# Analysis: skeleton notes 1: basics

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### The axioms for the real numbers

The set $\mathbf{R}$ of real numbers is defined by the following axioms.

The field axioms

$\mathbf{R}$ has operations of addition and multiplication satisfying:

F1 $x + y = y + x$ and $xy = yx$ for all $x, y \in \mathbf{R}$ (commutative laws);

F2 $x + (y + z) = (x + y) + z$ and $x(yz) = (xy)z$ for all $x, y, z \in \mathbf{R}$ (associative laws);

F3 $(x + y)z = xz + yz$ for all $x, y, z \in \mathbf{R}$ (distributive law);

F4 there are elements 0 and 1 of $\mathbf{R}$ such that $0+x = x = 1x$ for all $x \in \mathbf{R}$; moreover $0 \neq 1$ (identities);

F5 for each $x \in \mathbf{R}$ there is an element $-x \in \mathbf{R}$ such that $x + (-x) = 0$ and for each $y \in \mathbf{R}$ with $y \neq 0$ there is an element $y^{-1} \in \mathbf{R}$ such that $yy^{-1} = 1$ (inverses).

The order axioms

$\mathbf{R}$ has a binary relation $<$ satisfying:

O1 for each $x, y \in \mathbf{R}$ exactly one of the three statements $x = y$, $x < y$ and $y < x$ holds (trichotomy);

O2 if $x, y, z \in \mathbf{R}$, $x < y$ and $y < z$ then $x < z$ (transitivity);

O3 if $x, y, z \in \mathbf{R}$ and $x < y$ then $x + z < y + z$ (additivity);

O4 if $x, y, z \in \mathbf{R}$, $x < y$ and $0 < z$ then $xz < yz$ (multiplicativity).

The completeness axiom

C If $A$ is a nonempty subset of $\mathbf{R}$ and $A$ has an upper bound then $A$ has a least upper bound.

## Notation

These axioms don't mention the operations of subtraction and division. We therefore define $x - y$ as shorthand for $x + (-y)$ and $x/y$ as shorthand for $xy^{-1}$ (in the latter case only when $y$ is nonzero). They also define one inequality symbol $<$; the other familiar symbols $>$, $\leq$ and $\geq$ are defined in terms of $<$. So, $x > y$ means that $y < x$, $x \leq y$ means that either $x = y$ or $x < y$ and $x \geq y$ means that either $x = y$ or $y < x$.

We use standard notation for intervals in $\mathbf{R}$:

$$
\begin{aligned}
[a, b] &= \{x \in \mathbf{R} : a \leq x \leq b\}, \\
(a, b) &= \{x \in \mathbf{R} : a < x < b\}, \\
[a, b) &= \{x \in \mathbf{R} : a \leq x < b\}, \\
(a, b] &= \{x \in \mathbf{R} : a < x \leq b\}, \\
[a, \infty) &= \{x \in \mathbf{R} : a \leq x\}, \\
(a, \infty) &= \{x \in \mathbf{R} : a < x\}, \\
(-\infty, a] &= \{x \in \mathbf{R} : x \leq a\}, \\
(-\infty, a) &= \{x \in \mathbf{R} : x < a\}.
\end{aligned}
$$

Intervals $[a, b]$, $(a, b)$, $[a, b)$ and $(a, b]$ are all *bounded*. Intervals $[a, \infty)$, $(a, \infty)$, $(-\infty, a]$ and $(-\infty, a)$ are *unbounded*. Intervals $[a, b]$, $[a, \infty)$ and $(-\infty, a]$ are *closed*. Intervals $(a, b)$, $(a, \infty)$ and $(-\infty, a)$ are *open*.

## Commentary on the axioms

The axioms F1–F5 guarantee that one can make all the familiar algebraic manipulations involving real numbers that you are used to from school and the first year. In addition, axioms O1–O4 allows one to perform all operations involving inequalities you are familiar with.

We list some of these properties as a theorem.

**Theorem 1** *Let $x$, $y$, $z \in \mathbf{R}$. Then*

- *if $x + z = y + z$ then $x = y$;*
- *if $xz = yz$ and $z \neq 0$ then $x = y$;*
- $0x = 0$*;*

- *if* $x + y = x$ *then* $y = 0$;
- *if* $xy = x$ *and* $x \neq 0$ *then* $y = 1$;
- *if* $x + y = 0$ *then* $y = -x$;
- *if* $xy = 1$ *then* $x \neq 0$ *and* $y = x^{-1}$;
- $-(-x) = x$;
- *if* $x \neq 0$ *then* $x^{-1} \neq 0$ *and* $(x^{-1})^{-1} = x$;
- $(-1)x = -x$;
- $x(-y) = (-x)y = -(xy)$;
- $(-x)(-y) = xy$;
- *if* $x > 0$ *then* $-x < 0$ *and if* $x < 0$ *then* $-x > 0$;
- *if* $x < 0$ *and* $y < z$ *then* $xy > yz$;
- *if* $x \neq 0$ *then* $x^2 > 0$ *(where* $x^2$ *is short for* $xx$*)*;
- $1 > 0$;
- *if* $0 < x < y$ *then* $x^{-1} > y^{-1} > 0$.

**Proof** I'll only prove a handful of these, just to give you the flavour of such arguments.

To prove that $x + y = 0$ implies that $y = -x$, add $-x$ to both sides of $x + y = 0$ to give $(x + y) + (-x) = 0 + (-x)$. By F4, $0 + (-x) = -x$ and $(x + y) + (-x) = (y + x) + (-x) = y + (x + (-x)) = y + 0 = 0 + y = y$ using variously axioms F1, F2, F4 and F5. Hence $y = -x$.

To prove that $x + y = x$ implies $y = 0$ let's add $-x$ to both sides giving $(x+y)+(-x) = x+(-x)$. By axiom F5, $x+(-x) = 0$. Also $(x+y)+(-x) = (y + x) + (-x) = y + (x + (-x)) = y + 0 = 0 + y = y$ using variously axioms F1, F2, F4 and F5. Hence $y = 0$.

To prove that $0x = 0$, consider $(1 + 0)x = 1x + 0x = x + 0x$ by axioms F3 and F4. Also $(1 + 0) = (0 + 1)x = 1x = x$ using axioms F1 and F4. Thus $x = x + 0x$ using axiom F1. But since $x + y = x$ implies $y = 0$, then $0x = 0$.

To prove that $(-1)x = -x$ consider $x + (-1)x = 1x + (-1)x = (1 + (-1))x = 0x = 0$ by axioms F3, F4 and the previous part. As $x + (-1)x = 0$ then $(-1)x = -x$ from the fact that $x + y = 0$ implies that $y = -x$.

The remainder can be proved by this sort of hacking around. This is fun if you like this sort of thing, but it has little to do with the serious business of analysis. Indeed one might argue that if the facts in the theorem didn't follow from our axioms, we'd have included more axioms in order that these basic facts would then follow! $\square$

The completeness axiom is more significant. The field and order axioms are satisfied by the rational numbers $\mathbf{Q}$ (as well as many other structures). For instance there is no square root of 2 inside $\mathbf{Q}$. If we break up the positive rationals into two sets $A$ and $B$ with $A$ consisting of the rational $x$ with $x^2 < 2$ and $B$ consisting of the rational $x$ with $x^2 > 2$ then $A$ has no largest element and $B$ has no smallest element (I'll prove this in the lectures). There is a "hole" between the sets $A$ and $B$. The point of the completeness axiom is to ensure that the reals contain no such "holes".

To explain the jargon in the completeness axiom, an *upper bound* for a subset $A \subseteq \mathbf{R}$ is a number $b \in \mathbf{R}$ with the property that $a \leq b$ for all $a \in A$. A *least upper bound* is what it says on the tin — $c$ is a least upper bound for $A$ if $c$ is an upper bound of $A$ and $c \leq b$ whenever $b$ is an upper bound for $A$. There are similar notions of *lower bound* and *greatest lower bound*. I'll write $\operatorname{lub} A$ and $\operatorname{glb} A$ for the least upper bound and greatest lower bound of $A$, whenever these exist.

For example if $A = \{x \in \mathbf{Q} : x > 0, x^2 < 2\}$ then $A$ has an upper bound (for example 2 is one) but has no least upper bound in $\mathbf{Q}$ (so completeness fails in $\mathbf{Q}$), But $A$ has a least upper bound in $\mathbf{R}$, namely our old friend $\sqrt{2}$. (Proofs of these will be in lectures.)

One important consequence of completeness is the *Archimedean* property of $\mathbf{R}$.

**Theorem 2** *For each $x \in \mathbf{R}$ there is $n \in \mathbf{N}$ with $n > x$.*

**Proof** We argue by contradiction: if the statement is false there is a real number $x$ with the property that $n \leq x$ for all $n \in \mathbf{N}$. That, is $x$ is an upper bound of $\mathbf{N}$. Therefore as $\mathbf{N}$ has an upper bound then it has a least upper bound $a$ by the completeness axiom. Then $n \leq a$ for all $n \in \mathbf{N}$. If $m \in \mathbf{N}$ then $m + 1 \in \mathbf{N}$ and so $m + 1 \leq a$. But then $m \leq a - 1$ so that $a - 1$ is also an upper bound of $\mathbf{N}$. But $a - 1 < a$ contradicting $a$ being the least upper bound of $\mathbf{N}$. $\square$

A consequence of the Archimedean property is that $\mathbf{Q}$ is "dense" in $\mathbf{R}$, that is there is always a rational number between any two reals.

**Theorem 3** *Let $x$, $y \in \mathbf{R}$ with $x < y$. There is some $a \in \mathbf{Q}$ with $x < a < y$.*

**Proof** By the Archimedean property there is some $n \in \mathbf{N}$ with $n > 1/(y - x)$. It follows that $y - x > 1/n$. We claim there is $m \in \mathbf{Z}$ with $x < m/n < y$ (so we can take $a = m/n$). This is equivalent to saying $nx < m < ny$. Let $r$ be the largest integer with $r \leq nx$ $(*)$. Then $r + 1 > nx$ but $r + 1 \leq nx + 1 < ny$, so we can take $m = r + 1$. $\square$

In this proof I "waved my hands" at point $(*)$. There really is such an $r$ but proving this isn't quite immediate. You may like to justify $(*)$ as a follow-up exercise. As another follow-up you might prove that in fact for any $x$ and $y$ with $x < y$ there are *infinitely* many rational $a$ with $x < a < y$.