Infinities of stable periodic orbits near robust cycling in coupled cell systems

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We consider the dynamical behaviour of coupled cell systems with robust heteroclinic cycles to saddles that may be periodic or chaotic. We differentiate attracting cycles into types that we call phase-resetting and free-running depending on whether the cycle approaches a given saddle along one or many trajectories. At loss of stability of attracting cycling, we show in a phase-resetting example the existence of an infinite family of periodic orbits that accumulate on the cycling, whereas for a free-running example loss of stability of the cycling gives rise to a single quasiperiodic or chaotic attractor.

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Dynamical systems where an invariant subspace, or set of subspaces, are preserved because of symmetry or constraints on a system give rise to a number of new types of robust behaviour (i.e., behaviour that is robust to perturbations that preserve the structure) that would be highly degenerate for systems without these symmetries or constraints. There is an extensive literature discussing theory and examples of this for blowout, ruddling, bubbling and related phenomena [1].

The phenomenon of structurally stable heteroclinic cycles to equilibria for symmetric systems is well documented for flows and maps [2]. Heteroclinic cycles between chaotic saddles are also robust in systems with symmetry; these have been found in coupled cell systems [3] and models of planar magnetoconvection [4], the latter being a cycle alternating between equilibria and chaotic saddles. In general we expect this kind of behaviour in symmetric physical systems of sufficient complexity.

As expected by analogy with cycles to equilibria for flows, whether such cycling to periodic orbits or chaotic saddles is an attractor or not can be determined by examining the ratios of Lyapunov exponents at the saddles, at least for the model cases discussed in [4, 5], meaning that loss of stability will correspond to a resonance of Lyapunov exponents. Numerical simulations in [4] for a flow suggest that the resonance creates a large number of periodic attractors that branch from the cycling chaos. By contrast, for the skew-product example of cycling chaos examined in [5] the resonance was shown to give rise not to periodic orbits but to a chaotic attractor with average cycling chaos or quasiperiodicity that is intermittent (‘stuck on’) to the cycling chaos.

In this note, we aim to reconcile these differences by characterising them as examples of qualitatively different types of cycling. For what we call ‘phase-resetting’ cycling, there is only one approach trajectory towards each saddle within the cycle. For what we call ‘free-running’ cycling, there are multiple approaches to a single saddle.

We consider two systems of coupled iterated maps where the attraction of the cycling is determined by the strength of the coupling. At the bifurcation point where the cycling loses stability there is a resonance of Lyapunov exponents. These maps can be related to the case for flows in the usual way by considering them as arising as a Poincaré return map, and noting that equilibria and periodic points for a map both correspond to periodic orbits for a flow.

At the resonance bifurcation, for the ‘phase-resetting’ case we find a plethora of stable high-periodic orbits with an infinite number of stable periodic orbits accumulating at resonance. For the ‘free-running’ case the branching attractors are typically unique and quasiperiodic or chaotic. We observe no other scenarios for these models but believe there will be other scenarios for problems with higher dimensional saddles and connections.

Model I is a map of $[0, 1]^3$ with $Z_3$ symmetry given by

$$ (x_{n+1}, y_{n+1}, z_{n+1}) = F_I(x_n, y_n, z_n), $$

where

$$ F_I(x, y, z) = (f(x)e^{-y}, f(y)e^{-z}, f(z)e^{-x}), $$

and $f(z) = rz(1 - x)$ denotes the logistic map with parameter $r \in [0, 4]$. This map clearly preserves the coordinate planes $xyz = 0$. In each variable three distinct types of evolution are possible. For example, consider $x$: if $z \ll 1$ and $x \ll 1$ then $x$ grows approximately linearly - the growing phase. For $z \ll 1$ and $x \approx O(1)$, $x$ evolves according to logistic map dynamics - the active phase. Finally if $z \approx O(1)$ the dynamics in the $x$ direction is suppressed by the coupling term – the decaying phase.

Model II is identical to Model I except that the logistic map $f$ during a growing phase is replaced by $\hat{f}$:

$$ \hat{f}(x_n) = \begin{cases} f(x_n) & x_n < \epsilon \text{ or } x_n > f(\epsilon) \\ f^2(\epsilon) = \eta & x_n \in [\epsilon, f(\epsilon)]. \end{cases} $$

Each time a growing variable reaches the interval $[\epsilon, f(\epsilon)]$ (we use $\epsilon = 10^{-6}$), it is set to $\eta = f^2(\epsilon)$. This phase-
resetting’ is introduced to force trajectories leaving one saddle to approach the next one close to a single trajectory, as observed in cycling that alternates between equilibria and chaos for flows [4].

For both models the coupling is trivial when $\gamma = 0$. When the coupling parameter $\gamma$ is sufficiently strong the models both exhibit robust cycling between invariant sets. In this state, each variable cycles between the growing, active and decaying phases. We term a change in the phases a ‘switch’. More precisely, we say a switch occurs when the growing variable exceeds $\ln r/\gamma$. As in [4], for both models, the number of iterations between switches increases geometrically as trajectories approach the invariant subspaces. Increasing the coupling causes the rate of increase of the switching times to increase. Conversely, the rate of increase of switching times approaches zero as $\gamma$ approaches some critical value, which forms the limit of the cycling chaos stability.

The behaviour in the active phase is governed by $r$. For $r < 3$ the cycles are between period one points; as $r$ is increased, (after period doubling) we obtain cycles progressively between periodic orbits and then chaotic saddles. Since numerical simulations of this system need to resolve a neighbourhood of the invariant subspaces very clearly, we use logarithmic coordinates [4]. Fig. 1 shows a time series for Model I of three variables at parameters that produce attracting cycling chaos. The number of iterations spent in each phase grows and the orbit approaches the invariant subspace; Model II produces similar behaviour at this parameter value.

Suppose that cycling chaos loses stability on decreasing $\gamma$ through a critical value $\gamma_c$. We can compute $\gamma_c$ analytically, either from a resonance condition of Lyapunov exponents, or as follows. Suppose that a switch has just occurred and the growing variable is $z$, so $z \ll 1$, $z$ is $O(1)$ and $y$ is decaying. The evolution of $z$ is governed by $z_{n+1} = rz_n(1 - z_n)e^{-\gamma y_n}$, and this can be approximated by $z \to rz$. Starting at a switch at $z = z_0$, suppose that the number of iterations until the next switch is $N$. Then $z_N \approx r^N z_0$, and since $z_N$ is $O(1)$ at a switch, $N \approx -\ln z_0/\ln r$. While $z$ is growing, $y$ is decaying, and for critical $\gamma$ we require $y_N = z_0$. We approximate $y_N$ in a similar way, with $y_0$ an $O(1)$ number. Throughout the decay phase $y \ll 1$ but it is forced by the active variable $x$. Here we approximate by $y \to rye^{-\gamma y}$, and replace $x$ by its long-term average $A_\infty = (\lim_{m \to \infty} \frac{1}{m} \sum_{i=0}^{m-1} f^i(x_0))$ for each of the $N$ iterations, giving $y_N \approx r^N e^{-\gamma N A_\infty}$. Then substituting our expression for $N$, we have $\ln y_N \approx -\ln z_0 + (\gamma \ln z_0 A_\infty)/(\ln r)$. The critical value of $\gamma$ occurs when $y_N = z_0$, giving $\gamma_c = 2\ln r/A_\infty$. The average $A_\infty$ is easy to compute numerically, and so we obtain a curve of critical $\gamma$ shown in Fig. 2. The critical $\gamma$ for Model II can be found as for Model I because the dynamics in the invariant subspace $y = z = 0$ and its linearization about that subspace is identical to Model I.

One of the questions raised in [4] is what sort of attractors branch from cycling chaos at resonance. In that paper, numerical evidence was presented suggesting that the cycling chaos gives way to long-period periodic orbits made up of segments of chaotic trajectories that start in exactly the same way after every switch. In Model I this does not occur; for $\gamma < \gamma_c$ we find irregular cycling in which the number of iterations between switches behaves erratically. Model II, like Model I, exhibits attracting cycling above the resonance value $\gamma_c$. However, for $\gamma < \gamma_c$ we find existence of many periodic orbits, consisting of cycles between either periodic points or chaotic trajectories (depending on the value of $r$). We argue that this multistability of long period orbits is caused by, and is typical for, cycling with phase-resetting approach to chaotic saddles.
FIG. 3: Schematic diagram of a periodic orbit of period $3N$ for Model II; one third of a period is shown. This is a periodic orbit as the final and initial phases match up as shown. The iterate $k$ shows where the phases switch.

For the remainder of this note, we investigate these periodic orbits by carefully considering the evolution of the variables over one third of a periodic orbit as shown in Fig. 3. (Throughout, the period of the orbit will be $3N$.) We assume that $x$ has just reset to $x_n = \eta$ at $n = 0$, so that $y$ is the active variable and $z$ is in the decay phase. For a periodic orbit of period $3N$ to be possible, we require that $z_N = \eta$, i.e., that $z_{N-1} \in [\epsilon, f(\epsilon)]$. We take $y_k = \alpha$, where $\alpha$ is either some $O(1)$ number $\tilde{A}$ (for a rough estimate), or more precisely takes the value $f^{N+k}(\eta)$ (since $y_0 = x_N \approx f^N(\eta)$). There follow $N$ iterates of forced decay. We approximate this by $y_{N+k} = r^N y_k e^{-\gamma N \beta}$, where $\beta$ approximates the suppressing effect of the forcing. Again, for a rough estimate, we take $\beta$ to be the long-term average $A_{\infty}$, but for a more accurate estimate we take $\beta$ to be the $N$-average $A_N = \frac{1}{N} \sum_{i=0}^{N-1} f_i(f^k(\eta))$. Since this is a periodic orbit, $y_{N+k} = z_k = r^N \alpha e^{-\gamma N \beta}$. Finally we have $(N - k - 1)$ iterations of growth, approximated by $z \rightarrow rz$. This gives $z_{N-1} = r^{2N-k-1} \alpha e^{-\gamma N \beta}$. Taking logarithms, this estimate predicts that a periodic orbit will exist when

$$\ln \epsilon < (2N - k - 1) \ln r + \ln \alpha - \gamma N \beta < \ln \epsilon + \ln r;$$

that is, for the rough estimate $\alpha = \tilde{A}$, $\beta = A_{\infty}$, for

$$N \in [N_1, N_2] = \left[ \frac{a}{2 \ln r - \gamma A_{\infty}}, \frac{a + \ln r}{2 \ln r - \gamma A_{\infty}} \right],$$

where $a = \ln \epsilon - \ln \tilde{A} + (k + 1) \ln r$. This defines a pair of hyperbolas between which $N$ must lie for a periodic orbit to exist, and suggests that all $N \in [N_1, N_2]$ should be present. Both $\epsilon$ and $r$ are fixed, and $k$ (the number of iterations from the resetting point to the next switch) can be calculated. To estimate the latter, we consider the number of iterations to take $x_0 = \eta$ to $x_k > \ln r / \gamma$ under the approximation $x_{n+1} \approx r x_n$ giving $k \approx (\ln \ln r / \gamma) / \ln r$. In other words, the hyperbolas are governed by a single fitting parameter $\tilde{A}$. Note that the denominators in these expressions equal zero when $\gamma = 2 \ln r / A_{\infty} = \gamma_c$.

FIG. 4: Predicted and actual period $3N$ orbits for $r = 3.2$, $\gamma = 3.495$ for Model II. The squares indicate the location of periodic orbits; the line is the approximation of $z_N$; if it lies in the band $[\epsilon, r \epsilon]$ we predict that the resetting will lead to a stable periodic orbit. The inset shows the hyperbola (plotted as lines) predicted by the rough estimate, with actual periodic orbits (plotted as points) lying between them ($\tilde{A} = 2.8$).

FIG. 5: Predicted and actual periodic orbits for $r = 3.75$, $\gamma = 4.01$ for Model II. As in Fig. 4, squares represent stable periodic orbits and the line is the approximation of $z_N$.

This estimate works well for the case in which the active phase of the maps is a period one point – i.e., for $r < 3.0$. Numerically located periodic orbits lie within the predicted hyperbola. In particular, for a given $\gamma$ we obtain all periods for $N$ between $N_1$ and $N_2$ with a suitable choice of $\tilde{A}$. For more complicated behaviour within the invariant subspaces this estimate works less well. As $r$ is increased the logistic map undergoes period doubling. For values of $r$ in this region the numerically located periodic orbits still lie roughly between the predicted hyperbola. However we no longer find all periods for $N$ in $[N_1, N_2]$; some are not present near $N_1$ and $N_2$.

As $r$ increases further, the saddles become chaotic and the bifurcation diagram of periodic orbits gets more complicated. In this case we use the improved estimate with $\alpha = f^{N+k}(\eta)$, $\beta = A_N$. This gives the estimate

$$z_{N-1} = r^{N-k-1} f^{N+k}(\eta) e^{-\gamma N A_N}.$$

(1)
For fixed $r$, $\gamma$, $\epsilon$ and $\eta$, $z_{N-1}$ is a function only of $N$ and there are no free parameters. Fig. 4 plots $\ln z_{N-1}$ for different $N$ for $r = 3.1$, and shows how it successfully predicts periodic orbits when the line falls within the band defined by $[\ln \epsilon, \ln \epsilon + \ln r]$. The squares on the diagram represent actual periodic orbits. The inset shows the hyperbolic from the simple approximation working reasonably well.

For values of $r$ that give chaotic dynamics within invariant subspaces, the situation is more complicated, but the improved approximation still does a good job of predicting periodic orbits. The approximation for $z_{N-1}$ is plotted in Fig. 5 for $r = 3.75$ and $\gamma = 4.01$. Fig. 6 is a bifurcation diagram of periodic orbits present for the chaotic case $r = 3.75$, together with the predicted envelope. The actual periodic orbits fit well inside the prediction, with the exception of some longer period orbits lying above the envelope. These tend to be orbits which just fail to jump up and make a period $3N$ orbit, but instead become periodic after $6N$ iterations. Using this method to create an envelope not only gives a good way to predict the location of periodic orbits, but again makes clear that upon approaching $\gamma_c$ we expect to find periodic orbits of increasing period. For $\gamma = \gamma_c$ the chaotic curve of $\ln z_{N-1}$ against $N$ neither increases nor decreases on average (cf. Fig. 5), but the fluctuations, driven by the $NA_N$ term in Eq. (1), can be expected to increase. Hence we expect periodic orbits of arbitrarily high period as the curve repeatedly crosses the band $[\epsilon, r\epsilon]$. For $\gamma$ close to $\gamma_c$ (above or below) the fluctuations for $N$ large lead to possible long periodic orbits, but eventually the linear average behaviour leads the curve away from the band.

In the phase-resetting case (Model II), the qualitative dynamics is independent the value of $\eta$. The presence of the multiplicity of periodic orbits presents an intriguing parallel between this model and the persistent phenomenon of 'Newhouse sinks' in a neighbourhood of a homoclinic tangency [6]. One difference is that even in the simpler case of robust cycling between periodic points in Model II, the tangency between unstable and stable manifolds will be degenerate owing to the invariant manifolds containing the connections. Another difference is that the stable periodic orbits in this case are easy to locate numerically and indeed there appear to be no other attractors nearby. The mechanism that creates the periodic orbits in Model II resembles that found by Chawanya [7] near a robust heteroclinic network containing connections to a heteroclinic cycle. Model II is artificial in that it has a discontinuity at the phase resetting step. This means that the periodic orbits typically bifurcate from this discontinuity in a degenerate way. However, one can clearly remove this problem by smoothing out the discontinuity.

In summary, we have demonstrated how the absence (Model I) or presence (Model II) of phase-resetting in the connections between saddles of a cycling (robust heteroclinic) attractor can cause qualitatively different behaviours at loss of stability of the attractor by resonance of Lyapunov exponents, even though the behaviours for attraction are similar. These models are instructive in that they are simple enough to allow a precise estimation of the location of periodic orbits while having what we believe are the main features of robust types of dynamical behaviour in flows.

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