Tangency Properties of a Pentagonal Tiling Generated by a Piecewise Isometry.

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Abstract

Piecewise Isometries (PWIs) are known to have dynamical properties that generate interesting geometric planar packings. We analyse a particular PWI introduced by Goetz that generates a packing by periodically coded cells, each of which is a pentagon. Our main result is that the tangency graph associated with this packing is a forest (i.e. has no nontrivial cycles). We show however that this is not a general property of PWIs by giving an example that has an infinite number of cycles in the tangency graph of its periodically coded cells.

1 Introduction

Piecewise Isometries (PWIs) are natural generalisations of interval exchange transformations to dimensions higher than one. They appear in the theory of digital filters, Hamiltonian systems and polygonal dual billiards, and have been extensively studied using techniques from dynamical systems and ergodic theory; see for example [13]. In [8], a systematic study of Euclidean PWIs is proposed, focusing on the geometry and symbolic dynamics generated by piecewise isometric systems. Other work [9, 11] examines interesting examples of PWIs defined by piecewise rotations on polygonal regions in the plane, while in [6], it is shown that general PWIs defined on a finite union of polytopes have dynamics with zero topological entropy. In general a PWI will define a partition of phase space into a number of periodically coded cells (consisting of convex polygons or discs) and a remainder, referred to as the exceptional set. There have been several conjectures as to the nature of these periodic cells and the exceptional set that have only been settled for a very limited set of examples where one can perform a detailed renormalisation of the phase space as in [9, 11]. In [16] a general framework is introduced in which piecewise isometric systems can be fully or partially renormalised, but this only characterises a small region in phase space for most PWIs.

A particular conjecture is that typical PWIs have an exceptional set with positive measure but open dense complement [2]. Roughly speaking this suggests that the dynamically defined packing of phase space for typical PWIs is ‘loose’ in that it contains enough gaps for a set of positive measure to be in its complement. Some work towards this conjecture is [2] where it is shown that for typical PWIs in a particular one-parameter family, the tangency graph associated with this packing is trivial, i.e. has no tangencies. On the other hand, [17] shows that certain families of disc packing with infinite numbers of tangencies (Arbelos packings) cannot be generated by PWIs because of issues of finite generation of the tangency directions.
This paper continues this line of investigation by looking at a case where there are non-trivial tangencies. We consider a particular PWI introduced by Goetz [8, 9] on an isosceles triangle with rotations by multiples of $2\pi/5$. For this PWI it is known from previous work that its periodic cells are a full measure set consisting of an infinite number of pentagons whose sizes depend on the scaling properties associated with this PWI. The main goal of section 2 is show that the tangencies between these pentagons are, although nontrivial, quite sparse (we only count tangencies between midpoints of sides of the pentagons as these represent tangencies of inscribing discs that are stably periodically coded). More precisely we show that the tangency graph is such that every vertex is incident to four or five other vertices and is a forest, i.e. an infinite collection of trees with non nontrivial cycles. Nonetheless, we show that lack of cycles is not a necessary condition for tangency graphs of periodic cells for PWIs; in section 3, we introduce a related PWI, the Pie, whose tangency graph contains an infinite number of cycles.

We start with some definitions and properties of PWIs.

**Definition 1.** Let $X \subset \mathbb{R}^m$ be a bounded region and define $M$ to be a partition of $X$ if it is a collection $\{M_1, \ldots, M_n\}$ of disjoint open bounded convex sets such $X = \bigcup_{i=1}^n M_i$. We call an element $M_i$ of a partition an atom. A piecewise isometry in $\mathbb{R}^m$ is a pair $(T, M)$, where $T : X \to X$ is a map such that its restriction $T|_{M_i}$ to each atom $M_i$, $i = 1, \ldots, n$ is an isometry. If $T|_{M_i}$ is a rotation for all $i = 1, \ldots, n$, then $T$ is called a piecewise rotation.

We will be exclusively concerned with a case $m = 2$, meaning we can identify $X$ with a subset of the complex plane $\mathbb{C}$ and rotations can be written $R(z) = \rho z + w$ with $\rho, w \in \mathbb{C}$ and $|\rho| = 1$. The itinerary of a point $x \in X$ is the sequence of atoms visited by the forward orbit. This gives a way of coding the orbit of a point $x \in X$. We define a cell as the set of points with the same itinerary; those that have eventually periodic itinerary are called periodically coded cells or for short periodic cells. It is easy to see that aperiodically coded cells are convex and have zero measure; hence they must be at most a line segment. The exceptional set is the set of all points that are not periodically coded.

If $T : X \to X$ is a piecewise isometry and $S$ any polygonal subset of $X$ (this need not be an atom) we define the map that $T$ induces on $S$, $T_s : S \to S$ to be $T_s(x) = T^{k(x)}(x)$ where $k(x) \in \mathbb{N}$ is first return to $S$, i.e. the smallest value such that $T^{k(x)}(x) \in S$. The induced map of a PWI will be a PWI itself if $k(x)$ is bounded on $S$. The following result relates periodic cells of the induced map to the original.

**Lemma 1.** Suppose $(T, X)$ is a piecewise isometry and $X_i$ be any atom such that $T_{X_i}$ is a piecewise isometry. Let $P$ be a periodic cell for $T_{X_i}$. Then $P$ is a periodic cell for $T$.

**Proof.** If $P$ is a periodic cell for the coding of $T_{X_i} = \tilde{T}$ with period $p$, there must be a $k_0$ such that $\tilde{T}(P) = T^{k_0}(P)$. Similarly $\tilde{T}^2(P) = T^{k_0+k_1}(P)$, $\tilde{T}^3(P) = T^{k_0+k_1+k_2}(P)$ and so on. Thus we have

$$P = \tilde{T}^p(P) = T^{k_0+k_1+k_2+\cdots+k_{p-1}}(P)$$

and hence $P$ is a periodic cell for $T$. □

A standard argument (e.g. [2, 17]) shows that for typical piecewise rotations the periodic cells will be discs because the induced map on the cell is typically an irrational rotation of a convex invariant set. This means that the set of periodic cells for a typical PWI consists of an invariant set of non-overlapping discs; a disc packing. In our case rational rotations mean that the periodic cells are polygons. For an invariant disc packing we say two discs are tangent if their closures have a point in common. This defines a graph where vertices are the centres of discs and there is an edge precisely when two discs are tangent. Similarly, we say two pentagonal cells are tangent if their inscribed discs are tangent.
We say a PWI $T : X \to X$ has a renormalisation to $X' \subset X$ if the map induced by $T$ on $X'$ is conjugate to $T$ by a similarity, i.e. if there is a similarity $\pi$ with $\pi(X') = X$ such that $T_{X'}(x) = \pi^{-1} \circ T \circ \pi(x)$.

1.1 An Example of a Piecewise Isometry

In this section we describe the example of Goetz in [10] illustrated in Figure 1. Let $\alpha = e^{2\pi i/5}$ be a primitive fifth root of unity (note that $\alpha^5 = 1$ and $\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$). Define a triangle $M_0$ with vertices $(0, \alpha + \alpha^2 + \alpha^3, -1)$, a triangle $M_1$ with vertices $(0, -1, \alpha^3)$; one can verify that these describe points as shown in Figure 1.

Definition 2. We define a PWI on $X = M_0 \cup M_1$ by $T : X \to X$ where

$$T|_{M_0} = R_0, \quad T|_{M_1} = R_1 \quad (1)$$

where the rotations $R_0(z) = \alpha^2 z + (\alpha + \alpha^2 + \alpha^3)$ and $R_1(z) = \alpha^3 z + \alpha^3$.

This piecewise isometry has a clear self similar structure of cells that arises from the iteration of the map as depicted in Figure 2. The algebraic structure of the map allows a full understanding of its dynamics as characterised by the following theorem of Goetz.

Theorem 2. [9] Let $T$ be the piecewise rotation (1). Let $E = \alpha^3, \quad F = \alpha + \alpha^2 + \alpha^3$, then triangle $\triangle EF0$ is the union of an infinite number of periodic cells, an exceptional set $R_{\triangle EF0}$ of non-eventually periodic points, and a set of zero Lebesgue measure. The exceptional set has zero Lebesgue measure and its Hausdorff dimension lies strictly between 1 and 2.

A comparable result is obtained by Kahng [14] for a similar map on the torus. In [9], the numerical value of the Hausdorff dimension of the exceptional set is given as 1.00676. However, by considering the equation $k^s + 2(k^2)^s = 1$ associated with it (where $k$ is the inverse of the golden ratio, see [7] for more details), we note that the correct value is $-\log 2 / \log k = 1.44042$.

Theorem 2 implies in particular that the number of non-eventually periodic points is uncountable as, if that was not the case, then their Hausdorff dimension would be zero. The main result we will prove in this paper is the following.

Theorem 3. The tangency graph of the periodically coded pentagonal cells of the piecewise rotation $T$ is a forest.
Figure 2: The self similar structure of the periodic cells generated by the iteration of the map \( T \) on the triangle \( X \). Note that all periodic cells are regular pentagons.

Consequences of this include the observation that the tangency graph is bipartite simply because every tree is bipartite. We will gain a full description of this forest by application of a number of different renormalisations.

2 Renormalisation and tangency properties of the cells induced by \( T \)

We now discuss some dynamical properties of the map \( T \) by considering different renormalisations. The first is the ‘obvious’ renormalisation considered by Goetz [9]; the second is not so obvious but is needed in the proof of Theorem 3.

**Theorem 4.** The map \( T \) has a renormalisation to \( M_1 \).

**Proof.** We define a similarity \( \pi : \mathbb{C} \to \mathbb{C} \) by \( \pi(z) = (\alpha^3 + \alpha^2 + \alpha)(\overline{z} + 1) \) so that \( \pi(M_1) = X \). One can verify that the induced map \( \hat{T} = T_{M_1} \) is a piecewise isometry with atoms (see Figure 3) \( M_{11} \) and \( M_{10} \) that can be renormalised to \( M_1 \) and \( M_0 \) respectively, via the map \( \pi \). In fact the return map to \( M_1 \) on \( M_{11} \) is given by \( \hat{T}|_{M_{11}} = R_0 \circ R_0 \circ R_1 \). Similarly the return to \( M_1 \) on \( M_{10} \), this is given by \( \hat{T}|_{M_{10}} = R_1 \). By direct calculation one can verify that the vertices of \( M_{11} \) under \( \hat{T}|_{M_{11}} \) and \( M_{10} \) under \( \hat{T}|_{M_{10}} \) map around such that \( \hat{T} = \pi^{-1} \circ T \circ \pi \). \( \square \)

2.1 Renormalisation to 11 Triangles

We partition the triangle \( X \) into subsets consisting of 5 pentagons and 11 smaller triangles \( X_i \) obtained by renormalising the original triangle \( X \) such that the midpoints of the bases of the triangles \( X_i \) are the tangency points of the circles inscribed in the pentagons (see Figure 4).

Let \( \pi_i : X_i \to X \) be a similarity such that \( \pi_i(X_i) = X \) and \( \pi_i \) is surjective. In particular \( X \) is scaled by a factor which is \( \gamma^{-k} \) where \( \gamma = (1 + \sqrt{5})/2 \) is the golden ratio and for some
$$\alpha + \alpha + \alpha = \gamma$$

$$1 + \alpha + \alpha - 1 = 0$$

$$M_{11}$$

$$M_{10}$$

Figure 3: The return map induced on the atom $M_1$. In particular we illustrate $M_{11}$ and $M_{10}$ on the left and their images for the induced map on the right.

A positive integer $k$. Observe form Figure 4 that

$$X = \bigcup_{i=0}^{10} X_i \cup \bigcup_{i=0}^{4} \rho_i$$

and will show in Lemma 6 that we can obtain periodic cells by $\rho_0, \ldots, \rho_4$ under repeated application of similarities:

$$\pi_{i_n}^{-1} \circ \pi_{i_{n-1}}^{-1} \circ \cdots \circ \pi_{i_1}^{-1}(\rho_j),$$

where $0 \leq i_k \leq 10$, $1 \leq k \leq n$ and $j \in \{0, 1, 2, 3, 4\}$. We say that periodic cells of this type have level $n$.

**Theorem 5.** The triangle $X$ can be partitioned into the disjoint union of five pentagons $\rho_i$, $i = 0, \ldots, 4$ and eleven triangles $X_i$, $i = 0, \ldots, 10$ such that the map $T$ can be renormalised to each $X_i$ using a similarity $\pi_i$. The induced map on each pentagon $\rho_k$ is periodic, with the cells mapped around by $T(\rho_0) = \rho_0$, $T(\rho_1) = \rho_1$, $T(\rho_2) = \rho_3$, $T(\rho_3) = \rho_4$, and $T(\rho_4) = \rho_2$. Hence the return to each $\rho_k$ is a rotation of order five.

**Proof.** From the definition of the piecewise isometry $T$, one can verify that the itinerary of the triangles $X_i$ for $i = 0, \ldots, 10$ is as shown in Figure 5, where $X \rightarrow Y$ means $T(X) = Y$ and $T$ is surjective and $X \leftarrow Y$ means $T(X) \subset Y$ with $T$ not surjective.

Consider the return to $X_4$: note that $T^4(X_4) = X_2$; on the next iterate, one part of $X_2$ returns to $X_4$ while the other part returns after a further 7 iterates. Hence $T_{X_4}$ is a piecewise isometry with two atoms, namely the preimages of $T(X_7)$ and $T(X_2) \cap X_4$ (see Figure 6). One can verify that this map is conjugate to the original map $T$ via a similarity $\pi_i : X_i \rightarrow X$ such that $\pi_i(X_i) = X$. Also, one can verify that $T(\rho_i) = \rho_i$ meaning that $T|_{\rho_i}$ is a rotation of order five, for $i = 0, \ldots, 4$. 

\[\square\]
Figure 4: Partition of the triangle $X$ into triangles $X_0, X_1, \ldots, X_4$ (of equal size), triangles $X_5, \ldots, X_{10}$ (of equal size but smaller than the previous triangles) and 5 pentagons $\rho_0, \ldots, \rho_4$. Note that the sides of $X_i$ for $i = 0, 1, 2, 3, 4$ are scaled by $\gamma^{-3}$ and the sides $X_i$ for $i = 5, 6, \ldots, 10$ are scaled by $\gamma^{-4}$ with respect to those of $X$, where $\gamma$ is the golden ratio.

Figure 5: Diagram showing the itinerary of the eleven triangles $X_i$ of Theorem 5 under the map $T$. 
Figure 6: The itinerary of the triangles $X_4$ and $X_{10}$ under the PWI defined in Definition 2, defining the return map to $X_4 \cup X_{10}$. 
Lemma 6. Let $T$ be the piecewise isometry defined in Definition 2. Then all periodic cells are pentagons of the form:

$$\pi_{i_k}^{-1} \circ \pi_{i_{k-1}}^{-1} \circ \cdots \circ \pi_{i_1}^{-1}(\rho_j),$$

for $0 \leq i_l \leq 10$, $1 \leq l \leq k$ and $0 \leq j \leq 4$.

Proof. Suppose $T^k(\rho_j) = \rho_j$ and let $\pi = \pi_{i_k} \circ \pi_{i_{k-1}} \circ \cdots \circ \pi_{i_1}$ and $P = \pi^{-1}(\rho_j) = \pi_{i_k}^{-1} \circ \pi_{i_{k-1}}^{-1} \circ \cdots \circ \pi_{i_1}^{-1}(\rho_j)$, for $0 \leq i_l \leq 10$, $1 \leq l \leq k$ and $0 \leq j \leq 4$. Let $B = \pi^{-1}(X)$, then $T|_B = \pi^{-1} \circ T \circ \pi$ and note that

$$T|_B^k(\pi^{-1}(\rho_j)) = (\pi^{-1} \circ T \circ \pi) \circ (\pi^{-1} \circ T \circ \pi) \circ \cdots \circ (\pi^{-1} \circ T \circ \pi) \circ \pi^{-1}(\rho_j)$$

$$= \pi^{-1} \circ T^k(\rho_j) = \pi^{-1}(\rho_j).$$

But $\rho_j$ is a periodic cell for $T$, so $P = \pi^{-1}(\rho_j)$ is periodic with period $k$ for $T|_B$. By Lemma 1 we have that $P$ must also be periodic for $T$.

Let $A$ be the area of $X$; recall from (2) that all the pentagons $\rho_0, \rho_1, \ldots, \rho_4$ are of level 0 and those obtained by scaling them $n$-times are said to be of level $n$. By Theorem 5, the area of all the pentagons at level 0 is given by $A\beta$ for $0 < \beta < 1$ and similarly the area of those at level 1 are given by $A(1-\beta)\beta$ (as the area left is $A(1-\beta)$ and this area is covered by similar copies of $X$). So the total area covered by the pentagons up to and including level $n$ is

$$A\beta + A(1-\beta)\beta + A(1-\beta)^2\beta + \cdots + A(1-\beta)^n\beta$$

Therefore the total area covered by these cells is

$$A\beta \sum_{k=0}^{\infty} (1-\beta)^k = \frac{A\beta}{1-(1-\beta)} = A,$$

meaning that they have full measure $X$ and there are no further convex non-empty periodic cells.

2.2 Properties of the Tangency Graph

This section leads up to the proof of Theorem 3. We first introduce some more similarities as follows:

Definition 3. Define similarities $\Pi_1$, $\Pi_2$ and $\Pi_4$, such that

$$\Pi_1 (X_1 \cup X_5 \cup X_6 \cup \rho_2) = \Pi_2 (X_2 \cup X_7 \cup X_8 \cup \rho_3) = \Pi_4 (X_4 \cup X_{10} \cup X_9 \cup \rho_4) = X.$$

and $\Pi_{\rho_2}$ and $\Pi_{\rho_3}$, such that

$$\Pi_{\rho_2} (X_1 \cup X_5 \cup X_0 \cup X_6 \cup X_3 \cup \rho_2 \cup \rho_1) = \Pi_{\rho_3} (X_7 \cup X_8 \cup X_2 \cup X_3 \cup X_0 \cup \rho_3 \cup \rho_1) = X.$$

Note that we can write $\Pi_{\rho_2}(\rho_2) = \Pi_{\rho_3}(\rho_3) = \rho_1$ and $\Pi_{\rho_2}(\rho_1) = \Pi_{\rho_3}(\rho_1) = \rho_0$.

Lemma 7. Each of the similarities $\Pi_1$, $\Pi_2$, $\Pi_4$, $\Pi_{\rho_2}$ and $\Pi_{\rho_3}$ can be chosen such that there is a renormalisation, i.e. if $\tilde{X} = \Pi_i^{-1}(X)$ where $i \in \{1, 2, 4, \rho_2, \rho_3\}$, then $\tilde{T} = T_{\tilde{X}}$ satisfies $\tilde{T} = \Pi_i^{-1} \circ T \circ \Pi_i$. 

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Figure 7: The representation of the images of $X$ under the renormalisation defined in Definition 3, where (a) refers to the similarities $\Pi_1, \Pi_2, \Pi_4$, (b) to $\Pi_{\rho_2}$ and (c) to $\Pi_{\rho_3}$.
The periodic cells tangent to $\rho_1$ are determined by renormalising $\rho_1$ to the triangles $X_0, X_1, X_2$ and $X_3$, as described in Lemma 8.

**Proof.** Note from Theorem 5, the triangle $X$ can be renormalised to each $X_i$ via a similarity $\pi_i$ for $i = 0, \ldots, 10$ as graphically described in Figure 5. Each similarity in Definition 3 acts on unions of some $X_i$ for $i = 1, \ldots, 10$. One can thus explicitly find the return maps associated to the above similarities.

**Lemma 8.** The only periodic cells tangent to $\rho_1$ are $\pi_i^{-1}(\rho_1), \pi_1^{-1}(\rho_1), \pi_2^{-1}(\rho_1)$ and $\pi_3^{-1}(\rho_1)$ as shown in Figure 8.

**Proof.** As shown in Figure 8, the only cells tangent to $\rho_1$ are those that are located on the base of the triangles $X_1, X_2, X_3$ and $X_0$ such that the midpoints of the bases of the triangles touch the midpoints of the edges of $\rho_1$. We know that the all such triangles are defined as $X_i = \pi_i^{-1}(X)$ for $i = 0, 1, 2, 3$. We have that $X_i, i = 0, 1, 2, 3$ have pentagonal cells on their bases given by $\pi_i^{-1}(\rho_1)$ for $i = 0, 1, 2, 3$.

**Lemma 9.** The only cells that are tangent to $\rho_0$ are $\Pi_i^{-1}(\rho_1)$, for $i = 1, 2, 4$. Moreover $\Pi_{\rho_2}(\rho_1) = \Pi_{\rho_3}(\rho_1) = \rho_0$.

**Proof.** From Figure 4 the only pentagonal cells that are tangent to $\rho_0$ are those at the midpoint of the base of the triangles $(X_1 \cup X_5 \cup X_6 \cup \rho_2)$, $(X_2 \cup X_7 \cup X_8 \cup \rho_3)$ and $(X_4 \cup X_{10} \cup X_9 \cup \rho_4)$. These are renormalised to $X$ via the similarity $\Pi_i$ for $i = 1, 2, 4$. Therefore the tangent cells to $\rho_0$ are $\Pi_i^{-1}(\rho_1)$, for $i = 1, 2, 4$.

**Lemma 10.** Let $\rho_2$ and $\rho_3$ be the pentagonal cells as depicted in Figure 4; then they are of the form $\rho_2 = \Pi_{\rho_2}^{-1}(\rho_1)$ and $\rho_3 = \Pi_{\rho_3}^{-1}(\rho_1)$. Furthermore the only periodically coded cells tangent to $\rho_2$ and $\rho_3$ are

$$
\Pi_{\rho_2}^{-1}(\pi_i^{-1}(\rho_1)), \quad \Pi_{\rho_3}^{-1}(\pi_i^{-1}(\rho_1)),
$$

respectively, for $0 \leq i \leq 3$.

**Proof.** By Definition 3 we have that $\Pi_{\rho_1}(\rho_i) = \rho_1$ for $i = 1, 2$. By Lemma 8, the cells tangent to $\rho_1$ are $\pi_i^{-1}(\rho_1)$, for $i = 0, 1, 2, 3$. Therefore the only tangent cells to $\rho_j$, are $\Pi_{\rho_j}^{-1}(\pi_i^{-1}(\rho_1))$ for $j = 2, 3$, and $0 \leq i \leq 3$. 

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Lemma 11. Let \( \rho_4 \) be the pentagonal cell depicted in Figure 4; then it can be written as \( \Pi_4^{-1}(\rho_0) \). The only periodic cells tangent to \( \rho_4 \) are of the form

\[
\Pi_4^{-1}\left(\Pi_i^{-1}(\rho_1)\right)
\]

for \( i = 1, 2, 4 \).

Proof. From Definition 3, it is clear that \( \Pi_{\rho_2}(\rho_1) = \Pi_{\rho_3}(\rho_1) = \rho_0 \) and that \( \Pi_4^{-1}(\rho_0) = \rho_4 \).

We can immediately see that \( \rho_0 = \Pi_4(\rho_4) \), and by Lemma 9, the only pentagonal cells tangent to \( \rho_0 \) are \( \Pi_i^{-1}(\rho_1) \), for \( i = 1, 2, 4 \).

Therefore the only periodically coded cells tangent to \( \rho_4 \) are of the form

\[
\Pi_4^{-1}\left(\Pi_i^{-1}(\rho_1)\right)
\]

for \( i = 1, 2, 4 \), the result follows. \( \square \)

Definition 4. Let \( P \) be any pentagonal cell defined by the piecewise isometry \( T \) and \( Q \) is a pentagon of smaller size, such that the discs inscribed inside \( P \) and \( Q \) are tangent. Then \( Q \) is said to be a child of \( P \).

We will now show the following result

Theorem 12 (The Four Children Theorem). Any periodic cell for \( T \) has four children except those of the form \( \Pi_4^{-1}((\Pi_4^{-1})^k(\rho_0)) \) for \( k = 0, 1, 2, \ldots \) that have three children.

Note that the pentagons of the form \( (\Pi_4^{-1})^k(\rho_0) \) for \( k = 0, 1, 2, \ldots \) are precisely those cells that are tangent to two boundaries of \( X \).

Proof. We prove the statement by induction. As for the first inductive step, by Lemmas 8, 9, 10 and 11, we know that all the periodic cells tangent to the cells with level 0 are tangent to at most one bigger pentagonal cell. This means we know that the cells tangent to them (i.e. cells of level 2) can be described as follows:

Cells tangent to \( \Pi_i^{-1}(\rho_1) \): (which are tangent to \( \rho_0 \) by Lemma 9), for \( i = 1, 2, 4 \) they are of the form \( \Pi_i^{-1}(\pi_j^{-1}(\rho_1)) \) for \( 0 \leq j \leq 3 \).

Cells tangent to \( \Pi_i^{-1}(\Pi_i^{-1}(\rho_1)) \): (which are tangent to \( \rho_4 \) by Lemma 11), for \( i = 1, 2, 4 \) they are of the form \( \Pi_i^{-1}(\Pi_i^{-1}(\pi_j^{-1}(\rho_1))) \) for \( 0 \leq j \leq 3 \).

Cells tangent to \( \pi_i^{-1}(\rho_1) \): (which are tangent to \( \rho_1 \) by Lemma 8), for \( 0 \leq i \leq 3 \) they are of the form \( \pi_j^{-1}(\pi_i^{-1}(\rho_1)) \), for \( 0 \leq i, j \leq 3 \), and

Cells tangent to \( \Pi_i^{-1}(\pi_i^{-1}(\rho_1)) \): (which are tangent to \( \rho_1 \) by Lemma 10), for \( 0 \leq i \leq 3 \), \( l = 2, 3 \) they are of the form \( \Pi_l^{-1}(\pi_i^{-1}(\pi_j^{-1}(\rho_1))) \) for \( 0 \leq i, j \leq 3 \), \( l = 2, 3 \).

All the above renormalisations result in children that are not of the form \( (\Pi_4^{-1})^k(\rho_0) \) for \( k = 0, 1, 2, \ldots \) meaning they will also have four children. Thus we know that the first induction step is satisfied. Assume the theorem is true for all the cells with level \( n \). Consider any cell \( T \) of level \( n + 1 \): by Lemma 6, it can be written as \( \pi_{i_k}^{-1} \circ \pi_{i_{k-1}}^{-1} \circ \cdots \circ \pi_{i_1}^{-1}(\rho_j) \), for \( 0 \leq i_l \leq 10, 1 \leq l \leq k \) and \( 0 \leq j \leq 4 \). If \( T \) is tangent to at most one boundary of \( X \), then by Lemma 10, it can be written as one of the following

\[
\pi_{i_{n+1}}^{-1} \circ \pi_{i_n}^{-1} \circ \cdots \circ \pi_{i_1}^{-1}(\rho_1)
\]

\[
\pi_{i_{n+1}}^{-1} \circ \pi_{i_n}^{-1} \circ \cdots \circ \pi_{i_1}^{-1} \circ \Pi_{\rho_j}^{-1}(\rho_1) \text{ for } j = 2, 3
\]
Therefore the cells of level \( n+2 \) tangent to it are of form of one of the following

\[
\pi_{i_{n+1}}^{-1} \circ \pi_{i_n}^{-1} \circ \cdots \circ \pi_{i_1}^{-1}(\pi_k^{-1}(\rho_1))
\]

\[
\pi_{i_{n+1}}^{-1} \circ \pi_{i_n}^{-1} \circ \cdots \circ \pi_{i_1}^{-1} \circ \Pi_{\rho_i}^{-1}(\pi_k^{-1}(\rho_1))
\]

for \( j = 2, 3 \) and \( 0 \leq k \leq 3 \). Alternatively if \( T \) is tangent to two boundaries of \( X \), then \( T \) is of the form \((\Pi_4^{-1})^{k+1}(\rho_0)\) and the cells tangent to it are of the form \((\Pi_4^{-1})^{k+1}(\Pi_i^{-1}(\rho_1))\) for \( i = 1, 2, 4 \).

**Theorem 13.** Let \( G_{\rho_1} \) be the connected component of the tangency graph containing \( \rho_1 \). Then \( G_{\rho_1} \) is a tree where the vertex corresponding to \( \rho_1 \) has degree 4 and all the other vertices have degree 5.

**Proof.** Obviously \( \rho_1 \) is a cell that has level 0; by Lemma 8, the cells tangent to \( \rho_1 \) are \( \pi_i^{-1}(\rho_1) \) for \( 0 \leq i \leq 3 \) and these have level 1. We can now attach to each of the cells \( \pi_i^{-1}(\rho_1) \) for \( 0 \leq i \leq 3 \), four cells of level 2. This is possible because each \( X_i \), for \( i = 0, 1, 2, 3 \), can be renormalised to \( X \) via the similarity \( \pi_i \); then the cells of level 2 are of the form

\[
\pi_{i_2}^{-1} \circ \pi_{i_1}^{-1}(\rho_1)
\]

for \( 0 \leq \pi_i \leq 3 \).

We can continue the above reasoning and the cells within \( G_{\rho_1} \) will be of the form

\[
\pi_{i_k}^{-1} \circ \pi_{i_{k-1}}^{-1} \circ \cdots \circ \pi_{i_1}^{-1}(\rho_1)
\]

where \( 0 \leq i_l \leq 3 \), for \( 1 \leq l \leq k \).

Now suppose there is a cycle \( C \) in the connected component of the tangency graph \( G_{\rho_1} \); without loss of generality, pick a cell in the cycle that has smallest area of all in the cycle and this must have at least two neighbours that are equal or larger. This contradicts the four children result for this cell and so there can be no cycles. \( \square \)

**Theorem 14.** Let \( G_{\rho_0}, G_{\rho_2}, G_{\rho_3}, G_{\rho_4} \) be the connected components of the tangency graphs containing \( \rho_0, \rho_2, \rho_3 \) and \( \rho_4 \) respectively. Then they are all trees such that \( G_{\rho_0} \) and \( G_{\rho_4} \) where the vertices corresponding to \( \rho_0 \) and \( \rho_4 \) have degree 3; whereas \( G_{\rho_2} \) and \( G_{\rho_3} \) have degree 4 at the vertices to \( \rho_2 \) and \( \rho_3 \). All other vertices in any of the above graphs have degree 5.

**Proof.** We first consider \( G_{\rho_0} \). The pentagonal cell \( \rho_0 \) is tangent to 3 periodic cells such that they are located on the midpoint of one of the edges of the triangles \( X_1, X_2 \) and \( X_4 \) as depicted in Figure 4. These cells of level 1 can be written as \( \pi_1^{-1}(\rho_0), \pi_2^{-1}(\rho_0) \) and \( \pi_4^{-1}(\rho_0) \). By Definition 3, it is easy to see that \( \pi_i^{-1}(\rho_0) = \Pi_i^{-1}(\rho_1) \) for \( i = 1, 2, 4 \). Then by Theorem 13, all the cells within \( G_{\rho_0} \) are of the form

\[
\pi_{i_k}^{-1} \circ \pi_{i_{k-1}}^{-1} \circ \cdots \circ \pi_{i_1}^{-1}(\rho_1)
\]

for \( 0 \leq \pi_i \leq 3 \), and \( 1 \leq l \leq k \). Therefore all the cells within \( G_{\rho_0} \) are of the form

\[
\Pi_i^{-1} \left( \pi_{i_k}^{-1} \circ \pi_{i_{k-1}}^{-1} \circ \cdots \circ \pi_{i_1}^{-1}(\rho_1) \right)
\]

where \( i = 1, 2, 4, \) \( 0 \leq \pi_i \leq 3 \), and \( 1 \leq l \leq k \). Therefore \( G_{\rho_0} \) is a tree.

Now consider \( G_{\rho_2} \) and \( G_{\rho_3} \). Note that both \( G_{\rho_2} \) and \( G_{\rho_3} \) have the vertices corresponding to \( \rho_2 \) and \( \rho_3 \) with degree 4 whereas all the others have degree 5. Recall \( \Pi_i^{-1} \) from Definition 3. Note that \( X_i \cup X_5 \cup X_0 \cup X_6 \cup X_3 \cup \rho_2 \cup \rho_1 \) is equal to the atom \( M_0 \). Then clearly \( \Pi_i^{-1}(\rho_1) = \rho_i \), for \( i = 2, 3 \). Thus the cells within \( G_{\rho_2} \) and \( G_{\rho_3} \) are of the form

\[
\Pi_{\rho_i}^{-1} \left( \pi_{i_k}^{-1} \circ \pi_{i_{k-1}}^{-1} \circ \cdots \circ \pi_{i_1}^{-1}(\rho_1) \right),
\]
for \( i = 2, 3, 0 \leq i_l \leq 3, \) and \( 1 \leq l \leq k. \) Therefore, by Theorem 13, \( G_{\rho_2} \) and \( G_{\rho_3} \) are trees.

Finally consider \( G_{\rho_4}. \) Clearly \( G_{\rho_4} \) has the vertex corresponding to \( \rho_4 \) with degree 3 and all the other vertices have degree 5. Furthermore the cells within \( G_{\rho_4} \) are of the form

\[
\Pi^{-1}_4 \left( \Pi^{-1}_i \left( \pi^{-1}_{i_k} \circ \pi^{-1}_{i_{k-1}} \circ \cdots \circ \pi^{-1}_{i_1}(\rho_1) \right) \right)
\]

where \( i = 1, 2, 4, 0 \leq i_l \leq 3, \) and \( 1 \leq l \leq k. \) Therefore, by Theorem 13, \( G_{\rho_4} \) is a tree, and the result follows.

We can now conclude with the proof of our main result

**Proof of Theorem 3.** By Theorems 13 and 14, we know that \( G_{\rho_j} \) is a tree for \( 0 \leq j \leq 4. \) Note that each of the cells \( \rho_j \) for \( 0 \leq j \leq 4, \) are cells of level 0.

Define \( G^{(1)}_{\rho_j} \) to be the tangency graphs of the cells within \( \pi^{-1}_i(G_{\rho_j}) \), for \( 0 \leq j \leq 4 \) and \( 0 \leq i \leq 10. \) Note that some of the cells are already contained in \( G_{\rho_j} \) for \( 0 \leq j \leq 4. \)

Continue the above reasoning in order to get \( G^{(n)}_{\rho_j}. \) Then define the following

\[
G = \bigcup_{n \geq 0} G^{(n)}_{\rho_j}
\]

and note this is a (not necessarily disjoint) union of trees by Theorems 13 and 14. Pick any connected component \( \tilde{G} \) of \( G \) and consider the largest cell \( \rho \) in \( \tilde{G}. \) This must be a pentagon of the form \( \Pi^{-1}_{\rho_j} \) for some \( j \) and a similarity \( \Pi^{-1}. \) However the tangency graph from this pentagon will be of the form \( \Pi^{-1}(G_{\rho_j}) \) which is a tree. Hence \( G \) is a disjoint union of trees.

By Theorem 3, the tangency graph is a union of trees; hence it is bipartite (see e.g. [4]).

## 3 A Piecewise Isometry with Cycles in the Tangency Graph

In the previous section we analysed the tangency graph associated with a particular example of a piecewise isometry, showing that it is a forest. It is natural to ask whether this is a general property for any piecewise isometry. As we will see in this section, the answer is negative; in fact a counterexample is the Pie. We note that in this case, Theorem 12, no longer holds.

**Definition 5.** Let \( X \) be the triangle in Definition 2 generating the self similar structure depicted in Figure 2. Define the Pie as the PWI shown in Figure 9 acting on the union of 10 isometric copies of \( X. \)

Note that since the Pie is a union of copies of \( T \) acting on copies \( X, \) the periodically coded cells are just copies of those for \( T \) as depicted in Figure 10.

It is clear from Figure 10, that although one can verify that the tangency graph of the cells of the Pie is bipartite, it is by no means a forest and contains an infinite number of cycles:

**Theorem 15.** Consider the Pie PWI as defined in Figure 9. The tangency graph \( G_{\text{Pie}} \) contains an infinite number of cycles of length 10.

**Proof.** Note that each of the “slices” of the Pie is the isosceles triangle depicted in Figure 1. In particular the atoms in Figure 9 are of the form \( M_{0a} \) and \( M_{1b} \) where \( a, b \in \{0, 1, \ldots, 9\}; \) it is clear that the former correspond to the atom \( M_0 \) and the latter to \( M_1 \) as shown in Figure 1. Therefore \( A \) is a piecewise isometry.

In Figure 10 we clearly see the self similar structure of the periodic cells of the Pie. Note that this has a series of tangencies between each circle inscribed in the biggest pentagonal cell of each slice (i.e. the pentagon \( P_0 \) as shown in Figure 2). This is explicitly indicated in Figure 10 by the shaded pentagons. The result follows.
Figure 9: The piecewise isometry acting on the Pie as defined in Definition 5. Note the each ‘slice’ is an isosceles triangle identical to the original triangle $X$ depicted in Figure 1. The action of $A$ on the atoms $M_{1i}$ and $M_{0i}$ for $i = 0, \ldots, 9$, generates a self-similar structure as shown in Figure 10.
4 Other Geometric Properties of the Tangency Graph

PWIs have been extensively investigated in terms of their dynamical properties: they however generate interesting geometrical configurations which could be analysed per se. In particular their dynamical properties can be used to emphasise some properties otherwise difficult to pinpoint by using other techniques. In fact in sections 2.2 and 3, we have considered some topological properties of the tangency graphs: in order to prove Theorem 3, we have exploited the dynamics associated with the tangency graph. In section 4.2, we will investigate some further fractal properties of the tangency graph; in particular we will see that we can produce different fractal configurations such that their Hausdorff dimensions are different from the dimension of the exceptional set considered by Goetz [9].

4.1 Geometric Properties of the Edges of The Tangency Graph

In section 2.2 we have investigated some combinatorial properties of the tangency graph associated to the cell structure generated by the piecewise isometry (2): however this graph is embedded in the plane and has a well defined geometric structure. Furthermore it has the following property (see [15] for more details):

Lemma 16. Let $G$ be the tangency graph associated to the piecewise isometry (2). Then $G$ is an algebraic graph, i.e. the length of its edges is an algebraic number.

Proof. Since the vector between the vertices of any tangent pentagons is in $\mathbb{Q}[\alpha]$, then the length $\| \cdot \|$ of a vector $v_{i,j}$ between $P_i$ and $P_j$ is

$$d_{i,j} = \| v_{i,j} \| = (v_{i,j} \cdot v_{i,j})^{1/2}$$

Which is clearly real. Then from Proposition 1.1 page 224 in [12], $d_{i,j}^2$ is algebraic as $\mathbb{Q}[\alpha]$ is a finite extension of $\mathbb{Q}$; in other words it is the root of a polynomial in $\mathbb{Q}[x]$. In fact there is a polynomial

$$D(x) \in \mathbb{Q}[x]$$
such that \((x - d_{i,j}^2)\) is a factor of \(D(x)\). Then if we consider \(z^2 = x\) then \((z^2 - d_{i,j}^2) = (z - d_{i,j})(z + d_{i,j})\) is a factor of \(D(z^2) \in \mathbb{Q}[z]\). Hence the result follows.

\[\Box\]

4.2 Some Fractal Properties of the Adjacency Graph

In [9] the Hausdorff dimension of the exceptional set is investigated as reproduced in Theorem 2; however this is not the only interesting geometric subset defined by the piecewise isometry (2). We compute the Hausdorff dimension of the limit points of the tangency graph and we will investigate the Cantor set generated by removing the edges of the pentagons that lie on one side of another pentagon.

Consider the tangency graph associated to the circle packing generated by inscribing circles inside the pentagonal cells generated by the piecewise isometry in Definition 2, and pick a connected component of this. The centres of the tangent pentagons have a limit \(\Lambda_1\). On the other hand, if we examine one side of a pentagon and examine all parts that are not shared with a side of another pentagon, this defines another set \(\Lambda_2\); see Figure 11. For these Cantor sets one can compute

\[
\dim_H(\Lambda_1) = \frac{\log 4}{3 \log \gamma} = 0.9603, \quad \dim_H(\Lambda_2) = \frac{\log 2}{2 \log \gamma} = 0.4618.
\]

by noting that in the first case the dimension satisfies \(4\gamma^{-3s} = 1\) where \(\gamma = \frac{1 + \sqrt{5}}{2}\) while in the second case it satisfies \(2\gamma^{-2s} = 1\).

5 Conclusions

PWIs have been deeply investigated mainly using techniques taken from dynamical systems. In this paper we have analysed some combinatorial properties associated with the tangency graph for a specific example of a PWI: our main goal is to consider a property that is not usually linked with dynamics. In fact it suggests an interesting relationship between geometric dynamical systems and other topics such as packings and tilings, geometric graph theory and algebra. In particular the dynamical renormalisation structure associated with the PWI \(T\) is essential in analysing the combinatorial properties of its tangency graph.

It would be natural to try to extend our approach to more general piecewise isometric systems, however there are several problems. For general PWIs, the exact scaling properties of the cells are not well understood. In [11] a piecewise isometry acting on an isosceles triangle with angles \((\pi/7, 3\pi/7, 3\pi/7)\) is studied: it is by no means obvious whether it is possible to exploit the same method as in this paper to investigate the properties related to its tangency graph.
In fact there is generally a lack of general results when dealing with PWIs. However in [16], a more general renormalisation scheme is introduced which could help to analyse more examples and give more a useful insight into some general properties of PWIs.

In particular, what is the general structure of the tangency graph associated with general PWIs? In this paper we examine two examples: the first has no cycles in its tangency graph, whereas the second has an infinite number of them. In [2], Ashwin and Fu show that the tangency graphs associated with some PWIs are trivial (i.e. have an empty edge-set). This suggests that in general PWIs tend to have graphs with disconnected components. Is therefore the Pie one of the few examples where graphs exhibit a more complex structure? How do perturbations of a PWI influence its tangency graph? Would we always have a graph that is very poorly connected?

In this paper we have defined the tangency graph with respect to the tangencies between discs (or circles) inscribed inside the periodic cells. Is it possible to broaden this definition in order to embrace other tangency properties or to infer other dynamical or geometric properties of the exceptional set from this?

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References


