Analysis: Problem sheet 2

Assignment 1 consists of questions 1(b,e), 2(a,e), 4, 5, 7, 9, 12, 13(d), 14(e), 16(b), 20, 23(e), 25, 26(c) and 27(a); it is due in on 15 December 2011 [marks are indicated in square brackets]

1. In each case, determine whether $f(x)$ converges to a limit as $x \to a$. If not then determine whether $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$.
   (a) $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 4}$, $a = 2$,
   (b) $f(x) = \dfrac{\tan x}{x}$, $a = 0$ [5], (c) $f(x) = \dfrac{x^2 - 3x + 1}{x}$, $a = 0$,
   (d) $f(x) = \dfrac{1 - \cos x}{x^2}$, $a = 0$, (e) $f(x) = \dfrac{\sqrt{x} - 1}{x - 1}$, $a = 1$ [5].

2. In each case, determine whether $f(x)$ converges to a limit as $x \to \infty$. If not then determine whether $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to \infty$.
   (a) $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 4}$ [5], (b) $f(x) = \dfrac{\cos x}{x}$,
   (c) $f(x) = \dfrac{x^2 - 3x + 1}{x}$, (d) $f(x) = x^3 \cos x$,
   (e) $f(x) = \dfrac{1}{\sqrt{x+1} - \sqrt{x}}$ [5].

3. Prove that the equation $x^3 = e^{-x}$ has at least one solution with $x \geq 0$.

4. Prove that the equation $x = 8\cos x$ has at least **three** solutions with $x > 0$. [5]

5. Let $f : [a, b] \to \mathbf{R}$ be a continuous function with the property that $f(x) \in [a, b]$ for all $x \in [a, b]$. Prove that f has a *fixed point* c, that is $c \in [a, b]$ and $f(c) = c$. (You may assume the Intermediate Value Theorem). [5]

6. Prove that the cosine function is differentiable and that $\cos'(x) = -\sin(x)$.

7. For which positive real numbers α is the function

$$f_\alpha(x) = \begin{cases} |x|^\alpha \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

   continuous at 0? For which α is f_α differentiable at 0? Prove that f_3' is continuous at 0. [10]

8. Prove that if $0 < a < b < \pi/2$ then
$$(b-a)\sin a < \cos a - \cos b < (b-a)\sin b.$$

9. Prove that if $0 < a < b < 1$ then
$$\frac{b-a}{\sqrt{1-a^2}} < \cos^{-1} a - \cos^{-1} b < \frac{b-a}{\sqrt{1-b^2}}. \qquad [5]$$

10. Prove that the equation $xe^x = 2$ has a unique real solution.

11. Give a detailed argument to compute the derivative of the function $\cos^{-1} : [-1,1] \to \mathbf{R}$.

12. Consider the functions f_1, f_2 and f_3 defined by $f_1(x) = x - \sin x$, $f_2(x) = \cos x - 1 + x^2/2$ and $f_3(x) = \sin x - x + x^3/6$. Prove that all these functions are increasing on the interval $[0,\infty)$ and hence deduce that for $x \geq 0$ we have
$$x \geq \sin x \geq x - \frac{x^3}{6}. \qquad [10]$$

13. Compute the following limits:
(a) $\displaystyle\lim_{x\to 0} \frac{\sinh^3 x}{\sin^3 x}$, (b) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{(\log(1-x))^2}$, (c) $\displaystyle\lim_{x\to 0} \frac{x - \sinh x}{x \sinh(x^2)}$,
(d) $\displaystyle\lim_{x\to 1/2} \frac{(1-x)^{20} - x^{20}}{(1-x)^{11} - x^{11}}$ [5].

14. Find the radius of convergence of each of the following power series:
(a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2011^n}$, (b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n!}{n^n} x^n$, (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3+1}{n^2+1} x^n$,
(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n - 2^n}{2^n - 1} x^n$, (e) $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n)!} x^n$ [5], (f) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{(n!)^3} x^n$.

15. For $z = x + yi$, find the real and imaginary parts of:
(a) z^3, (b) $1/z^2$, (c) $(1+z)/(1-z)$.

16. Find all complex solutions of the following equations:
(a) $z^6 = -i$, (b) $z^3 = 2 - 2i$ [5], (c) $z^4 = -4$.
(In each case express the solution in the form $x + yi$.)

17. Prove that if $z, w \in \mathbf{C}$ then (a) $|z-w| \geq |z| - |w|$ and (b) $|z/w| = |z|/|w|$ (provided, of course, that $w \neq 0$).

18. Write each of the following in the form $x + yi$:
(a) e^{3-2i}, (b) $\cos(2 - i)$, (c) $\sin(1 + i)$, (d) $\log(3i)$, (e) $\log(-3 + i)$.

19. Let (z_n) be a sequence of complex numbers. Prove that $\lim_{n\to\infty} z_n = z$ if and only if $\lim_{n\to\infty} |z_n - z| = 0$.

20. Prove that for real x and y,

$$|\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y$$

and deduce that $|\sin(x + iy)| \leq \cosh y$. [10]

21. Prove that the set $A = \{z \in \mathbf{C} : \mathrm{Re}(z) < 0\}$ is open and connected.

22. In each case, prove that the function f is continuous on $\mathbf{C}$:
(a) $f(z) = \overline{z}$, (b) $f(z) = |z|$.

23. For each function f, determine where f is analytic, and compute its derivative there:
(a) $f(z) = (z + 1)^4$, (b) $f(z) = (1 - z)/(1 + z)$, (c) $f(z) = \sin(1/z^2)$, (d) $f(z) = e^{-z^2}$, (e) $f(z) = e^z/(e^{2z} + 1)$ [5].

24. In each case, verify the Cauchy-Riemann equations for the following functions (by calculating the appropriate partial derivatives):
(a) $f(z) = z^4$, (b) $f(z) = 1/z^2$, (c) $f(z) = e^{z^2}$.

25. Let $f(x+yi) = u(x, y) + iv(x, y)$ where u and v are real-valued. If f is analytic on $\mathbf{C}$, $u(x, y) = x^2 + xy - y^2$ and $f(0) = 0$ what is $v(x, y)$? [5]

26. Evaluate the following contour integrals:

    (a) $\int_\gamma \mathrm{Im}(z)\, dz$ where γ is the line segment from 0 to i followed by the line segment from i to $1 + i$;

    (b) $\int_\gamma \sin z\, dz$ where γ is the line segment from $-1 + 2i$ to $1 + 2i$;

    (c) $\int_\gamma (1 - z)^2/z\, dz$ where γ is the unit circle (centre 0, radius 1), traversed anticlockwise; [5]

    (d) $\int_\gamma (1 - z)^2/z\, dz$ where γ is the circle with centre 2 and radius 1, traversed anticlockwise;

    (e) $\int_\gamma \overline{z}\, dz$ where γ is the unit circle, traversed anticlockwise.

27. (a) Let γ be the arc of the circle with centre 0 and radius 2 lying in the right half-plane $\{z \in \mathbf{C} : \mathrm{Re}(z) \geq 0\}$. Prove that

$$\left| \int_\gamma \frac{e^z\, dz}{z^3 + 1} \right| \leq \frac{2\pi e^2}{7}.$$ [5]

(b) Let γ be the arc of the unit circle contained in the upper half plane $\{z \in \mathbf{C} : \mathrm{Im}(z) \geq 0\}$. Prove that

$$\left| \int_\gamma \frac{\cos z}{z^4} \, dz \right| \leq \frac{\pi}{2}(e + e^{-1}).$$

(c) Let γ be the circle with centre 0 and radius R. Prove that

$$\left| \int_\gamma \frac{z^2 - 2z - 1}{z^4 + z^2 + 1} \, dz \right| \leq \frac{2\pi R(R+1)^2}{R^4 - R^2 - 1}$$

provided that R is sufficently large.

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