MAS2010

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTER SCIENCE AND MATHEMATICS

MATHEMATICAL SCIENCES

May/June 2007

ANALYSIS

Module Leader: Professor A. Langer

Duration: 2 HOURS.

The mark for this module is calculated from 75% of the percentage mark for this paper plus 25% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a CLOSED BOOK examination.

SECTION A

1. (a) Find the set of real numbers x satisfying

$$\frac{3}{x-4} < -x.$$

(5)

(b) Find the set of real numbers x satisfying

$$|x^2 + x - 4| = 2.$$

(5)

(c) Show that if (x_n) is a sequence with $x_n \leq b$ for all n and $\lim_{n\to\infty} x_n = L$, then $L \leq b$. (4)

(d) Compute the supremum and infimum of the function

$$f(x) = \frac{x}{1+|x|}, \text{ with } x \in \mathbb{R}.$$

(6)

(e) Determine whether or not the following series are convergent

$$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{\sin(2^n)}{2^n} \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \frac{n-3}{\sqrt{2+9n^6}}.$$

(10)

(f) Calculate $\lim_{n\to\infty} x_n$ in each of the following cases

$$\text{(i)} \quad x_n = \sqrt{n+1} - \sqrt{n} \qquad \text{(ii)} \quad x_n = \frac{(n+3)!}{n!n^3}.$$

(8)

(g) Calculate the radius of convergence R of the power series $\sum_{n=1}^{\infty} a_n z^n$ in each of the following cases

$$\text{(i)} \quad a_n = \frac{3n+4}{2^n} \qquad \text{(ii)} \quad \frac{(2n)!}{n^n}.$$

(12)

[50]

SECTION B

2. (a) Show that the product of a nullsequence and a bounded sequence is again a nullsequence. (5)

(b) Find examples of sequences $(a_n), (b_n)$ such that $b_n > 0$ for all n, $a_n \longrightarrow 0$, $b_n \longrightarrow 0$ and

(i) $\dfrac{a_n}{b_n} \longrightarrow l$ where l is a prescribed number;

(ii) $\dfrac{a_n}{b_n} \longrightarrow \infty$;

(iii) $\dfrac{a_n}{b_n} \longrightarrow -\infty$;

(iv) the sequence $\left(\dfrac{a_n}{b_n}\right)$ is bounded but does not converge. (12)

(c) Prove that one of the following statements is true and that the other is false.

(i) If $x_n \longrightarrow 1$ as $n \longrightarrow \infty$, then $(x_n^n) \longrightarrow 1$ as $n \longrightarrow \infty$.

(ii) If $0 < r < 1$ and $x_n \longrightarrow r$ as $n \longrightarrow \infty$, then $(x_n^n) \longrightarrow 0$ as $n \to \infty$. (8)

[25]

3. (a) (i) State (without proof) Rolle's Theorem for differentiable functions on closed bounded intervals.

(ii) Investigate the number of (real) roots of each of the polynomials

$$p(x) = x^3 + 3x + 1 \text{ and } q(x) = x^3 - 3x + 1.$$

(12)

(b) Let

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{for } x < 0 \\ \cos x & \text{for } x \geq 0. \end{cases}$$

Show that f is continuous at 0 (and hence continuous everywhere). (8)

(c) Let f be a differentiable function on (a, b). Show that if $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant function. (5)

[25]

4. (a) Find the derivatives of the functions

$$\text{(i)} \quad f(x) = x^x \text{ for } x > 0 \qquad \text{(ii)} \quad x^{10} \sin \frac{1}{x} \text{ for } x > 0.$$

(8)

(b) Prove L' Hôpital's rule which states that if f and g are functions which are differentiable on an open interval I containing a such that $f(a) = g(a) = 0$ and $g'(x) \neq 0$ except perhaps at a, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the second limit exists. (9)

(c) Use L'Hôpital's rule to compute

$$\text{(i)} \quad \lim_{x \to 0} \frac{1 - \cos x}{x^2} \qquad \text{(ii)} \quad \lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}.$$

(8)

[25]