Analysis: skeleton notes 7: complex numbers and functions

Robin Chapman

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The set $\mathbf{C}$ of *complex numbers* is defined as $\mathbf{C} = \{x + yi : x, y \in \mathbf{R}\}$ where $i^2 = -1$. Addition, subtraction and multiplication of complex numbers (using $i^2 = -1$) is straightforward.

We represent complex numbers as points in the *Argand diagram* of "complex plane". The complex number $z = x + yi$ is identified with the point whose Cartesian coordinates are (x, y).

The *real part* of $z = x + yi$ is $\operatorname{Re} z = x$, its *imaginary part* is $\operatorname{Im} z = y$ and its *complex conjugate* is $\overline{z} = x - yi$. It's straightforward to prove that $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{z - w} = \overline{z} - \overline{w}$, $\overline{zw} = \overline{z}\overline{w}$ and $\overline{\overline{z}} = z$. Also $z\overline{z} = x^2 + y^2 \geq 0$ and $z\overline{z} = 0$ if and only if $z = 0$. The *absolute value* of z is $|z| = \sqrt{z\overline{z}}$. If $z \neq 0$ and $w = \overline{z}|z|^{-2}$ then $zw = 1$ so that z has a reciprocal (and $\mathbf{C}$ is a field). Note that $|z-w|$ is the distance between points z and w in the Argand diagram.

One basic theorem in complex numbers is the *triangle inequality*: $|z+w| \leq |z| + |w|$.

If z is a nonzero complex number then $w = z/|z|$ satisfies $|w| = 1$. So w lies on the *unit circle* in the Argand diagram, that is the circle with centre 0 and radius 1. It follows that there is some real number θ with $w = \cos\theta + i\sin\theta$. We write $e^{i\theta}$ for $\cos\theta + i\sin\theta$ and note that the addition identities for sine and cosine imply that $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$. We can then write $z = re^{i\theta}$ where $r = |z| > 0$ and θ in $\mathbf{R}$. Such a number θ is called an *argument* of z. The argument of z is not unique since $e^{i\theta} = e^{i(\theta+2\pi)}$. However, z has a unique argument θ in the interval $(-\pi, \pi]$ which we call the *principal argument* and denote by $\arg z$. The general argument of z is $\arg z + 2k\pi$ where $k \in \mathbf{Z}$.

We define the complex exponential by $\exp(x + iy) = e^x e^{iy} = e^x(\cos y + i\sin y)$ for $x.\ y \in \mathbf{R}$. Then $\exp(z + w) = \exp(z)\exp(w)$. For non-zero z, the equation $e^w = z$ has the general solution $w = \log|z| + i\arg z + 2k\pi i$ (where $k \in \mathbf{Z}$). Then $\log|z| + i\arg z$ is defined to be the *principal logarithm* $\log z$

of z.

Convergence of sequences and series of complex numbers are defined in much the same way as those of real numbers, and the basic theorems are the same, so we shall not dwell on the details. A sequence (z_n) of complex numbers converges to a limit z if for all $\varepsilon > 0$ then eventually $|z_n - w| < \varepsilon$. Also $\lim_{n\to\infty} z_n = w$ if and only if both $\lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} w$ and $\lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} w$. Sums, differences, products and quotients (under the usual caveats) of convergent complex sequences are convergent. Again, a series $\sum_{n=1}^{\infty} z_n$ converges to z if and only if $\lim_{N\to\infty} \sum_{n=1}^{N} z_n = w$. As with real series, absolute convergence implies convergence.

We also consider complex functions: maps $f : A \to \mathbf{C}$ where $A \subseteq \mathbf{C}$ Limits and continuity for complex functions are defined in the same way as for real functions. For instance $f : A \to \mathbf{C}$ is continuous at $a \in A$ if and only if $\lim_{n\to\infty} f(a_n) = f(a)$ for all sequences (a_n) of points in A with $a_n \to a$. Again, continuity satisfies the same basic properties as for real functions: for example, sums, differences, products, quotients and composites of continuous functions (subject to the usual caveats) are continuous. As a consequence, polynomial functions are continuous, and so are rational functions where they are defined (where the denominator is nonzero).

The complex exponential function exp is continuous on $\mathbf{C}$. Indeed $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$. For real x, as $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$ then $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $i \sin x = \frac{1}{2}(e^{ix} - e^{-ix})$. We define the complex sine and cosine function using these formulae:

$$\cos z = \frac{\exp(iz) + \exp(-iz)}{2}, \qquad \sin z = \frac{\exp(iz) - \exp(-iz)}{2i}.$$

Then $\cos iz = \frac{1}{2}(\exp(z) + \exp(-z)) = \cosh z$ and $\sin iz = -i\frac{1}{2}(\exp(-z) - \exp(z)) = i \sinh z$. This shows that although the sine and cosine are bounded on $\mathbf{R}$ they are not bounded functions on $\mathbf{C}$.

To study differentiability it is convenient to restrict the domains of our functions to sets "without boundary points". The *open disc* of centre $a \in \mathbf{C}$ and radius $r > 0$ is

$$D(a, r) = \{z \in \mathbf{C} : |z - a| < r\}.$$

A subset $U \subseteq \mathbf{C}$ is *open* if it contains an open disc centred at each of its points; more formally U is open if for each $a \in U$ there is $r > 0$ such that $D(a, r) \subseteq U$. Informally speaking a set U is open if none of its "boundary points" are elements of U. As behaviour of a function at boundary points can be complicated, it is convenient when we don't have any! Using the triangle inequality one can prove that each open disc $D(a, r)$ is an open set.

An open set U is *connected* if any two points in U can be joined by a polygonal path lying inside U. A *polygonal path* from z_0 to z_n is a set $[z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, z_n]$ where $[z, w] = \{z + t(w - z) : 0 \le t \le 1\}$ is the line segment joining z and w. A connected open set if often called a *region* or *domain*. The open disc $D(a, r)$ is a connected open set. Connected open sets play in complex analysis an analogous role to open intervals in real analysis.

When $f : U \to \mathbf{C}$ is a function on a connected open set U and $a \in U$, we say that f is *differentiable* at a if

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists; if so this number is called $f'(a)$. If f is differentiable at all points of U then f is an *analytic* function on U. Analytic functions are the main objects of study in complex analysis. As ever, sums, differences, products, quotients and composites are analytic functions (subject to the usual caveats) and their derivatives are given by the familiar formulae from calculus.

We can express a complex function by two real-valued functions of two real variables:

$$f(x + iy) = u(x, y) + iv(x, y).$$

If f is analytic the real functions u and v satisfy the *Cauchy-Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

There is a converse (which I won't prove) to the effect that if u and v satisfy the Cauchy-Riemann equations **and** their partial derivatives are continuous, then f is analytic.

We now turn to integration of complex functions. Our integrals are analogues of line integrals in vector calculus. Before proceeding to complex integrals, note that differentiation and integration of complex-valued functions of a real variable is straightforward. Let $f : [a, b] \to \mathbf{C}$ be a function. Then $f(t) = u(t) + iv(t)$ where u and v are real-valued functions on $[a, b]$. We define $f'(t)$ to be $u'(t) + iv'(t)$ for any t where u and v are both differentiable. We also define

$$\int_a^b f(t)\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt$$

as long as both real integral exist (they certainly will if f is continuous). As an exercise, you might prove that $\int_a^b (f(t) + g(t))\, dt = \int_a^b f(t)\, dt + \int_a^b g(t)\, dt$ and $\int_a^b \alpha f(t)\, dt = \alpha \int_a^b f(t)\, dt$ for $\alpha \in \mathbf{C}$.

A *path* is a continuous map $\gamma : [a, b] \to \mathbf{C}$. It is *smooth* if its derivative γ' exists and is continuous on $[a, b]$. A *contour* is a finite sequence of paths joined end-to-end; formally it is a sequence $\gamma = (\gamma_1, \ldots, \gamma_n)$ of smooth paths $\gamma_k : [a_k, b_k] \to \mathbf{C}$ with $\gamma_k(b_k) = \gamma_{k+1}(a_{k+1})$ for $1 \le k \le n-1$. This contour is *closed* if $\gamma_n(b_n) = \gamma_1(a_1)$. We define the *path integral* of a function f along the smooth path γ as

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

For this to make sense, f must be defined at all points $\gamma(t)$ of the path. If f is continuous on a region containing these points then certainly this integral makes sense. The *contour integral* of f along the contour $\gamma = (\gamma_1, \ldots, \gamma_n)$ is defined as

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz.$$

The *length* of a smooth path $\gamma : [a, b] \to \mathbf{C}$ is

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\,dt$$

but perhaps a better way to think about it is as the distance travelled by a point moving in the Argand diagram with position $\gamma(t)$ at time t, between times a and b. Naturally, the length of a contour $\gamma = (\gamma_1, \ldots, \gamma_n)$ is $\ell(\gamma) = \ell(\gamma_1) + \cdots + \ell(\gamma_n)$. There is a useful method of bounding a contour integral:

$$\left|\int_\gamma f(z)\,dz\right| \le M\ell(\gamma)$$

where M is a number with the property that $|f(z)| \le M$ for all z on the contour γ. (If f is continuous, there will always be such an M.)

Many contour integrals can be evaluated using the complex version of the Fundamental Theorem of Calculus. This states that if $f(z) = g'(z)$ in a region U and γ is a contour in U then

$$\int_\gamma f(z)\,dz = g(\beta) - g(\alpha)$$

where α and β are the starting and ending points of the contour γ. When γ is a closed contour, then $\alpha = \beta$ and then $\int_\gamma f(z)\,dz = 0$ (provided that f is the derivative of an analytic function). If $\gamma : [a, b] \to \mathbf{C}$ is a smooth path we prove the fundamental theorem for γ simply by calculating

$$\int_\gamma f(z)\,dz = \int_a^b g'(\gamma(t))\gamma'(t)\,dt = \int_a^b \frac{d}{dt} g(\gamma(t))\,dt = g(\gamma(b)) - g(\gamma(a)) = g(\beta) - g(\alpha).$$

Consider the integral,

$$\int_\gamma \frac{dz}{z} = 2\pi i$$

where γ is the unit circle. This is an integral over a closed contour, but is nonzero. This shows that $f(z) = 1/z$ **cannot** be expressed as a derivative $1/z = g'(z)$ in any region U containing the unit circle. with more precise analysis one can prove that $1/z$ is not a derivative in any region that "winds around" 0. In particular there is no analytic function on $\mathbf{C} \setminus \{0\}$ which corresponds to a logarithm. This is a very significant fact about the arithmetic and geometry of the complex numbers.