Analysis: Problem sheet 1

Assignment 1 consists of questions 2, 5, 7, 9, 15, 18(b,c,f), 19, 21, 22(c,e,h,j,l), 23; it is due in on 10 November 2011 [marks are indicated in square brackets]

1. In each case, decide whether the set A has an upper bound and whether it has a lower bound. If it has an upper bound determine the least upper bound and whether this is the greatest element of A, Likewise if it has a lower bound, determine the greatest lower bound and whether this is the least element of A:

   (a) $A = \{x \in \mathbf{R} : -1 < x \leq 1\}$;

   (b) $A = \{(-1)^n/n : n \in \mathbf{N}\}$;

   (c) $A = \{n + (-1)^n/n : n \in \mathbf{N}\}$;

   (d) $A = \{x \in \mathbf{R} : x^2 - 4x + 3 < 0\}$.

2. Prove that if x^2 is irrational, then x is irrational. Hence prove that $\sqrt{3} + \sqrt{5}$ is irrational. (You may assume that if $n \in \mathbf{N}$ isn't a square then $\sqrt{n}$ is irrational). [5]

3. Is it always true that if x and y are irrational then $x+y$ is also irrational? What about xy?

4. Let A and B be two sets of real numbers, and suppose that A has least upper bound α and B has least upper bound β. Prove that the set $A + B$ defined by

   $$A + B = \{a + b : a \in A, b \in B\}$$

   has least upper bound $\alpha + \beta$. (Hint: first show that $\alpha + \beta$ is an upper bound of $A + B$, and then show that if $\gamma < \alpha + \beta$ then there are $a \in A$ and $b \in B$ with $\gamma < a + b$.)

   But if we define

   $$AB = \{ab : a \in A, b \in B\}$$

   show, by example, that AB need not have an upper bound, and even if it does, its least upper bound might not equal $\alpha\beta$.

5. Let A be a non-empty set of **positive** real numbers. Defining

   $$A^{-1} = \{a^{-1} : a \in A\}$$

   prove that if A has least upper bound α then A^{-1} has greatest lower bound α^{-1}. [5]

6. Let us define a sequence of numbers $a_1, a_2, \ldots$ by $a_1 = 3$, $a_2 = 3$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove, by induction, that $a_n = 2^n - (-1)^n$.

7. Prove, by induction, that

$$\sum_{k=1}^{n} kx^k = x + 2x^2 + 3x^3 + \cdots + nx^n = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

(as long as $x \neq 1$). [5]

8. Prove, by induction, that if $0 < x < y$ then $x^n < y^n$ for all $n \in \mathbf{N}$.

9. Prove, by induction, that if $0 < x < 1$ then $(1-x)^n \geq 1 - nx$ for all $n \in \mathbf{N}$. [5]

10. The *triangle inequality* states that

$$|x + y| \leq |x| + |y|$$

for all real numbers x and y. Prove it.

Using the triangle inequality, prove, by induction, that

$$|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$$

for all real numbers $x_1, \ldots, x_n$.

Also, prove that

$$|x + y| \geq |x| - |y|.$$

11. Prove that if x and y are real numbers with $x < y$ then there is an **irrational** number α with $x < \alpha < y$. (Hint: I think it's easier to exploit the corresponding result for rationals then to prove from scratch.)

12. In the lectures, in the proof that $\sqrt{2}$ exists, I used the fact that if $\alpha^2 > 2$ and $\alpha > 0$ then $\beta = \frac{1}{2}(\alpha + 2/\alpha)$ satisfied $0 < \beta < \alpha$ and $\beta^2 > 2$. Suppose I wanted to prove the existence of $\sqrt{3}$ instead. How should I define β to ensure that if $\alpha^2 > 3$ and $\alpha > 0$ then $0 < \beta < \alpha$ and $\beta^2 > 3$: as (a) $\beta = \frac{1}{2}(\alpha + 2/\alpha)$, (b) $\beta = \frac{1}{3}(\alpha + 3/\alpha)$ or (c) $\beta = \frac{1}{2}(\alpha + 3/\alpha)$? Hence give a proof that there is a positive real solution of $x^2 = 3$.

13. Let (a_n) be a sequence of nonzero real numbers. Prove that if $a_n \to \infty$ (or if $a_n \to -\infty$) then the sequence $(1/a_n)$ converges to 0.

    If (b_n) is a sequence of nonzero real numbers converging to 0, it is necessarily true that $1/b_n \to \infty$ or that $1/b_n \to -\infty$?

14. Let (a_n) and (b_n) be sequences converging to a and b respectively. Prove that the sequence $(a_n - b_n)$ converges to $a - b$.

15. The "squeezing" or "sandwich" principle asserts that if (a_n), (b_n) and (c_n) are three sequences of real numbers satisfying $a_n \le b_n \le c_n$ for all n and if (a_n) and (c_n) both converge to the same limit L, then (b_n) also converges to L. Prove it. [10]

16. Let (a_n) be an increasing sequence that is not bounded above. Prove that $a_n \to \infty$ as $n \to \infty$.

17. Prove that a convergent sequence has a **unique** limit. That is, if (a_n) converges both to a and to b, then $a = b$. (Hint: if not assume that $a \neq b$ and prove that for each $\varepsilon > 0$ then (a_n) is eventually ε-close to both a and b, but that there is a value of ε for which this is impossible.)

18. Determine whether each the following sequences (a_n) are convergent. When so, determine their limit, when not determine whether they tend to ∞ or to $-\infty$:

$$\text{(a)} \quad a_n = \frac{n^3 \cos(n\pi/4)}{n^4 + 1} \;; \quad \text{(b)} \quad a_n = \frac{n^3 \cos(n\pi/4)}{n^3 + 1} \quad [5];$$

$$\text{(c)} \quad a_n = \frac{n - \cos n}{\sqrt{n^2 + 1}} \quad [5]; \quad \text{(d)} \quad a_n = \cos(1/n^3);$$

$$\text{(e)} \quad a_n = \frac{1}{\sqrt{n^2 + 1} - n}; \quad \text{(f)} \quad a_n = \frac{2n^2 + n \sin n}{n^2 + ne^{-n} + \cos n} \quad [5].$$

(You may use the fact that, which I haven't yet proved in lectures, that if (b_n) is a sequence of positive terms and $b_n \to b$ then $\sqrt{b_n} \to \sqrt{b}$.)

19. Let us define the sequence (a_n) by $a_1 = 3$ and $a_{n+1} = \frac{1}{2}(a_n + 3/a_n)$ for $n \ge 1$. Prove that (a_n) is decreasing and is bounded below. Also find $\lim_{n\to\infty} a_n$, justifying your answer. [10]

20. Let us define the sequence (a_n) by $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n^2/3}$ for $n \ge 1$. Prove that $a_n^2 < 3/2$ and $a_{n+1}^2 > a_n^2$ for all n. Deduce that (a_n) is convergent, and find its limit.

21. Let us define the sequence (a_n) by $a_1 = 0$ and

$$a_{n+1} = \frac{3a_n + 1}{a_n + 2}$$

for $n \ge 1$. Prove that $0 \le a_n < 3$ and that $a_{n+1} > a_n$ for all n. Deduce that (a_n) is convergent, and find its limit. [10]

22. Determine whether each of the following series are convergent:

$$\text{(a)} \quad \sum_{n=0}^{\infty} \frac{2n-1}{n^2+2}; \quad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{2^n-1}{3^n-2^n}; \quad \text{(c)} \quad \sum_{n=1}^{\infty} \frac{\log n}{n}; \quad [5]$$

$$\text{(d)} \quad \sum_{n=0}^{\infty} \frac{999^n n^{1000}}{1000^n}; \quad \text{(e)} \quad \sum_{n=0}^{\infty} \frac{2^n n^3+3^n}{3^n n+2^n} \quad [5]; \quad \text{(f)} \quad \sum_{n=2}^{\infty} (-1)^n \frac{n}{n-1};$$

$$\text{(g)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}; \quad \text{(h)} \quad \sum_{n=1}^{\infty} \sin((-1)^n/n) \quad [5]; \quad \text{(i)} \quad \sum_{n=1}^{\infty} \sin(1/n);$$

$$\text{(j)} \quad \sum_{n=0}^{\infty} 20^n \frac{(n!)^3}{(3n)!} \quad [5]; \quad \text{(k)} \quad \sum_{n=0}^{\infty} \frac{2^n n!}{n^n}; \quad \text{(l)} \quad \sum_{n=0}^{\infty} \frac{3^n n!}{n^n} \quad [5].$$

(You may use the fact, which I haven't proved yet in lectures, that $\lim_{n\to\infty}(1+n^{-1})^n) = e$.

23. Theorems have hypotheses!

Give an example of a **divergent** alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ where each $a_n > 0$ and (a_n) is decreasing. [5]

Give an example of a **divergent** alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ where each $b_n > 0$ and $b_n \to 0$. [5]

24. The *Cauchy condensation test* states that:

Let (a_n) be a decreasing sequence of positive numbers. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{m=1}^{\infty} 2^m a_{2^m}$ converges.

Use the Cauchy condensation test to determine for which positive real numbers α is the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}}$$

convergent.

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