## ECM2701

## UNIVERSITY OF EXETER

# COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES 

## MATHEMATICS

## January 2012

Analysis

## Module Leader: Robin Chapman

## Duration: 2 HOURS.

The mark for this module is calculated from $80 \%$ of the percentage mark for this paper plus $20 \%$ of the percentage mark for associated coursework.

Answer Section A (50\%) and any TWO of the three questions in Section B (25\% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a CLOSED BOOK examination.

## SECTION A

1. (a) Prove that $\sqrt{5}+\sqrt{10}$ is irrational. (You may assume, without proof, that $\sqrt{2}$ is irrational.)
(b) Find

$$
\begin{equation*}
\lim _{m \rightarrow \infty}\left(\lim _{n \rightarrow \infty} \frac{m}{m+n}\right) \quad \text { and } \quad \lim _{n \rightarrow \infty}\left(\lim _{m \rightarrow \infty} \frac{m}{m+n}\right) \tag{6}
\end{equation*}
$$

(c) Let $\left(a_{n}\right)$ and $\left(b_{n}\right)$ be null sequences (sequences that converge to 0 ). Prove directly from the definition of convergence, that $\left(a_{n}+b_{n}\right)$ is a null sequence.
(d) Determine whether the series

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{2^{n} n+1}{\sqrt{n^{4}+4^{n} n^{3}}} \tag{10}
\end{equation*}
$$

converges.
(e) Prove that the equation

$$
\begin{equation*}
e^{x}=2-4 x+5 x^{2} \tag{8}
\end{equation*}
$$

has at least two solutions in the interval $[0,1]$.
(f) Compute

$$
\begin{equation*}
\lim _{x \rightarrow 0} \frac{8-9 \cos x+\cos 3 x}{\sin ^{4} 2 x} \tag{8}
\end{equation*}
$$

(g) Find all complex solutions of the equation

$$
\begin{equation*}
z^{6}=-64 \tag{8}
\end{equation*}
$$

expressing each in the form $z=x+i y$.

## SECTION B

2. (a) Using the completeness axiom, prove that each bounded increasing sequence is convergent.
Hence prove the following form of the comparison test:
If $\left(a_{n}\right)$ and $\left(b_{n}\right)$ are sequences with $0 \leq a_{n} \leq b_{n}$ for all $n$, and if $\sum_{n=1}^{\infty} b_{n}$ is convergent, then $\sum_{n=1}^{\infty} a_{n}$ is convergent.
(b) Consider the sequence ( $a_{n}$ ) defined recursively by $a_{1}=1$ and

$$
a_{n+1}=\frac{5 a_{n}+2}{2 a_{n}+1}
$$

for $n \geq 1$. Prove that the sequence $\left(a_{n}\right)$ is convergent and find its limit.
(c) Determine the radius of convergence of the power series

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{(2 n)!}{n^{n} n!} x^{n} . \tag{6}
\end{equation*}
$$

(You may assume, without proof, that $e^{t}=\lim _{n \rightarrow \infty}(1+t / n)^{n}$.)
3. (a) Let

$$
f(x)=x^{3}+a x^{2}+b x+c
$$

where $a, b$ and $c$ are real numbers and also $b>0$. Assume also that $-a \neq-c / b$. Prove that the equation $f(x)=0$ has a solution in the closed interval with endpoints $-a$ and $-c / b$.
(You may assume the intermediate value theorem.)
(b) Let $g$ be a differentiable function on the interval $[a, b]$ with $g^{\prime}(a)>0$ and $g^{\prime}(b)<0$. Prove that the maximum value of $g$ on $[a, b]$ is attained at a point $c \in(a, b)$ and deduce that $g^{\prime}(c)=0$. (You may assume the boundedness theorem.)
(c) Using the mean value theorem, or otherwise, prove that

$$
\begin{equation*}
2 x-\pi / 2 \leq \tan x-1 \leq 4 x-\pi \tag{7}
\end{equation*}
$$

whenever $\pi / 4<x<\pi / 3$.
4. (a) Recall that if a complex function

$$
f(x+i y)=u(x, y)+i v(x, y)
$$

is analytic (where $u$ and $v$ are real-valued functions) then $u$ and $v$ satisfy the Cauchy-Riemann equations

$$
\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text { and } \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}
$$

Using this notation, suppose that $f$ is analytic on $\mathbf{C}, u(x, y)=$ $e^{y} \sin x+x^{2}+x y-y^{2}$ and $f(0)=0$. Find $v(x, y)$.
(b) Evaluate the integral

$$
\int_{\gamma} \frac{(1-z)^{3}}{z^{2}} d z
$$

where (i) $\gamma$ is the circle with centre 0 and radius 1 , and where (ii) $\gamma$ is the circle with centre 2 and radius 1 .
(c) Prove that

$$
\begin{equation*}
\left|\int_{C} \frac{e^{-z}}{3 z^{3}+2} d z\right| \leq 2 \pi e \tag{10}
\end{equation*}
$$

where $C$ is the unit circle (with centre 0 and radius 1 ).

