

Limits and bounds

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19 October 2011

When computing the least upper bound (or greatest lower bound) of a subset A of \mathbf{R} , an alternative to arguing from first principles is to use sequences. We have the following result.

Lemma. *Let A be a subset of \mathbf{R} and suppose that A is bounded above.*

- *There is a sequence (a_n) of elements of A converging to $\text{lub } A$.*
- *If (b_n) is a sequence of elements of A converging to b , and b is an upper bound of A then $b = \text{lub } A$.*

Proof Let $a = \text{lub } A$. For $n \in \mathbf{N}$, $a - 1/n$ is not an upper bound for A so there is some $a_n \in A$ with $a_n > a - 1/n$. But $a_n \leq a$ as a is an upper bound of A . Hence $a - 1/n < a_n \leq a$ and by the squeeze principle (as $a - 1/n \rightarrow a$), $a_n \rightarrow a$.

Now suppose that $b_n \in A$ and $b_n \rightarrow b$ which is an upper bound of A . Let c be any upper bound of A . Then $b_n \leq c$ for all n , and as $b_n \rightarrow b$ then $b \leq c$. Therefore b is the least upper bound of A . \square

Of course this lemma applies *mutatis mutandis* to greater lower bounds.

As an example, consider the set $A = [0, 1) = \{x \in \mathbf{R} : 0 \leq x < 1\}$ which I treated in the lectures. It is clear that 1 is an upper bound of A ; also for $n \in \mathbf{N}$, $1 - 1/n \in A$ and as $1 - 1/n \rightarrow 1$ then by the lemma, 1 is the least upper bound of A .