MAS1006

UNIVERSITY OF EXETER

SCHOOL OF MATHEMATICAL SCIENCES

ADVANCED CALCULUS

4 June 1999

2:15 p.m. – 5:15 p.m.
Duration: 3 hours

Examiner: Dr J. Brodzki

The marks from Section A (40%) and the best THREE questions in Section B (20% for each) will be recorded.

Marks shown in questions are merely a guideline.

Approved calculators of the following type may be used Casio fx82 series, Sharp EL521 or EL531 Series and Texas TI-30X or TI-36X.

SECTION A

1. (a) Find the general solution of each of the following ordinary differential equations:

(i) $\dfrac{dy}{dx} - xy = xe^{x^2}$; (6)

(ii) $x^2 + y^2 = 2xy\dfrac{dy}{dx}$; (6)

(iii) $\dfrac{y}{x}\dfrac{dy}{dx} = \dfrac{1+x}{1+y}$. (5)

(b) Evaluate the limit

$$\lim_{x\to 1}\left(\frac{1}{x-1} - \frac{1}{\ln x}\right) .$$

(5)

(c) Evaluate the gradient, ∇f, of the function $f(x, y) = y^2 \ln x$. Find its directional derivative at the point $(1, 4)$ in the direction of the vector $\mathbf{a} = -3\mathbf{i} + 3\mathbf{j}$. (5)

(d) Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{5^n n} .$$

(4)

(e) Find dz/dx given that

$$z = \int_0^{\sqrt{x}} \sin t^2 \, dt .$$

(4)

(f) Using the binomial series, or otherwise, obtain the Maclaurin series for $f(x) = 1/(1 + x^2)$. Use this series to obtain the Maclaurin series for $\tan^{-1} x$. (5)

[**40**]

SECTION B

2. (a) Find the general solution of each of the following ordinary differential equations:

   (i) $y'' - 3y' + 2y = xe^{3x}$; (5)

   (ii) $y'' + 9y = \sin 3x$. (6)

   (b) Test each of the following series for convergence. State clearly which tests you are using.

   (i) $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$; (ii) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$; (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n}$. (9)

   [**20**]

3. (a) Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$$

   converge? Does it converge absolutely? Justify your answers. (6)

   (b) Show that if $z = x + f(u)$, where $u = xy$, then

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = x.$$

   (4)

   (c) Given that $w = f(u, v)$, where $u = (x^2 - y^2)/2$ and $v = xy$, obtain expressions for the operators $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$. Show that

$$w_{xx} + w_{yy} = (x^2 + y^2)(f_{uu} + f_{vv}).$$

   (10)

   [**20**]

4. (a) Find and classify the critical points of the function

$$f(x, y) = 4xy - x^4 - y^4.$$

(8)

(b) Evaluate the integral

$$\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dzdydx\ .$$

(5)

(c) Find the volume of the solid that lies under the paraboloid $z = x^2+y^2$ and above the triangle in the (x, y)-plane enclosed by the lines $y = x$, $x = 0$, $x + y = 2$. (7)

[**20**]

5. (a) Find the maximal and minimal values of the function $f(x, y) = x^2 + y^2$ on the curve $x^2 + xy + y^2 = 1$. (8)

(b) Reverse the order of integration to evaluate the integral

$$\int_0^1 \int_y^1 x^2 e^{xy}\, dxdy\ .$$

(5)

(c) Change the cartesian coordinates to polar coordinates to evaluate the integral

$$\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln{(x^2 + y^2 + 1)} dxdy\,.$$

(7)

[**20**]