ECM2701

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTING AND MATHEMATICS

MATHEMATICAL SCIENCES

May/June 2010

Analysis

Module Leader: Dr M. Saïdi

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) Show that if x is rational and y is irrational then $x+y$ is irrational. (8)

   (b) Determine whether the following series converges:

$$\sum_{n=1}^{\infty} \frac{2^n\sqrt{n}}{3^n}.$$

   Justify your answer. (6)

   (c) Evaluate the following limit

$$\lim_{x\to 0} \frac{\cos x - 1}{2x^2}.$$

   (8)

   (d) Say whether the following set is closed:

$$D = \{z \in \mathbb{C} \mid |z| \geq \frac{1}{3}\}.$$

   Justify your answer. (8)

   (e) Find in polar form all the solutions to the equation $z^4 - 16i = 0$. (10)

   (f) State the *Cauchy-Riemann equations.* Verify them for the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = z^2 - 2iz.$$

   (10)

[50]

SECTION B

2. (a) Define the sequence $(a_n)_{n\geq 1}$ inductively by $a_1 = \frac{3}{2}$, and

$$a_{n+1} = \frac{1}{4}(a_n^2 + 3)$$

for all $n \geq 1$. Prove that $1 < a_n < 3$ for all $n \geq 1$. Prove that the sequence (a_n) is monotonic decreasing. Deduce that the sequence (a_n) converges and find its limit. (10)

(b) Prove that the equation $2\sin x = x^2 - 1$ has a solution x which satisfies $1 < x < 2$. Show that this solution is unique. (10)

(c) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{(n+1)^2} x^n.$$

(5)

[25]

3. (a) Let A and B be non-empty sets of real numbers with supremums $\alpha = \sup A$, and $\beta = \sup B$, respectively. Define the set

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Show that the set $A + B$ has a supremum. Find the supremum of $A + B$. (10)

(b) State the *Intermediate Value Theorem.* Let $f : [0, 1] \to \mathbb{R}$, and $g : [0, 1] \to \mathbb{R}$, be continuous functions. Assume that $f(0) < g(0)$ and $f(1) > g(1)$. Show that there exists $0 < x < 1$ such that $f(x) = g(x)$. (10)

(c) Find the derivative of the function

$$f(x) = \sin[\log(x + x^2)].$$

(5)

[25]

4. (a) Evaluate the integral

$$\int_\gamma \sin 2z \, dz$$

where γ is the line segment joining $i + 1$ to $-i$. (10)

(b) Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ be a polynomial with real coefficients, that is $a_i \in \mathbb{R}$ for all $0 \leq i \leq n$. Prove that if $z_0 \in \mathbb{C}$ satisfies $P(z_0) = 0$ then $P(\overline{z_0}) = 0$. (5)

(c) Let γ be the arc of the circle $\{|z| = 2\}$, traversed once anti-clockwise, that lies in the first quadrant $\{z \in \mathbb{C} \mid Im(z) \geq 0,\ Re(z) \geq 0\}$. Show that

$$\left| \int_\gamma \frac{dz}{z^2+1} \right| \leq \frac{\pi}{3}.$$

(10)

[25]