Extreme sensitivity to detuning for globally coupled phase oscillators

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We discuss the sensitivity of a population of coupled oscillators to differences in their natural frequencies, i.e. to detuning. We argue that for three or more oscillators, one can get great sensitivity even if the coupling does not vanish. For N globally coupled phase oscillators we find cases of bifurcation to extreme sensitivity, where frequency locking can be destroyed by arbitrarily small detuning. This extreme sensitivity is absent for N = 2, appears at isolated parameter values for N ≥ 3, and can appear robustly (i.e. for open sets of parameter values) for N = 5 for more oscillators.

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Coupled oscillators and their synchronization properties have been studied in models of coupled neurons [1, 2], multimode lasers [3] and a wide variety of other systems [4]. Of particular interest in many applications is the boundary between in-phase and antiphase oscillation. For example, work starting with that of Kuramoto [5] (see also the review [6]) has looked at the interplay between coupling strength and frequency spread for coupled phase oscillators of the form

\[ \dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} g(\theta_i - \theta_j), \]

where \( (\theta_1, \ldots, \theta_N) \in T^N \) and \( T^1 = [0, 2\pi) \). We examine coupling as in [1] given by

\[ g(\phi) := -\sin(\phi + \alpha) + r \sin(2\phi). \]

Many studies have considered the case \( r = 0 \) and \( \alpha = 0 \) in the thermodynamic limit \( N \to \infty \), and with frequencies \( \omega_i \) chosen from a Gaussian distribution with some given variance [6, 7]. Some details of the very rich dynamics that can appear in a system of a few coupled phase oscillators at loss of stability of a synchronized attractor have recently been examined in Ref. [8]. Also, chaotic dynamics of the phases has been found and studied in the Kuramoto model for weak and intermediate coupling strength [9], while Ref. [10] investigates the dynamics of ‘repulsive interaction’ in the case \( r = 0 \).

For the Kuramoto model [7] (i.e. for system (1) with \( r = 0 \) and \( \alpha = 0 \)) reducing the coupling strength will result in greater sensitivity of synchronized behaviour to variation in the frequencies. In other words, for smaller coupling strength, a smaller mismatch (detuning) in the natural frequencies of the oscillators will result in desynchronized dynamics. We will explore cases where a parameter varies that affects the type but not necessarily the strength of coupling. In particular, we show that it is possible to get great sensitivity to detuning without decoupling.

In this Letter we uncover the sensitivity in system (1) to detuning in frequencies of the oscillators. We show that even for large coupling strengths an arbitrarily small mismatch in oscillator frequencies can destroy synchrony in the ensemble. We argue that this extreme sensitivity is a mechanism that is typical for \( N \geq 3 \) oscillators related to the appearance of ‘slow oscillations’ (attracting heteroclinic cycles) in these systems.

We consider system (1) with \( r \) and \( K \) fixed and \( \alpha \) varying. It is known that this system displays a variety of attractors and bifurcations, including cluster states, periodic orbits, tori and heteroclinic cycles that may be robust [1]. Other coupling functions of the form \( g(\theta_i, \theta_j) \) can permit, for example, the possibility of oscillation death (e.g. [11]) but system (1) is justifiable for weak to moderate values of coupling [12] and so we remain with this model. The dynamics of this system was investigated in Refs. [1, 14, 15] for identical oscillators with the same natural frequencies \( \omega_i = \omega \). For \( N \geq 4 \) oscillators there can be robust slow oscillations between cluster states, caused by the presence of attracting heteroclinic cycles. We show how these heteroclinic cycles can be used to explain the emergence of extreme sensitivity to detuning in system (1).

To quantify the sensitivity to detuning we define a vector \( \delta = (\Delta_1, \ldots, \Delta_{N-1}) \) by \( \Delta_i = \omega_{i+1} - \omega_i \). Fixing all parameters in system (1) except \( \delta \) we define sensitivity to detuning to be the supremum of \( \Omega \) such that \( \| \delta \| < \Omega \) implies all attractors for the system of oscillators are phase locked in the sense of bounded phase differences between oscillators. If \( \Omega \) is small, the system has a lower threshold of sensitivity to detuning. In the case \( \Omega = 0 \) we say the system shows extreme sensitivity to detuning.

We find that loss of stability of in-phase oscillation \( \theta_1 = \cdots = \theta_N \) for identical oscillators (1) is associate with a minimum of the sensitivity \( \Omega \) on fixing \( r \), \( K \) and varying \( \alpha \). The nature of this minimum depends critically on the number of oscillators. For \( N = 2 \) we find that the minimum is positive while for \( N \geq 3 \) the minimum is zero (extreme sensitivity) and remains exponentially close to zero for parameters values near loss of stability of in-phase oscillation. For \( N \geq 5 \) we see that it can become zero for a whole interval of parameter values, indicating...
robust extreme sensitivity.

We say two oscillators $\theta_i$ and $\theta_j$ have bounded phase differences on a given trajectory if $|\theta_i(t) - \theta_j(t)|$ is bounded uniformly for $t > 0$. This implies that $\theta_i$ and $\theta_j$ are frequency synchronized; i.e. $\lim_{t \to \infty} \frac{\theta_i(t) - \theta_j(t)}{t} = 0$. If $\omega_i = \omega$ (so $\Delta_i \equiv 0$) then the sets $S_{ij}$, defined by those phases where $\theta_i = \theta_j$ are invariant for all $i \neq j$ and form codimension one barriers to the dynamics. Hence, without loss of generality, we can consider the flow in the so-called canonical invariant region [12] defined by $\theta_1 \leq \theta_2 \leq \theta_3 \ldots \leq \theta_N \leq \theta_1 + 2\pi$. All oscillators are phase locked in this case regardless of the dynamics, and regardless of $N$. This argument does not hold as soon as the $\Delta_i$ are different.

The model (1) with $r = 0$, $\alpha = 0$ and $\omega_i = \omega$ has only a degenerate bifurcation on changing $K$; for $K > 0$ the in-phase oscillation is stable and antiphase states (where the order parameter $R = N^{-1} \sum_{k=1}^{N} \exp(i \theta_k)$) are unstable. For $K < 0$ they simply swap stability giving rise to attractors that are degenerate manifolds of neutrally stable periodic solutions [10]. For $K = 0$ the system is effectively uncoupled and foliated with neutrally stable periodic orbits.

We now consider the more general case with $r \neq 0$ and $\alpha \neq 0$ but still identical oscillators ($\Delta_i = 0$), first for the cases $N = 2$ and $3$. The complexity of the bifurcations increases dramatically with $N$ and so for $N = 4, 5$ we focus only on robust heteroclinic attractors that can appear in these cases. We set $K = 1$ from hereon and note that other cases of $K$ can be treated by a simple scaling of time.

For $N = 2$ and $\phi := \theta_2 - \theta_1$, Eq. (1) reduces to
\[
\dot{\phi} = \Delta_1 - \sin \phi \cos \alpha + r \sin 2\phi, \tag{2}
\]
from which one can determine that for fixed $\Delta_1 = 0$ and $0 < r < 0.5$ there are solutions $\phi = 0$ (in-phase) and $\phi = \pi$ (antiphase) for all $\alpha$. There are also branches of solutions with $\phi = \pm \arccos(\frac{2r}{\Delta_1})$ that appear at $\alpha = \arccos(2r)$ and disappear at $\alpha = \arccos(-2r)$ via supercritical pitchfork bifurcations as illustrated in Fig. 1(a). One can check that qualitatively the same bifurcation behaviour is found for all $0 < r < 0.5$. At $r = 0$ the branches are vertical and hence degenerate. If $N = 2$ and $0 < r < 0.5$ then for any $\alpha$ one can find the sensitivity $\Omega = \min(\|F_+\|, \|F_-\|)$, where $F_+$ (resp. $F_-$) is the global maximum (resp. minimum) of $F(\phi) = -\sin \phi \cos \alpha + r \sin 2\phi$. One can observe that these extrema occur at a value of $\pm \arccos \left( \frac{\cos \alpha \pm \sqrt{\cos^2 \alpha + 4r^2}}{2r} \right)$.

The sensitivity $\Omega$ as a function of $\alpha$ is shown in Fig. 4 for $r = 0.2$, including the case $N = 2$. One can see that (i) this has minima at $\alpha = \pi/2$ (where $|F_+| = |F_-|$) corresponding to the sin $\phi$ term vanishing in Eq. (2) and (ii) the minimum value is given by $r$ and so is non-zero.

For $N = 3$ we can reduce system (1) to a two dimensional flow on a torus, given by an ODE for the phase differences $\phi_1 = \theta_2 - \theta_1$ and $\phi_2 = \theta_3 - \theta_2$. Fixing $r = 0.2$ and $\Delta_1 = \Delta_2 = 0$, the bifurcation diagram on varying $\alpha$ is a lot more complicated than that for $N = 2$. The bifurcations of the system, as found by numerical path following with xppaut [16] are shown in Fig. 1(b), with the locations of bifurcations $A \to F$ listed. Inclusion of the term $r \neq 0$ breaks much but not all of the degeneracy. In particular, the Hopf bifurcation $C$ and transcritical bifurcation $D$ occur at the same parameter value, $\alpha = \arccos(2r)$.

Figure 2 plots the phase differences for $N = 3$ using $\xi = \theta_1 + e^{2\pi i} \theta_2 + e^{4\pi i/3} \theta_3$ (permutations of the oscillators correspond to rotations and reflections in the phase space that preserve an equilateral triangle [13]). The lift of this torus of phase differences to $\mathbb{R}^3$ can be rep-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Schematic bifurcation diagrams for system (1) with $r = 0.2$, $K = 1$ and (a) $N = 2$, (b) $N = 3$. For $N = 2$ in (a), obtained analytically, there are pitchfork bifurcations at $A$ and $B$ located at $\alpha = \arccos(\pm 2r)$ and branches of attracting symmetry-broken equilibria connecting $A$ and $B$. For $N = 3$ in (b), obtained using path-following, the antiphase state bifurcates at $C$ to give a branch of unstable periodic orbits (thin dotted curve). This has a saddle node at $F$ with a stable periodic orbit (bold dotted curve) which terminates at a global saddle-node heteroclinic bifurcation at $E$. For limit cycles the minimum value over the cycle is plotted. Lines show equilibria; bold indicates that they are attracting. Bifurcations illustrate: $A$ ($\alpha = 0.7691$): Pitchfork; $B$ ($\alpha = 1.1269$): Saddle-node of equilibria; $C$ ($\alpha = 1.1593$): Hopf from antiphase state; $D$ ($\alpha = 1.1593$): $S_\lambda$ transcritical bifurcation; $E$ ($\alpha = 1.2574$): Global saddle-node heteroclinic; $F$ ($\alpha = 1.4452$): Saddle-node of limit cycles. Branch $AEBD$ lies within $\phi_2 = 0$ and we note there are symmetrically related branches in $\phi_1 = 0$ and $\phi_1 = \phi_2$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{(a) Flow near the global saddle-node heteroclinic bifurcation scenario for $N = 3$ with $r = 0.2$ shown schematically in the plane $\xi = \theta_1 + e^{2\pi i/3} \theta_2 + e^{4\pi i/3} \theta_3$. The phase portraits show (a) before the bifurcation for $\alpha$ just less than $\alpha_0 = 1.2574$, (b) at the bifurcation for $\alpha = 1.2574$, and (c) after the bifurcation for $\alpha$ just greater than $\alpha_0$ there is an attracting limit cycle $L$ that is sensitive to detuning.}
\end{figure}
resented by reflecting triangles from Fig. 2 along their edges to build a triangulation of the plane. The sides of the triangles represent the invariant subspaces \( \phi_1 = 0 \), \( \phi_2 = 0 \) and \( \phi_1 = \phi_2 \) bounding the canonical invariant region for \( \Delta_1 = \Delta_2 = 0 \). The bifurcation at \( E \) [Fig. 1(b)] at \( \alpha = 1.2574, r = 0.2 \) is of particular interest as it gives extreme sensitivity to detuning (a qualitatively similar sequence of bifurcations occurs for \( 0.156 < r < 0.375 \)). This is shown in Fig. 2; for \( \alpha < \alpha_0 \) nearby there is a stable node \( P \) and saddles \( Q \) and \( R \) on the invariant subspaces [Fig. 2(a)]. At \( \alpha_0 \), a saddle-node bifurcation of \( P \) and \( R \) occurs [Fig. 2(b)]. Finally, at \( \alpha > \alpha_0 \) there is a stable limit cycle \( L \) created near the union of unstable manifolds of the remaining saddles \( Q \) [Fig. 2(c)]. This limit cycle is very close to the invariant subspaces, meaning that small detuning results in a nearby attractor that is not phase locked.

At the bifurcation Fig. 2(b) we assume the eigenvalues at \( Q \) are \( \lambda_1 < 0 \) and \( \lambda_2 > 0 \) while at \( P \) they are \( \lambda_3 < 0 \) and 0. We derive an approximating map \( F : \Pi_1 \to \Pi'_1 \) with \( F(a) = c \), for \( 0 < a \ll 1 \), where the symbols are as shown in Fig. 3. Since \( \Pi_1 \) and \( \Pi'_1 \) are symmetrically related, the return map to \( \Pi_1 \) is given simply by \( F^3 \). Consider a parameter \( \nu \) that unfolds the saddle-node at \( P \), such that there are two hyperbolic fixed points near \( P \) for \( \nu < 0 \) and none for \( \nu > 0 \). Clearly for \( \nu \leq 0 \) trajectories starting close to the invariant subspaces, i.e., having a small enough, will not intersect \( \Pi'_1 \) as they will be trapped at \( P \). For \( \nu > 0 \) suppose a trajectory starting on \( \Pi_1 \) at distance \( a \) from the stable manifold of \( Q \) returns to \( \Pi_1 \) at a distance \( \tilde{a} \) from the stable manifold. Approximation by a linear flow near \( Q \) gives to leading order

\[
\tilde{a} = K_1 \exp[-K_2 \nu^{-\frac{1}{2}}]
\]

independent of \( a \) to leading order in \( |\alpha - \alpha_0| \). The period \( T \) of the cycle scales as \( T = K_3|\alpha - \alpha_0|^{-1/2} \) implying that the period goes to infinity very slowly as we approach the bifurcation point illustrated in Fig. 2 (as \( \alpha \) tends to \( \alpha_0 \) from above).

At the saddle-node heteroclinic bifurcation illustrated in Fig. 2 one finds a web of interconnecting invariant manifolds that wind nontrivially around the torus. These arise on lifting the invariant manifolds shown in Fig. 2(b) (where the arrows denote the invariant manifolds of \( P \) and \( Q \) and their symmetric images) to the torus. This web of connections causes extreme sensitivity to detuning, i.e. arbitrarily small detuning \( \Delta \) will give attractors near this web that have unbounded phase differences. Moreover, for \( \alpha > \alpha_0 \) the attracting periodic orbits that are exponentially close to the invariant manifolds (see Fig. 2(c)) mean that we can estimate

\[
\Omega = O(\exp[-K|\alpha - \alpha_0|^{-1/2}])
\]

for \( \alpha > \alpha_0 \). On the other side of the bifurcation, \( \alpha < \alpha_0 \) one will have

\[
\Omega = O(|\alpha - \alpha_0|)
\]

due to the appearance of a saddle-node pair. This is in excellent agreement with behaviour of the sensitivity \( \Omega \) near \( \Delta = 0 \) at \( \alpha = \alpha_0 \) for the case \( N = 3 \) in the subplot of Fig. 4 [18].

For \( N = 4 \) the bifurcation diagram is considerably more complicated and one can find open sets of parameter values where the only attractor is a robust heteroclinic cycle showing slow oscillations [1] between cluster states of the form \( (\theta, \theta, \phi, \phi) \) with \( \theta \neq \phi \). Rather than discuss this in detail, we refer the reader to Refs. [1, 14]. These attractors consist of a saddle periodic orbits and their one-dimensional unstable manifolds. Typical initial conditions are attracted to this union of connecting orbits, meaning that they oscillate between symmetrically related saddle periodic orbits while showing ‘slowing down’
of the oscillation period. Our numerical calculations of sensitivity as a function of $\alpha$ for $N = 4$ and $r = 0.2$ are given in Fig. 4 and we find a bifurcation giving extreme sensitivity at $\alpha_1 = 1.15928$.

The bifurcation scenario illustrated in Fig. 2 and the transcritical homoclinic bifurcation of [13] lead to extreme sensitivity at the point of bifurcation for the same reason; there are attracting heteroclinic networks that allow winding of trajectories around the torus. The dynamics in [10] shows extreme sensitivity to detuning for larger $N$; however in this case the exceptional nature of the coupling gives non-generic behaviour; there are manifolds of neutrally stable periodic orbits in the dynamics for $N \geq 4$. In our system, the scenarios giving rise the sensitivity are non-degenerate, given the assumption of global coupling.

For $N = 5$ a new phenomenon appears, illustrated in Figure 5. Fig. 5(b) shows that (as found in Ref. [17]) the robust heteroclinic attractor connects one fundamental region of the torus to the next, allowing unbounded growth of phase differences for arbitrarily small detuning. One can contrast this with $N = 4$ where the robust heteroclinic attractor is contained within one fundamental region. Numerical calculations of the system with $N = 5$, $K = 1$, and $r = 0.2$ show that for an interval of values $\alpha \in [1.21, 1.41]$ the trajectories of the system for small detuning $\Delta$ are spuriously attracted by a saddle fixed point [Fig. 5(a)]. This gives non-zero but incorrect values of the sensitivity $\Omega$ that are much reduced by higher precision calculations [Fig. 5(b)]. Hence, for $N = 5$ one can find regions of robust extreme sensitivity that are nonetheless difficult to resolve numerically. We suspect that open regions of robust extreme sensitivity are typical for higher $N$.

In conclusion, we show in this Letter that for moderate $N$ the phenomenon of ‘slow switching’ (robust heteroclinic attractors) is associated with the appearance of extreme sensitivity to detuning. This means that global coupled oscillators, under certain generic types of coupling, will have attractors that respond very sensitively to differences in the oscillators, whether these are differences in detuning, signal or noise inputs, and even if the coupling is neither weak nor degenerate. There are a variety of different bifurcation scenarios that can lead to extreme sensitivity; these are clearly of interest in their own right.

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18. For $N = 2$ the sensitivity $\Omega$ was obtained analytically as detailed in the text. For $N = 3, 4, 5$ the calculations were performed by setting $\Delta_1 = \Delta$ and starting from $\Delta = 0$, where the initial conditions are chosen to show in-phase dynamics (e.g., $\phi_1 = 0, \phi_2 = 3.15$). Increasing the parameter $\Delta$ the synchronized state is followed up to the moment when at least one of the phase differences $\phi_{1,2}$ starts to exhibit unbounded rotations. Such bifurcation values of $\Delta$ are depicted by the appropriate curve.