of the motivation and the steps involved in the reduction are given below and in Sec. IV with further details in the Appendix.
#1 Author, please check your corrections carefully. I am not sure I read some of the corrections on the proof correctly so please double-check them. Thank you very much.