MAS2101

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTER SCIENCE AND MATHEMATICS

MATHEMATICAL SCIENCES

January 2008

ANALYSIS

Module Leader: Dr M. Saïdi

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) Prove that if x is irrational and y is rational then $x+y$ is irrational. (6)

(b) Prove that the following series converges:

$$\sum_{n=1}^{\infty} \frac{2^n \sqrt{n}}{3^n}.$$

(7)

(c) Find the limit of

$$f(x) = \frac{x \cos x}{x^3 + 1}$$

when x tends to infinity. (6)

(d) Let $f : [a, b] \to \mathbb{R}$ be a function which is continuous and differentiable on $[a, b]$. Suppose that f attains its maximum at a point c, that is $f(c) = \sup\{f(x) \mid x \in [a, b]\}$. Prove that $f'(c) = 0$. (6)

(e) Find the radius of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{(n+1)^2} x^n.$$

(7)

(f) Say whether the following set is open:

$$D = \{z \in \mathbb{C} \mid |z| < 1\}.$$

Justify your assertion. (6)

(g) Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial with real coefficients, that is $a_i \in \mathbb{R}$ for all $0 \le i \le n$. Prove that if $z_0 \in \mathbb{C}$ satisfies $P(z_0) = 0$ then $P(\overline{z_0}) = 0$. (6)

(h) State the *Cauchy-Riemann equations*. Verify them for the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = iz^2 + 2z$. (6)

[50]

SECTION B

2. (a) Define the sequence $(a_n)_{n\geq 1}$ inductively by $a_1 = \frac{3}{2}$, and

$$a_{n+1} = \frac{1}{4}(a_n^2 + 3)$$

for all $n \geq 2$. Prove that $1 < a_n < 3$ for all $n \geq 1$. Prove that the sequence (a_n) is monotonic decreasing. Deduce that the sequence (a_n) converges and find its limit. (10)

(b) Prove that the equation $2\sin x = x^2 - 1$ has a solution x which satisfies $1 < x < 2$. Prove that this solution is unique. (10)

(c) Compute the derivative of the function $f(x) = \sin^{-1}(e^{-x^2-1})$, where $\sin^{-1} : [-1, 1] \to [\frac{-\pi}{2}, \frac{\pi}{2}]$ is the inverse of the sine function. (5)

[25]

3. (a) Evaluate the following limit:

$$\lim_{x\to 1} \frac{(x-1)^3}{\log x}.$$

(6)

(b) Let A and B be two nonempty subsets of $\mathbb{R}$ that are bounded above, and write $A + B = \{a + b \mid a \in A, b \in B\}$. Prove that $A + B$ is bounded above and that $\sup(A + B) = \sup A + \sup B$. (10)

(c) State the *Mean Value Theorem.* Prove that if a function

$$f : [a, b] \to \mathbb{R}$$

is differentiable, and $f'(x) > 0$ for all $x \in [a, b]$, then f is a strictly increasing function on $[a, b]$ (Hint: use the Mean Value Theorem).

(9)

[25]

4. (a) Find, in polar form, the complex numbers $z \in \mathbb{C}$ satisfying the equation: $z^3 = 1 - i$. (8)

(b) Prove the following: if $f : A \to \mathbb{C}$, where $A \subset \mathbb{C}$ is an open subset, is continuous at $z_0 \in \mathbb{C}$ and $f(z_0) \neq 0$ then there exists $\delta > 0$ such that $f(z) \neq 0$ for all $z \in D(z_0, \delta) \cap A$. (9)

(c) Prove that

$$\left| \int_\gamma \frac{dz}{3 + z^3} \right| \leq \frac{\pi}{2},$$

where γ is the upper half of the unit circle traversed once anticlockwise. (8)

[25]