MAS2010

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTER SCIENCE AND MATHEMATICS

MATHEMATICAL SCIENCES

ANALYSIS

May/June 2006

Time allowed: 2 HOURS.

Examiner: Professor A. Langer

This is a **CLOSED BOOK** examination.

The mark for this module is calculated from 75% of the percentage mark for this paper plus 25% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

SECTION A

1. (a) Find the set of real numbers x satisfying

$$\frac{x}{x-1} \leq \frac{2x+3}{x+3},$$

expressing your answer in interval notation. (5)

(b) Find positive numbers k and N with

$$\left| \frac{n-3}{3n^2-1} \right| < \frac{k}{n} \text{ for all } n \geq N.$$

(4)

(c) State the limit of the sequence $\left(\frac{2n^2+4n+2}{3n^2-2n-2} \right)$.
Prove that your answer is indeed the limit:

(i) from the definition; and

(ii) using the Theorem on the Algebra of Limits of Sequences. (10)

(d) Find the supremum and infimum, where they exist, of the set

$$\{x + |x-1| : x \in \mathbb{R}\}.$$

(6)

(e) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(n!)^2 x^{2n}}{(2n)!}.$$

(8)

(f) Use appropriate tests to decide which of the following series is convergent and which is divergent.

$$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{1+\sin(n)}{3^n+2n^3}, \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \frac{3^n}{2^n+1}.$$

(12)

(g) For the following statements either give a general proof or give a counterexample.

(i) If a function is continuous then it is differentiable.

(ii) If a sequence is bounded then it converges. (5)

[50]

SECTION B

2. (a) State the definition of "$a_n \to l$ as $n \to \infty$", and the definition of "the sequence (a_n) is bounded".

   Prove from these definitions that a convergent sequence is bounded. (5)

   (b) Prove that

   (i) if $|x| < 1$ then $x^n \to 0$ as $n \to \infty$;

   (ii) if $x = 1$ then $x^n \to 1$ as $n \to \infty$;

   (iii) if $x = -1$ then the sequence (x^n) diverges;

   (iv) if $|x| > 1$ then the sequence (x^n) diverges. (10)

   (c) Prove that one of the following statements is true and the other is false.

   (i) If $x_n \to 1$ as $n \to \infty$, then $(x_n^n) \to 1$ as $n \to \infty$.

   (ii) If $0 < r < 1$ and $x_n \to r$ as $n \to \infty$, then $(x_n^n) \to 0$ as $n \to \infty$. (10)

   [25]

3. (a) (i) State (without proof) the Intermediate Value Theorem for continuous functions on closed bounded intervals.

   (ii) Prove that the equation $f(x) = 2x\cos x - \sin x = 0$ has at least 5 solutions in $[-2\pi, 2\pi]$. (Hint: Consider $f(k\pi)$ for $k = -2, -1, 0, 1, 2$). (10)

   (b) State, and prove, the Fixed Point Theorem for a continuous function from the closed interval $[0,1]$ to itself. (9)

   (c) Let $f : [0,1] \to [0,1]$ be a continuous function. Explain (no proofs required) why the function $g(x) = (f(x))^2$ is a continuous function from $[0,1]$ into $[0,1]$. Hence use the Fixed Point Theorem to show that there is a number c with $0 \leq c \leq 1$ and $f(c) = \sqrt{c}$. (6)

   [25]

4. (a) State (without proof) Rolle's Theorem. Use this to prove Cauchy's Mean Value Theorem, which states that if f and g are differentiable on (a, b) and continuous on $[a, b]$ then, provided $g'(x) \neq 0$ for all $x \in (a, b)$, there exists a $c, a < c < b$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(8)

(b) Prove L'Hôpital's rule, which states that if f and g satisfy the hypotheses of Cauchy's Mean Value Theorem, and if x_0 satisfies $a < x_0 < b$ and $f(x_0) = g(x_0) = 0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \text{ provided the latter limit exists.}$$

(6)

(c) Let $p(x)$ be a polynomial satisfying $p(-1) = 2$, $p(0) = -1$, $p(1) = 0$, $p'(2) = p(2) = 0$, and $p(3) = 0$.
Show that there are three distinct numbers $c_1, c_2, c_3 \in (-1, 3)$ with $p^{(2)}(c_i) = 0$ for $i = 1, 2, 3$. (6)

(d) Use L'Hôpital's rule to evaluate

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \cot\ x \right).$$

(5)

[25]