MAS2010

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTER SCIENCE & MATHEMATICS

DEPARTMENT OF MATHEMATICAL SCIENCES

ANALYSIS

6 June 2005

9:30 a.m. – 11:30 a.m.
Duration: 2 hours

Examiner: Professor A. Langer

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline.

Calculators labelled as approved by the Department of Mathematical Sciences may be used.

SECTION A

1. (a) Find the set of real numbers x satisfying

$$\frac{-3}{x-4} \leq x.$$

(5)

(b) Find the set of real numbers x satisfying

$$|2x+1| \leq |3x-6|.$$

(5)

(c) If $\sum_{n=1}^{\infty} a_n z^n$ has finite radius of convergence R, what is the radius of convergence of $\sum_{n=1}^{\infty} a_n z^{2n}$? Give a proof of your answer. (4)

(d) Compute the supremum and infimum of the function

$$f(x) = \frac{x^2}{1+x^2} \quad \text{on } \mathbb{R}.$$

(6)

(e) Determine whether or not the following series are convergent

$$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}, \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \frac{(2+n)}{\sqrt{4n^2-1}}.$$

(10)

(f) Calculate $\lim_{n\to\infty} x_n$ in each of the following cases

$$\text{(i)} \quad x_n = \frac{\cos(n^2)}{\sqrt{n^2+n}}, \qquad \text{(ii)} \quad x_n = \frac{(3n+1)^2}{\sqrt{4n^4+1}}.$$

(8)

(g) Calculate the radius of convergence R of the power series $\sum_{n=1}^{\infty} a_n z^n$ in each of the following cases:

$$\text{(i)} \quad a_n = \frac{(2n)!}{(n!)^2}, \qquad \text{(ii)} \quad a_n = \frac{2^n}{3^n+1}.$$

(12)

[50]

SECTION B

2. (a) Show that a convergent sequence (a_n) is bounded. (5)

   (b) (i) Let $a_n = 1 + (-1)^n$. Show that (a_n) diverges.

   (ii) Let $(a_n), a_n > 0$ be a sequence with $a_n \longrightarrow a$. Show that the sequence $(\sqrt{a_n})$ is convergent and determine its limit.

   (iii) Let $(a_n), (b_n)$ be sequences with $a_n \longrightarrow a$, $b_n \longrightarrow b$. Show that the sequences $(\max(a_n, b_n)), (\min(a_n, b_n))$ and $(|a_n|)$ are convergent and $\max(a_n, b_n) \longrightarrow \max(a, b)$, $\min(a_n, b_n) \longrightarrow \min(a, b)$, $|a_n| \longrightarrow |a|$. (15)

   (c) Prove that $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} (a_n) \cdot \lim\limits_{n\to\infty} (b_n)$ provided that (a_n), (b_n) are convergent sequences with $\lim\limits_{n\to\infty} (a_n) = a$, $\lim\limits_{n\to\infty} (b_n) = b$. (5)

[25]

3. (a) (i) State (without proof) the Intermediate Value Theorem for continuous functions on closed bounded intervals.

   (ii) Prove that the equation

$$f(x) = x \cdot \sin^2 x - \cos x = 0$$

   has at least 4 solutions in $[-2\pi, 2\pi]$. (12)

   (b) Show that a function that is differentiable at a, is also continuous at a. Give a counterexample to show that the converse statement (continuous$\Rightarrow$ differentiable) is not true. (8)

   (c) Show that the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = |x|^n$$

   is differentiable at 0 for all $n \in \mathbb{N}, n > 1$. (5)

[25]

4. (a) Compute the derivatives of the following functions

$$\text{(i)}\quad f(x) = \exp\left(\frac{1-x^2}{1+x^2}\right), \qquad \text{(ii)}\quad g(x) = \log(\log x), x > 1.$$

(8)

(b) Prove L' Hôpital's rule which states that if f and g are functions which are differentiable on an open interval I containing a such that $f(a) = g(a) = 0$ and $g'(x) \neq 0$ except perhaps at a, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the second limit exists. (9)

Use L'Hôpital's rule to compute

(i) $\displaystyle \lim_{x\to 0} \frac{3x - \sin x}{x}$,

(ii) $\displaystyle \lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x}$.

(8)

[25]