**MAS1006**

UNIVERSITY OF EXETER

**SCHOOL OF MATHEMATICAL SCIENCES**

# ADVANCED CALCULUS

8 June 2000

2:15 p.m. – 5:15 p.m.
Duration: 3 hours

Examiner: Dr J. Brodzki

*The marks from Section A (40%) and the best THREE questions in Section B (20% for each) will be recorded.*

*Marks shown in questions are merely a guideline.*

*Approved calculators of the following type may be used Casio fx-83wa series, Sharp EL521 or EL531 Series and Texas TI-30X or TI-36X.*

## SECTION A

1. (a) Find the general solution of each of the following ordinary differential equations:

   (i) $e^y(1+x^2)\dfrac{dy}{dx} - 2x(1+e^y) = 0;$ (6)

   (ii) $(x+y)\dfrac{dy}{dx} - y = 0;$ (6)

   (iii) $\dfrac{dy}{dx} - 2xy = x - x^3.$ (6)

   (b) Define what it means for a sequence $\{a_n\}$ of real numbers to converge to the limit $L$. (3)

   (c) Find the Maclaurin series for $f(x) = 1/(1+x)$ using the definition of the Maclaurin series. Use your result to obtain the Maclaurin series for $\ln(1+x)$. (7)

   (d) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$$

   converges or diverges. (3)

   (e) Show that the function $u(x,y) = e^x \sin y$ satisfies the partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

   (5)

   (f) Evaluate the limit

$$\lim_{x\to 0} \frac{e^{3x} - 3x - 1}{\sin^2 5x}.$$

   (4)

   [**40**]

## SECTION B

2. (a) Find the general solution of each of the following ordinary differential equations:

   (i) $y'' - 4y = 8e^{2x}$; (4)

   (ii) $y'' - 2y' + 2y = 2\cos x + \sin x$. (5)

   (b) Test each of the following series for convergence. State clearly which tests you are using.

   (i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$; (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{2^n}$ (6)

   (c) Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

   converge? Does it converge absolutely? (5)

**[20]**

3. (a) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (3x)^n}{\sqrt{n+1}}$$

(5)

   (b) Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$. (4)

   (c) Find the gradient of the function $f(x, y) = x^2y^3 - 4y$. Find the directional derivative of $f$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$. (5)

   (d) Let $u = f(x, y)$ where $x = e^s \cos t$, $y = e^s \sin t$. Show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]$$

(6)

**[20]**

4. (a) Find and classify the critical points of the function:

$$f(x, y) = xye^{-(x^2+y^2)/2}.$$

(8)

(b) Evaluate the integral

$$\int\int_T (x^2y + y^2)dA$$

over the triangle $T$ with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$. (6)

(c) Sketch the region of integration in the iterated integral

$$I = \int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dydx$$

Reverse the order of integration and evaluate the integral $I$. (6)

[**20**]

5. (a) Find the minimal value of the function $f(x, y) = x^2 + y^2$ on the curve $x^2y = 16$. (7)

(b) Find the volume of the solid that lies under the plane $z = 1 + y$ and above the region $D$ in the $(x, y)$-plane bounded by lines $y = 0$ and $y = 1 - x^2$. (7)

(c) Let $R$ be the region in the first quadrant of the $(x, y)$-plane bounded by the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 4$ and the lines $y = 0$ and $y = x$. Evaluate the integral

$$\int\int_R \frac{y^2}{x^2} dA$$

by changing from cartesian coordinates to polar coordinates. (6)

[**20**]