Analysis: skeleton notes 4: functions

Robin Chapman

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We study functions of the real variable, and discuss their limiting values and continuity.

A *real function* is some mapping $f : A \to \mathbf{R}$ where A is a subset of $\mathbf{R}$. In most of our examples A will be an interval (or maybe an interval with a few points removed) so it will cause little harm to think of A as an interval.

As with sequences there is a notion of monotonicity. A function $f : A \to \mathbf{R}$ is *increasing* if $f(a) \le f(b)$ whenever $a < b$ and is *strictly increasing* if $f(a) < f(b)$ whenever $a < b$. Similarly we can define decreasing and strictly decreasing functions. A function is *monotone* if it is either increasing or decreasing.

Similarly to sets and sequences a function may be bounded. $f : A \to \mathbf{R}$ is *bounded above*, if there is $C \in \mathbf{R}$ such that $f(x) \le C$ for all $x \in A$. Again there is a corresponding notion of being *bounded below* and f is *bounded* if f is both bounded above and below.

One of the most important concepts in analysis is that of a limit. Let $f : A \to \mathbf{R}$ be a function with $A \subseteq \mathbf{R}$ and let $b \in \mathbf{R}$. Assume there is a sequence (a_n) with each $a_n \in A$, each $a_n \ne b$ and with $a_n \to b$. We say that *f has the limit c at b* if for every sequence (a_n) with

- $a_n \in A$ and $a_n \ne b$ for all n, and $a_n \to b$

then $f(a_n) \to c$.

If this is the case we write $f(x) \to c$ as $x \to b$, or $\lim_{x\to b} f(x) = c$. Note that whether or not the limit exists does not depend on the value $f(b)$ nor even whether $f(b)$ exists (the limit $\lim_{x\to b} f(x)$ may exist even when $b \notin A$).

There are variants on the notion of limit involving "infinity". For instance $\lim_{x\to\infty} f(x) = c$ means that $f(a_n) \to c$ for all sequences (a_n) in A diverging to ∞. Another example: $f(x) \to -\infty$ as $x \to b$ means that for all sequences (a_n) with $a_n \in A$ and $a_n \ne b$, and with $a_n \to a$ then $f(a_n)$ diverges to $-\infty$. I won't exhaustively list all such possibilities; I hope common sense should tell you what's meant in each case.

Limits obey the "algebra" of limits. Let $f : A \to \mathbf{R}$ and $g : A \to \mathbf{R}$ and suppose that $f(x) \to c$ as $x \to b$ and $g(x) \to d$ as $x \to b$. Then

- $f(x) + g(x) \to c + d$ as $x \to b$,
- $f(x) - g(x) \to c - d$ as $x \to b$,
- $f(x)g(x) \to cd$ as $x \to b$, and
- provided that $g(x) \neq 0$ for all $x \in A$ and $d \neq 0$ then $f(x)/g(x) \to c/d$ as $x \to b$.

All of these follow immediately from the corresponding results for sequences.

One important example of a limit, which doesn't follow from the algebra of the limits is

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

I will give a geometric proof of this in the lectures.

A function $f : A \to \mathbf{R}$ is *continuous at* $a \in A$ if whenever (a_n) is a sequence of points in A converging to a then $f(a_n) \to f(a)$. This is equivalent to saying that $\lim_{x \to a} f(x) = f(a)$ (whenever the limit makes sense). A function $f : A \to \mathbf{R}$ is *continuous* if f is continuous at every $a \in A$. The identity function $x \mapsto x$ and all constant functions $x \mapsto c$ (for $c \in \mathbf{R}$) are continuous. There is an "algebra of continuity": if f, g are continuous on A then so are $f + g$, $f - g$, fg and (provided that g is never zero) f/g. As a consequence all *polynomials*

$$p : x \mapsto a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

are continuous, and all *rational functions*

$$r : x \mapsto \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m}$$

are continuous where defined (that is, except where the denominator is zero).

There are some big theorems concerning continuous function. The Intermediate Value Theorem states:

> Let f be a continuous function on the interval $[a, b]$. If $c \in \mathbf{R}$ and either $f(a) < c < f(b)$ or $f(a) > c > f(b)$ then there is $t \in (a, b)$ with $f(t) = c$.

Intuitively this means that a continuous path between two points on different sides of a line must pass through that line. Proving such results is essential in checking whether our naive intuitions regarding continuity really do square with our mathematical definitions. In particular the analogue of the intermediate value theorem over the rationals is false! I wrote up the proof I gave last year in my “proofs of major theorems” document that is on the website, but I’ll give a different prove in this year’s lectures, just for variety.

Another equally important theorem lacks a standard name, but I call it the Boundedness Theorem. It states:

> Let f be a continuous function on the interval $[a, b]$. There are c, $d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.

This is often stated that a continuous function on a closed bounded interval is bounded and it “attains its bounds”. For a proof see “proofs of major theorems”. Together the intermediate value theorem and boundedness theorem amount to saying that if $f : [a, b] \to \mathbf{R}$ is continuous, then the range $f([a, b])$ of f is a closed bounded interval.

The final big theorem involving continuous functions is the Inverse Function Theorem. This states that

> Let $f : I \to \mathbf{R}$ be a strictly increasing or strictly decreasing function on an interval I. Then its range $J = f(I)$ is an interval, and the inverse function f^{-1}, characterised by $f^{-1}(f(x)) = x$, is continuous on J.

From the inverse function theorem we obtain results such as the continuity of the n-th root function. More precisely, let $I = [0, \infty)$ and n be a positive integer and define $f(x) = x^n$. Then the inverse function $f^{-1} : I \to \mathbf{R}$ is given by $f^{-1}(y) = \sqrt[n]{y} = y^{1/n}$ and is continuous.

Another example is to note that since the exponential function is continuous then the logarithm function from $(0, \infty)$ to $\mathbf{R}$ is continuous. Similarly the continuity of the trigonometric functions implies that of the inverse trigonometric functions as defined in the usual way on the appropriate intervals.