ECM2701

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTING AND MATHEMATICS

MATHEMATICAL SCIENCES

January 2009

Analysis

Module Leader: Dr M. Saïdi

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) Determine the supremum and infimum of the following set

$$S = \{1 + (-1)^n \frac{1}{n},\ n \geq 1,\ n\ integer\}.$$

(8)

(b) Determine whether the following series converges

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}.$$

Justify your answer. (6)

(c) Evaluate the following limit

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{\sin x^2}.$$

(8)

(d) Say whether the following set is open:

$$D = \{z \in \mathbb{C} \mid |z| < \frac{1}{2}\}.$$

Justify your answer. (8)

(e) Find in polar form all the solutions to the equation $z^3 = 1 + i$. (10)

(f) State the *Cauchy-Riemann equations*. Verify them for the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = 2z^2 + iz.$$

(10)

[**50**]

SECTION B

2. (a) Define the sequence $(a_n)_{n\geq 0}$ inductively by $a_0 = 0$, and

$$a_{n+1} = \frac{3a_n + 1}{a_n + 3}$$

for all $n \geq 0$. Prove that $0 \leq a_n < 1$ for all $n \geq 0$. Prove that the sequence (a_n) is monotonic increasing. Deduce that the sequence (a_n) converges and find its limit. (10)

(b) Let f be continuous on $[a, b]$, and suppose that $f(c) \neq 0$ for some $c \in [a, b]$. Show that there exists a $\delta > 0$ with the property that $f(x) \neq 0$ for all $x \in [a, b]$ such that $|x - c| < \delta$. (10)

(c) Compute the derivative of the function

$$f(x) = \sin^{-1}(e^{-x^2-1})$$

where $\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the inverse of the sine function. (5)

[25]

3. (a) Let A be a set of positive real numbers with infimum $\beta > 0$. Define

$$B = \{\frac{1}{a} \mid a \in A\}.$$

Show that B has a supremum. What is its value? Justify your answer. (10)

(b) State the *Mean Value Theorem*. Prove that if a function

$$f : [a, b] \to \mathbb{R}$$

is differentiable, and $f'(x) < 0$ for all $x \in [a, b]$, then f is a strictly decreasing function on $[a, b]$. (10)

(c) Find the derivative of the function

$$f(x) = \cos[\log(1 + x^2)].$$

(5)

[25]

4. (a) Show that the set

$$\{z \in \mathbb{C} \mid Im(z) > 0\}$$

is an open set. (9)

(b) Evaluate the integral

$$\int_\gamma \sin 2z \; dz$$

where γ is the line segment joining $i+1$ to $-i$. (6)

(c) Let γ be the arc of the circle $\{|z| = 2\}$, traversed once anti-clockwise, that lies in the first quadrant $\{z \in \mathbb{C} \mid Im(z) \geq 0,\ Re(z) \geq 0\}$. Show that

$$\left| \int_\gamma \frac{dz}{z^2+1} \right| \leq \frac{\pi}{3}.$$

(10)

[**25**]