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RESEARCH PAPER

Optimal design of periodic structures using evolutionary topology optimization

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Abstract This paper presents a method for topology optimization of periodic structures using the bi-directional evolutionary structural optimization (BESO) technique. To satisfy the periodic constraint, the designable domain is divided into a certain number of identical unit cells. The optimal topology of the unit cell is determined by gradually removing and adding material based on a sensitivity analysis. Sensitivity numbers that consider the periodic constraint for the repetitive elements are developed. To demonstrate the capability and effectiveness of the proposed approach, topology design problems of 2D and 3D periodic structures are investigated. The results indicate that the optimal topology depends, to a great extent, on the defined unit cells and on the relative strength of other non-designable part, such as the skins of sandwich structures.

Keywords Topology optimization · Periodic structure · Sandwich structure · Bi-directional evolutionary structural optimization (BESO)

1 Introduction

Structural topology optimization has become an effective design tool for obtaining more efficient and lighter structures. In recent years, various optimization methods such as the solid isotropic mircostructure with penalization method (Bendsøe 1989; Zhou and Rozvany 1991; Mlejnek 1992) and the evolutionary structural optimization (ESO) method (Xie and Steven 1993; 1997) have found increasing applications. The ESO method was originally introduced by Xie and Steven (1993) to obtain the optimum shape and topology of continuum structures. In this method, inefficient material is slowly removed from the design domain. Bi-directional evolutionary structural optimization (BESO) is an extension of ESO, which allows for material to be added to the structure where it is most needed at the same time as the inefficient material is being removed. The BESO method proves to be more robust than the ESO method (Yang et al. 1999; Huang and Xie 2007).

Periodic structures, e.g. the honeycomb core of a sandwich plate, are widely used in the structural designs because of their lightweight and ease of fabrication (Wadley et al. 2003). A lightweight cellular material usually possesses periodic mircostructures. An inverse homogenization method was proposed by Sigmund (1994) and Sigmund and Torquato (1997) using periodic boundary conditions to tailor effective properties of cellular materials. As a result, the cellular material with extreme macroscopic properties has been found. However, the design of macrostructures with periodic geometries, e.g. the core of a lightweight sandwich for the mean compliance minimization, is different from the pure material design of microstructures. Recently, Zhang and Sun (2006) investigated scale-rated effects of the cellular material by combining the macroscopic design aimed at finding a preliminary layout of materials and the refined design to determine locally the optimal material microstructure. The method can be directly applied to the design of periodic structures, but the computational cost is expensive because it has to perform two finite-element analyses in each iteration, one for the macroscale optimization problem and the other for the microscale sub-optimization problem. For the design of periodic structures, the macroscopic distribution of the

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designable material must be periodic, although the stress/strain distribution may not exhibit any periodic characteristics. Therefore, a general macroscopic optimization method with additional periodic constraints needs to be established to solve the optimization problems of periodic structures efficiently.

This paper proposes a method for topology optimization of periodic structures under given boundary and loading conditions using the BESO method. Sensitivity numbers that consider the periodic constraint for the repetitive elements are developed. Then, the optimal topology of the unit cell is determined by gradually removing and adding material according to the sensitivity numbers. Finally, several 2D and 3D examples are presented.

2 Problem statement and sensitivity number

2.1 Topology optimization problem for periodic design

The objective of the present optimization problem is to find an optimal periodic topology of the structure for a given amount of material. Thus, the resulting structure will have the maximum stiffness to carry the prescribed force under the given boundary conditions. To consider the periodicity of a structure in a given designable domain, for example, for 2D cases, the designable domain is divided into $m=m_1\times m_2$ unit cells where m_1 and m_2 denote the numbers of unit cells along direction x and direction y, respectively, as shown in Fig. 1. The total number of the unit cell is usually prescribed by the client's or end-user's design specifications. It can be seen that the limit case of $m=1\times 1$ corresponds to the conventional topology optimization problem. Thus, we can formulate the optimization problem related to the overall mean compliance minimization in terms of the binary design variable $X_{i,j}$, where i and j denote the cell number and element number in the cell, respectively (see Fig. 1), as

$$\text{Minimize } C=\frac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{u} \tag{1a}$$

$$\text{Subject to}: V^{*}-mV_i=0 \tag{1b}$$

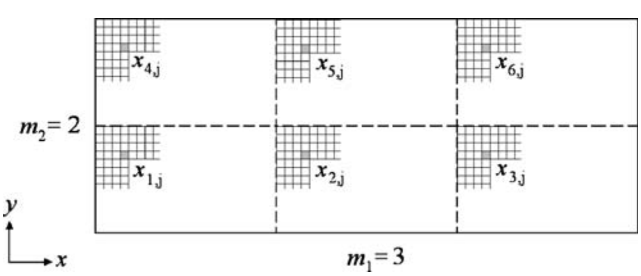


Fig. 1 A 2D design domain with m=6 unit cells (where $m=m_1\times m_2$ and m_1 and m_2 denote the number of unit cells along the x and y direction; $X_{i,j}$ is the design variables where i and j denote the cell number and element number in the cell, respectively)

$$V_i=\sum_{j=1}^{n}V_{i,j}x_{i,j} \tag{1c}$$

$$x_{1j}=x_{2j}=\cdots=x_{mj} \tag{1d}$$

$$x_{i,j}\in\{0,\ 1\}\quad (j=1,\ 2,\ \cdots,\ n)\ (i=1,\ 2,\ \cdots,\ m) \tag{1e}$$

where $\boldsymbol{f}$ and $\boldsymbol{u}$ are the applied load and displacement vectors and C is known as the mean compliance. V_i is the total volume of ith unit cell, V_{ij} the volume of the jth element in the ith unit cell and V^{*} the prescribed total structural volume. n is the total element number in a unit cell. The binary design variable $X_{i,j}$ declares the absence (0) or presence (1) of an individual element. Equation (1d) denotes that the status (1 or 0) of elements for the same position in all unit cells remains to be the same so as to ensure the periodicity of the design as shown in Fig. 1.

2.2 Elemental sensitivity numbers

In the conventional evolutionary optimization method, the sensitivity number is defined by the first-order variation of the mean compliance because of removing one element from the design domain. When ith solid element (1) is removed from a structure, the change of the mean compliance is approximately equal to the elemental strain energy as (Chu et al. 1996)

$$\Delta C_i=\frac{1}{2}u_i^{\mathrm{T}}\mathbf{K}_iu_i \tag{2}$$

where $\boldsymbol{u}_i$ is the nodal displacement vector of the ith element and $\mathbf{K}_i$ is the element stiffness matrix. The original ESO procedure is directly driven by gradually removing the lowest sensitivity elements. Similarly, when the jth element in the ith unit cell is removed, the change of the overall mean compliance is

$$\Delta C_{i,j}=\frac{1}{2}u_{i,j}^{\mathrm{T}}\mathbf{K}_{i,j}u_{i,j} \tag{3}$$

Because of the periodicity of the cells, the status (1 or 0) of the jth element in all cells should be the same. In other words, these elements should be removed or added simultaneously. Therefore, the optimization process can be conducted in a representative unit cell, which can be selected from any unit cell by the user. According to (3), the change of the overall mean compliance because of the removal of a series of elements is approximately equal to the sum of the total strain energy stored in these elements. Therefore, the sensitivity number of the jth element in the representative unit cell is defined by the changes of the overall mean compliance because of the removal of jth elements in all unit cells together

$$\alpha_j = \sum_{i=1}^{m} \Delta C_{i,j} \tag{4}$$

The mesh of the representative unit cell can be selected from any unit cell in the structure. Topology of the structure can be solely expressed by the representative unit cell because the whole structure is divided into m-independent and identical unit cells. Therefore, the following optimization algorithm can only be applied in the representative unit cell.

3 Filtering and update sensitivity numbers

Topology optimization can often exhibit an instability for which the resulting topology contains checkerboard patterns of solid and void elements. It has been established that the appearance of the checkerboard patterns because of the mixed variable problem does not represent an optimal topology (Sigmund and Petersson 1998). A most popular heuristic treatment for preventing checkerboards is filtering sensitivity numbers, which is based on image-processing techniques and works as a low-pass filter eliminate features below a certain length scale in the optimal designs.

First, the nodal sensitivity numbers $\left(\alpha_i^n\right)$, which do not carry any physical meaning on their own, are defined by averaging the sensitivity numbers of connected elements. Then, these nodal sensitivity numbers must be converted back into elements before the topology can be determined. In this process, a filter scheme is used to carry it out. The defined filter functions are based on a length scale $r_{\min}$. The primary role of the scale parameter $r_{\min}$ in the filter scheme is to identify the nodes that influence the sensitivity of ith element. This can be visualized by drawing a circle of radius $r_{\min}$ centred at the centroid of ith element, thus generating the circular sub-domain Ω_i. Nodes located inside Ω_i contribute to the computation of the improved sensitivity number of jth element as

$$\alpha_j^1 = \frac{\sum_{i=1}^{M} w\left(r_{ij}\right)\alpha_i^n}{\sum_{i=1}^{M} w\left(r_{ij}\right)} \tag{5}$$

where α_j^1 denotes the sensitivity number after filtering, M is the total number of nodes in the sub-domain Ω_i and $w(\mathrm{r}_{ij})$ is the linear weight factor defined by

$$w\left(r_{ij}\right) = r_{\min} - r_{ij} \quad (i = 1,\ 2,\ \cdots,\ M) \tag{6}$$

where r_{ij} is the distance between the centre of the element j and node i. It can be seen that the above filter scheme is similar to the mesh-independency filter used in Sigmund and Petersson (1998) except that the nodal sensitivity numbers are used. If the filter is based on the elemental sensitivities, the BESO method will be converted into the ESO method without the capability of adding material when the filter domain only covers one element. As a result, the checkerboard pattern would be unavoidable. The main function of the filter is to smooth the sensitivity number within the design domain of the base cell. It should be pointed out that Sigmund and Petersson (1998) considers the elemental density in the filter, thus the sensitivity number will be infinite for void elements. However, the present filter does not consider the element states (0 or 1), and the initial sensitivity number for void (0) elements is zero. Using the above filter technique, non-zero sensitivity numbers for void (0) elements are obtained as a result of filtering the sensitivity numbers of solid (1) elements. Some void (0) elements may be added in the later design.

It can be seen that the sensitivity numbers of the solid (1) and void (0) elements are based on the different status (1 or 0) of the element. Therefore, there is an abrupt change of the sensitivity number when an element changes the status. As a result, the objective function and topology are unstable and hard to converge. Computational experience has shown that averaging the sensitivity number with its historical information is an effective way to avoid this problem (Huang and Xie 2007). It can be simply expressed by

$$\alpha_j^2 = \frac{\alpha_{j,k}^1 + \alpha_{j,k-1}^1}{2} \tag{7}$$

where k is the current iteration number. Thus, the updated sensitivity number includes all sensitivity information in the previous iterations.

Note that the sensitivity number is a measure of the approximate variation of the objective function because of element removal or addition in the representative unit cell. To minimize the mean compliance, the evolutionary process will be conducted by removing elements with the smallest sensitivity numbers and adding the elements with the highest ones. Mathematically, such a procedure is known as the 'hill-climb' method or the 'steepest descent' algorithm.

4 Numerical implementation

The evolutionary iteration procedure for solving the optimization problems for periodic structures can be outlined as follows:

1. Create the analysis model, and apply the boundary and loading conditions.
2. Discretize the model using an finite-element mesh, and assign the initial property values (0 or 1) of elements to construct the initial design.

3. Perform finite-element analysis on the current design to obtain elemental sensitivity numbers in (3).
4. Identify the group of elements according to the periodic constraint and obtain elemental sensitivity numbers of the representative unit cell using (4).
5. Conduct filtering and averaging sensitivity numbers with its history information in the representative unit cell.
6. Determine the target volume for the next iteration by gradually decreasing or increasing material as

$$V_{i+1} = V_i(1 \pm \mathrm{ER}) \quad (i = 1,\ 2,\ 3\cdots) \tag{8a}$$

where ER is called the evolutionary volume ratio. Once the objective volume is reached, the volume will be kept constant for the remaining iterations as

$$V_{i+1} = V^* \tag{8b}$$

7. Reset the property values of elements for the representative unit cell. For solid elements, the property value is switched from 1 to 0 if the following criterion is satisfied.

$$\alpha_i \leq \alpha_{\mathrm{th}} \tag{9a}$$

For void elements, the property value is switched from 0 to 1 if the following criterion is satisfied.

$$\alpha_{\mathrm{i}} > \alpha_{\mathrm{th}} \tag{9b}$$

Where α_{th} is the threshold of the sensitivity number that is determined by the target material volume in each iteration. For example, if the target volume for the present iteration is 70% of the unit cell, then elements for which sensitivity numbers are ranked at the top 70% of all element in the unit cell will remain to be solid or be added, and all other elements will be removed or remain to be void.
8. Construct a new representative unit cell using elements with property value 1.
9. Reassign property values of elements in all unit cells according to the representative unit cell. As a result, a new design is constructed for the next finite-element analysis.
10. Repeat steps 3–9 until a convergent solution is obtained. The following convergence criterion will be applied after the volume constraint is satisfied.

$$\tau = \frac{\left|\sum_{i=1}^{N}\left(C_{k-i+1} - C_{k-N-i+1}\right)\right|}{\sum_{i=1}^{N} C_{k-i+1}} \leq \tau_{\max} \tag{10}$$

where k is the current iteration number, $\tau_{\max}$ is a allowable convergence error and N is integral number. $\tau_{\max}$=0.1% and N=5 are selected throughout this paper, which means a stable compliance in ten successive iterations.

5 Examples and discussion

5.1 Numerical verification of the optimal design for a periodic structure

The topology optimization for a continuum structure is complex because of a large number of design variables. Evolutionary structural optimization offers an alternative method for solving various topology optimization problems

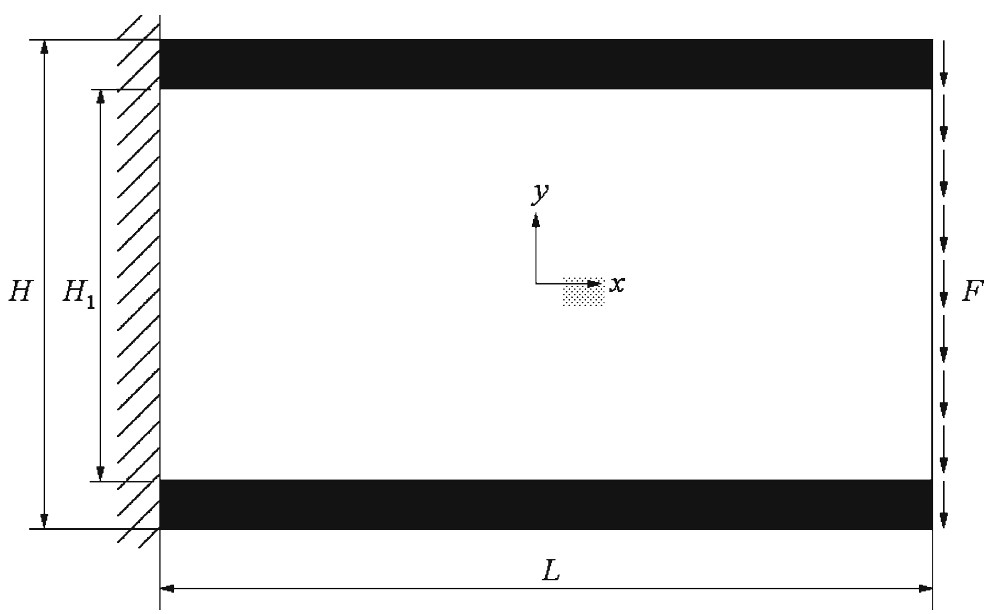


Fig. 2 Design domain of the optimization problem in Zhang and Sun (2006)

of continuum structures (Xie and Steven 1997). Theoretically, it is noted that the sequential linear programming-based approximate optimization method followed by the Simplex algorithm is equivalent to ESO/BESO (Tanskanen 2002). However, had the optimization problem been solved with a periodic constraint, the structure would still evolve to an optimal solution.

To verify the proposed method, an optimization problem in Zhang and Sun (2006) is tested. The 2D rectangular domain of the problem with L=32 and H=20 is shown in Fig. 2. The designable domain refers to the inner core with H_1=16. The plate is fixed on the left end and loaded vertically with F=100 (force/length) on the right end. Young's modulus and Poisson's ratio of material are E=1000 and ν=0.3. The objective of the problem is to find the optimal topology of the core with a volume fraction of 50% over the core area. To avoid the singularity of the problem, a small non-designable elastic portion is added artificially along the right edge to transfer the applied load. Four cases for m=2×1, 4×2, 8×4 and 16×8 will be studied and compared. In consistence with the periodic pattern in Zhang and Sun (2006), the symmetric unit cells about the neutral axis are arranged for later three cases.

The filter radius is selected to be $r_{\min}$=1.5, 1.0, 0.5 and 0.3 for m=21, 4×2, 8×4 and 16×8, respectively. BESO starts from the full design and gradually decreases the volume fraction with the parameter ER=2% until the constraint of the volume fraction 50% is achieved. Then, the volume keeps constant until the defined convergent criterion is satisfied. Figure 3 shows the evolutionary histories of topology, volume fraction and objective function (mean compliance) for m=2×1. It can be seen that the topology volume fraction and objective function are all convergent at the end of the optimization process. Figure 4 shows the final optimal topologies for all four cases, which are very similar to the topologies of Zhang and Sun (2006). The mean compliance are 72,899.3, 73,843.3, 77,617.1 and 79,706.2 for m=2×1, 4×2, 8×4 and 16×8, which are much lower than the corresponding solutions of Zhang and Sun (2006), which are 82,530.6, 84,012.9, 88,308.3 and 90,547.5, respectively. This difference mainly attributes to the effect of the soft material whose strain energy is artificially increased because of the penalized parameter, p>1.

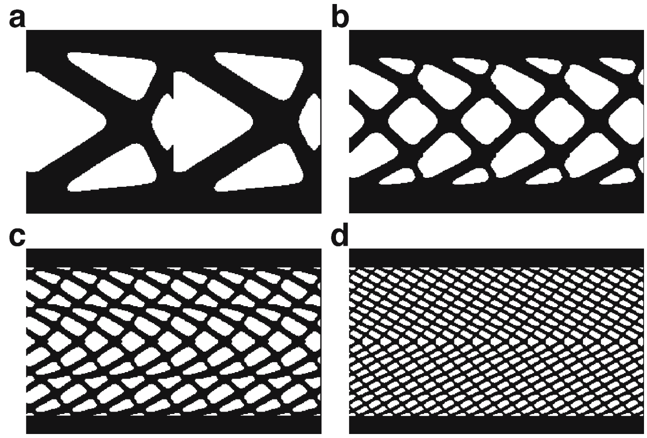


Fig. 4 Optimal designs for various periodic constraints **a** m=2×1, **b** m=4×2, **c** m=8×4 and **d** m=16×8

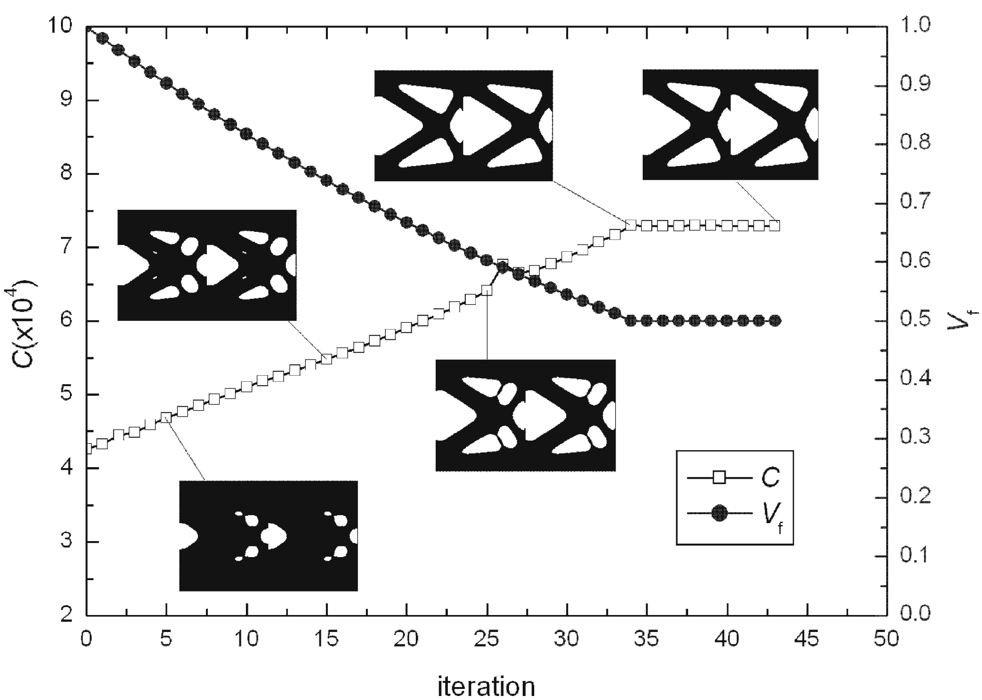


Fig. 3 Evolutionary histories of volume fraction, mean compliance and topology for the periodic condition with m=2×1

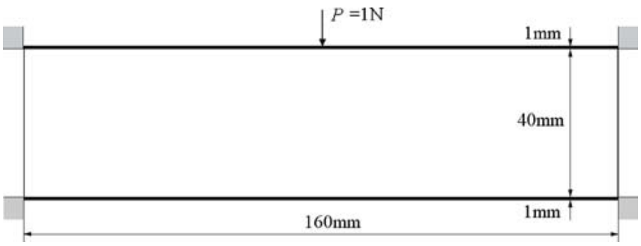


Fig. 5 Design domain of 2D sandwich structure

5.2 Example 1: 2D sandwich design

The proposed method can be applied to the cellular core design of a sandwich structure as shown in Fig. 5 where the sandwich structure is fixed at both ends of skins. The designable core is a rectangle of size 160 by 40 mm, which is divided into 320×80 four-node plane stress elements, and the two skins have 1 mm thickness, which is divided into 320 beam elements. It is assumed that the skins and core are tied together. A vertical point force P=1 N is applied at the middle point of the top skin. The materials of the skins and core are assumed with Young's modulus 100 GPa and Poisson's ratio 0.3 and Young's modulus 1 GPa and Poisson's ratio 0.3, respectively. Assume that only 30% of the material is available to construct the final design. The filter radius and evolutionary ratio are selected to be r_m=2 mm and ER=1%.

Figure 6 shows the evolutionary histories of topology, volume fraction and objective function for m=2×1. It can be seen that all topologies satisfy the defined periodic conditions and the mean compliance of the final topology is 8.33 Nm. Table 1 lists the optimal topologies and their mean compliances for various number of unit cells (m). A typical unit cell is given inside dash lines except for m=1×1. Related mean compliance values are plotted against the number of the unit cells in Fig. 7. In general, the mean compliance increases with the number of unit cells because the number of constraints associated with the design variables increases. Therefore, the solution of the conventional BESO method corresponding to a limit case with m= 1× has the minimum mean compliance. On the other hand, the optimal topology depends on the aspect ratio of the unit cell. For example, the optimal topologies for m=2×1 and m=1×2 are different, although their total numbers of unit cells are equal.

The present BESO method may starts from an initial guess design, which may be highly non-optimal. As an example, Fig. 8 shows the evolutionary histories of topology, volume fraction and mean compliance for m= 2×1. The compliance and volume are both decreasing at the beginning, which indicates that the overall stiffness of the structure has improved because of the optimization algorithm, although the volume of the structure becomes less. The volume fraction remains constant after the seventh iteration, and the compliance continues to decrease until it converges to a constant value, 8.33 Nm, which is equal to the solution shown in Fig. 7. The advantage of this

Fig. 6 Evolutionary histories of volume fraction, mean compliance and topology for 2D sandwich structure with m=2×1 when BESO starts from the initial full design

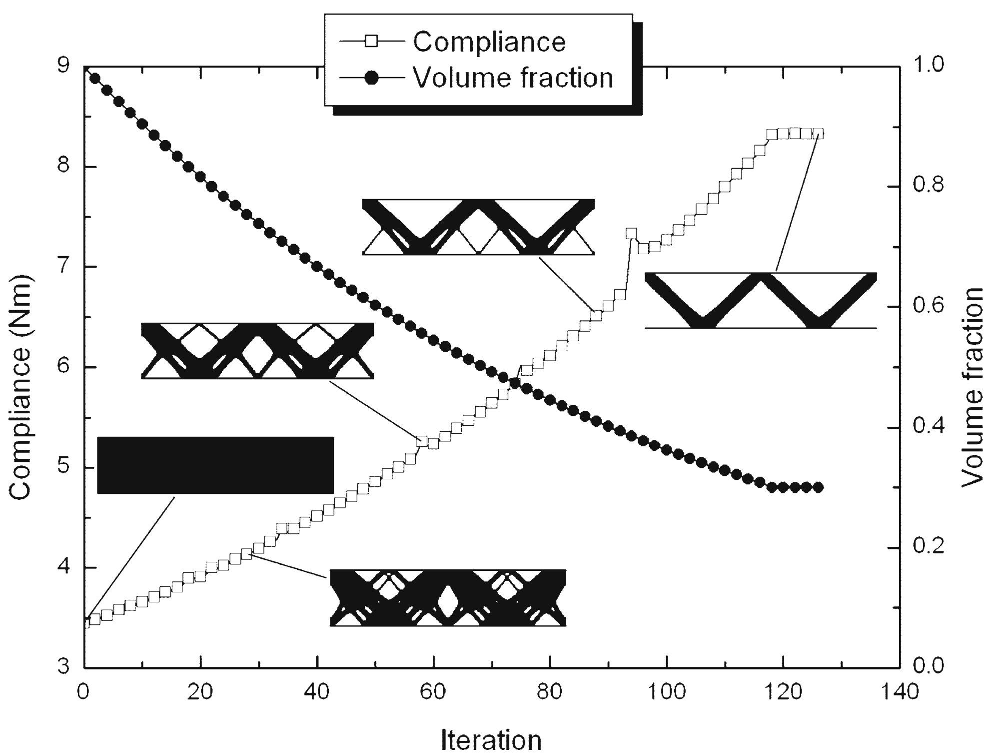
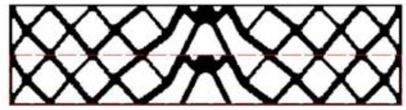
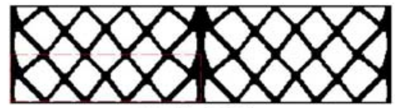
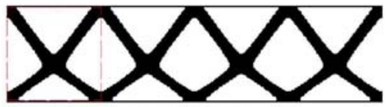
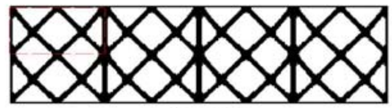
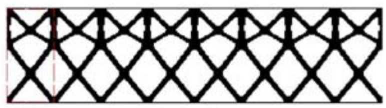
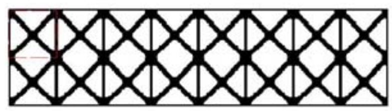


Table 1 Optimal designs and their mean compliance for 2D sandwich structures under various periodic constraint

	$m2=1$	$m2=2$
$m1=1$	 $C=7.97$Nm	 $C=10.47$Nm
$m1=2$	 $C=8.33$Nm	 $C=10.66$Nm
$m1=4$	 $C=10.38$Nm	 $C=11.31$Nm
$m1=8$	 $C=12.20$Nm	 $C=12.93$Nm

procedure is that the optimization process is straightforward from the history of the objective function. Another advantage is that this procedure possesses a high computational efficiency because only a portion of all elements in the full model is involved in the finite-element analysis. Nevertheless, the BESO starting from an initial guess design may converge to a local optimal solution because some void elements in the initial guess design may never be included in the finite-element analysis during the whole optimization process. To eliminate or reduce the likelihood of a local optimum, sometimes it might be necessary that BESO should start from the initial full design; thus, all elements are involved in the finite-element analysis at least once (Huang and Xie 2007).

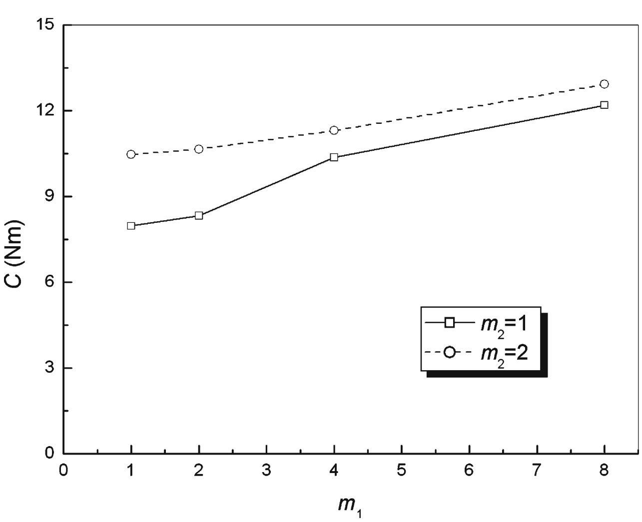


Fig. 7 Variation of mean compliance against the number of total unit cells, m

5.3 Example 2: 2D bridge design

Figure 9 shows an optimization problem for designing a bridge. The design domain is a rectangle of size $L=240$, $H=60$ and thickness $t=1$, the bottom deck with length $L=240$, height $h=5$ and thickness $t=1$ is supported at two bottom corners, and a vertical force $P=100$ is applied at the middle point of the bottom deck. The design domain is discretized by a 240×60 four-node quadrilateral element mesh, and the non-designable deck is meshed with 240 beam elements. The nodes of the beam elements are connected with those of the plate elements at the bottom side of the design domain. The materials for the design domain and the deck are same to be Young's modulus $E=200$ and Poisson's ratio $\nu=0.3$. Suppose only 30% of the designable domain material is available for constructing the final structure. The evolutionary ratio ER=2% is used in this example.

Figure 10a shows the optimal design with $m=1\times1$, which is the same to the conventional optimal design, and the mean compliance of the design is 1.12. When the design domain is divided with unit cells, the final designs are shown in Fig. 10b and c for $m=4\times1$ and $m=6\times1$, respectively. Their mean compliances are 1.53 and 1.78, which are higher than that of the conventional design. Similar to the above example, the mean compliance increases with the total number of unit cells.

To study the influence of the deck, the above problem with the height of the deck $h=50$ is solved with $m=4\times1$.

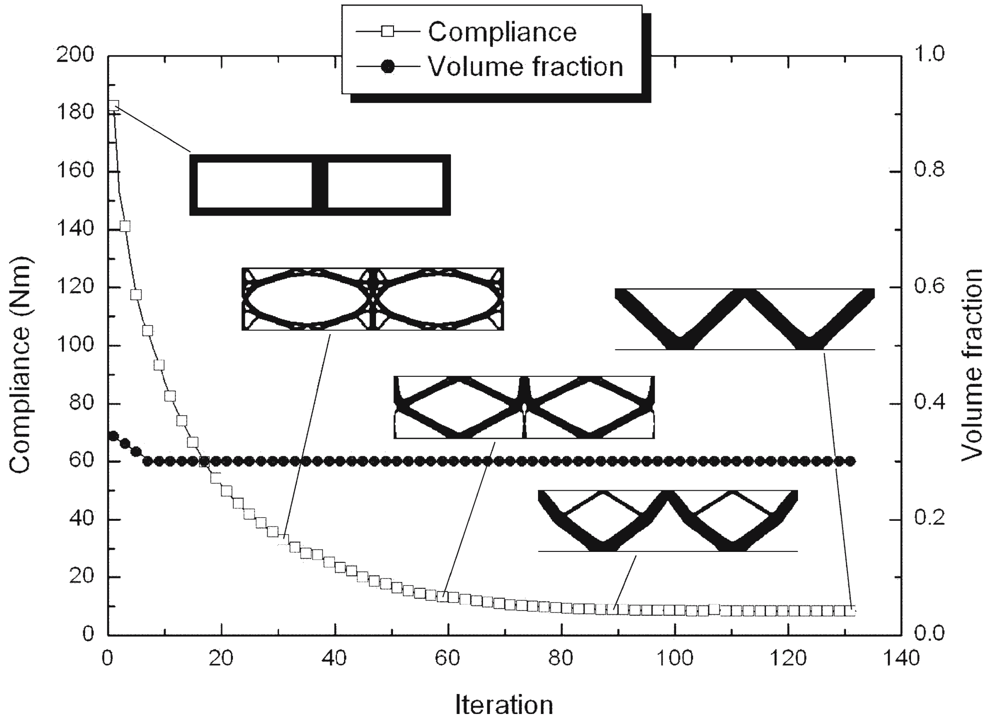


Fig. 8 Evolutionary histories of volume fraction, mean compliance and topology for 2D sandwich structure with m=2×1 when BESO starts from the initial guess design

The final design is shown in Fig. 11, which totally differs from the above one shown in Fig. 10b. It demonstrates that the optimal topology may also depend on the strength of other non-designable parts.

5.4 3D sandwich designs

The proposed optimization approach can also be extended for designing 3D periodic structures. Figure 12 shows a sandwich cantilever undergoing four vertical concentrated loads P=1. The designable core of the size 100×20×40 is divided into a 100×20×40 mesh using eight-node cubic elements, and both non-designable skins with a unit thickness are divided into a 100×20 mesh using four-node plate elements. The materials are assumed with Young's modulus E=1, Poisson's ratio ν=0.3 for the core and E=100, ν=0.3 for both skin plates. The objective is to obtain the optimal periodic layout of the core while minimizing the mean compliance of the design and at the same time satisfying the requirement of allowable material volume, 10% of the designable domain. The evolutionary ratio ER=2% is used in this example.

Figure 13a shows that the optimal topology with m=1×1×1 and its mean compliance is 40.9. Figure 13b and c show that the final topologies for the number of unit cells m=4×1×1 and m=4×2×1 and their mean compliance are 45.4 and

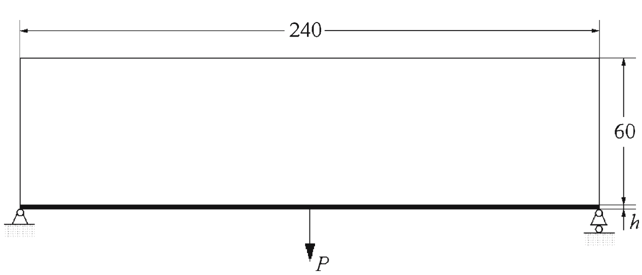


Fig. 9 The optimization problem for 2D bridge structure

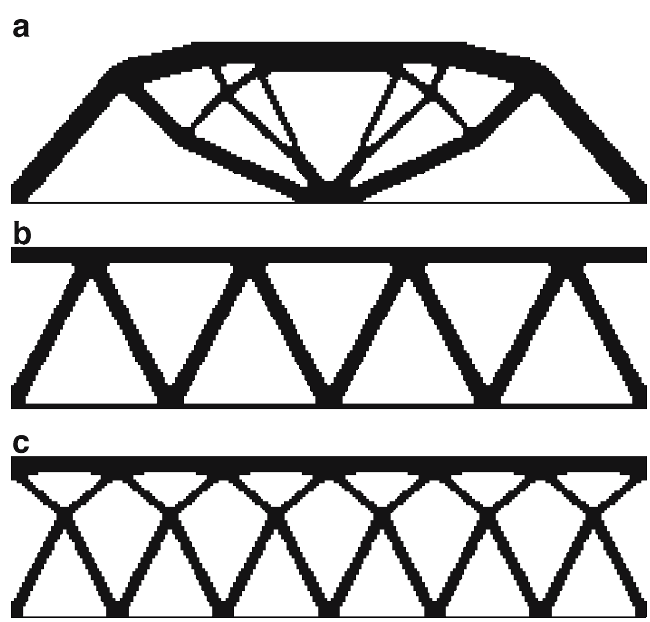


Fig. 10 Optimal designs and their mean compliance for 2D bridge structure **a** m=1×1, C=1.12, **b** m=4×1, C=1.53, **c** m=6×1, C=1.78

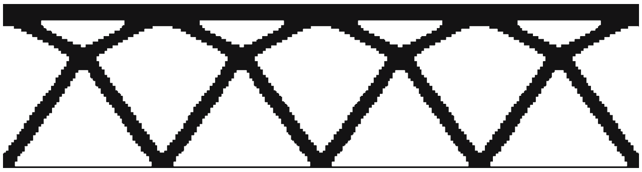


Fig. 11 Optimal designs with $m=4\times1$ for 2D bridge structure with strong deck $h=50$

49.5, respectively. It is also found that the mean compliance increases with the total number of 3D unit cells.

6 Conclusions

A method for topology optimization of periodic structures has been developed in this paper. Additional periodic constraint has been added to the optimization formulation to ensure that the structure comprises a prescribed number of identical unit cells. The optimal topology of the unit cell is found by gradually removing and adding material using the BESO method. Several 2D and 3D examples are presented. The following conclusions can be drawn:

1. The proposed method can be used to effectively optimize 2D and 3D periodic structures. With a given amount of material, the optimal design from the proposed method will have a substantially lower mean compliance than the initial, guess design.
2. The optimal topology highly depends on the total number and the aspect ratio of the unit cells.
3. The value of the objective function (mean compliance) becomes higher when the total number of unit cells increases. Thus, the solution of the conventional topology optimization corresponding to the limit case with $m=1\times1$ has the lowest mean compliance. However, the advantage of a periodic design is that the manufacturing or construction cost could be much reduced.
4. The optimal topology of the designable domain may also depend on the relative strength of other non-designable parts, such as the skins of the sandwich structures.

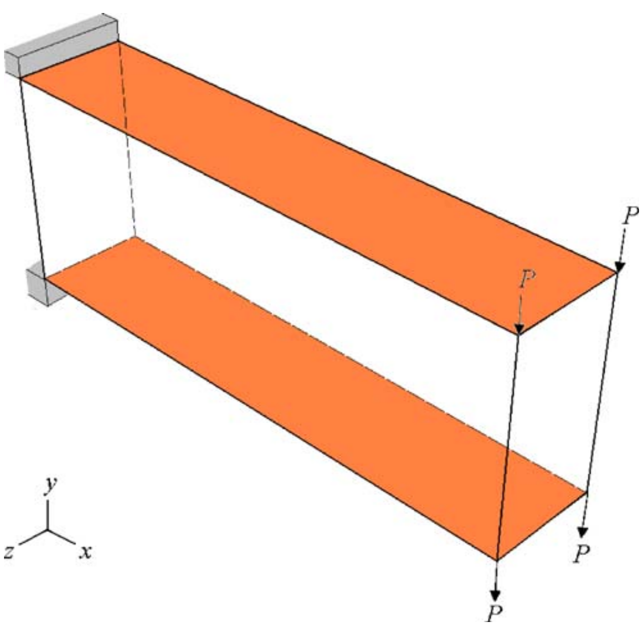


Fig. 12 The optimization problem for 3D sandwich structure

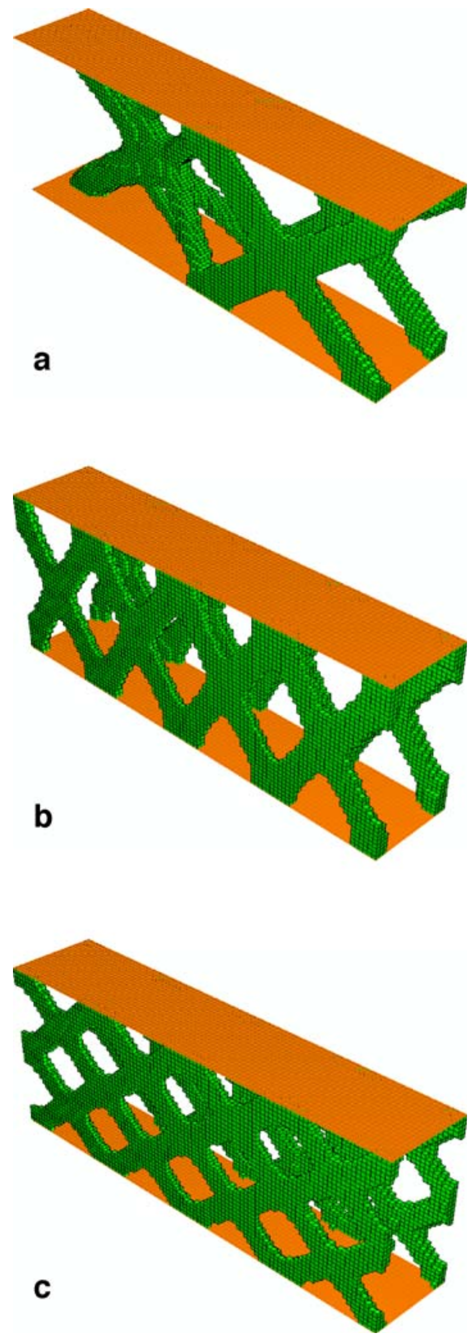


Fig. 13 Optimal designs for 3D sandwich structure **a** $m=1\times1\times1$, $C=40.9$, **b** $m=4\times1\times1$, $C=45.4$, **c** $m=4\times2\times1$, $C=49.5$

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