ECM2701

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

January 2011

Analysis

Module Leader: Robin Chapman

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) Find irrational numbers a and b such that $a+b$ is rational. Also find irrational numbers c and d such that cd is rational. (4)

(b) Find the greatest lower bound and the least upper bound (if they exist) of the set

$$A = \{(-1)^n/(n^2+1) : n \in \mathbf{Z}\}.$$

(8)

(c) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n+\sin n}{\sqrt{n^4+1}}$$

converges. (8)

(d) Prove that the equation

$$e^x = 2\cos x$$

has exactly one solution in the interval $[0, \frac{\pi}{2}]$. (10)

(e) Compute

$$\lim_{x\to 0} \frac{x-\sin x}{\sinh^3 x}.$$

(10)

(f) Find all complex solutions of the equation

$$z^8 = 16$$

expressing each in the form $z = x + iy$. (10)

[**50**]

SECTION B

2. (a) What is an *absolutely convergent* series? Prove that each absolutely convergent series is convergent. (You may assume the comparison test for series with non-negative terms). (9)

   (b) Define a sequence (a_n) recursively by $a_1 = 2$ and

$$a_{n+1} = 4 - \frac{3}{a_n}.$$

   Prove that $1 < a_n < 3$ for all n and that (a_n) is increasing. Hence prove that (a_n) is convergent and find its limit. (8)

   (c) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n!(2n)!}{(3n)!} x^n.$$

(8)

[25]

3. (a) State *Rolle's theorem* and the *mean value theorem*. Assuming Rolle's theorem, prove the mean value theorem. (10)

   (b) Using the mean value theorem, or otherwise, prove that

$$1 - \cos t < t \sin t$$

   whenever $0 < t < \frac{\pi}{2}$. (5)

   (c) Prove that the function f defined by

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0 \end{cases}$$

   is differentiable for all x but that f' is not continuous at 0. (10)

[25]

4. (a) Let f be a complex function and write

$$f(x+iy) = u(x,y) + iv(x,y)$$

where u and v are real-valued functions. Prove that if f is analytic then the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold. (11)

(b) Evaluate the integral

$$\int_\gamma \cos z \, dz$$

where γ is the line segment from $\frac{\pi}{4} - i$ to $\frac{\pi}{4} + i$. (7)

(c) Prove that

$$\left| \int_C \frac{e^z}{4z^2+1} \, dz \right| \leq \frac{2\pi e}{3}$$

where C is the unit circle (with centre 0 and radius 1). (7)

[**25**]