Analysis: harder questions

This sheet is entirely optional; it consists of more challenging examples (more challenging than I would expect to set on an exam)

1. Let A be a nonempty subset of $\mathbf{N}$. Prove that A has a least element. (It's clear that A has a greatest lower bound a; you need to show that $a \in A$.)

2. Determine whether each of the folllowing series are convergent:

$$\text{(a)} \quad \sum_{n=1}^{\infty}(1-\cos(1/n)); \qquad \text{(b)} \quad \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2};$$

$$\text{(c)} \quad \sum_{n=0}^{\infty}\frac{e^n n!}{n^n}; \qquad \text{(d)} \quad \sum_{n=0}^{\infty}\frac{n^n}{e^n n!}.$$

3. Prove that the sequence $(\sin n)$ is divergent.

4. Prove that the sequence (a_n) defined by

$$a_n = \left(1+\frac{1}{n}\right)^n$$

   is increasing and bounded, and so convergent. (It converges to e).

5. The *Cauchy condensation test* states that:

   Let (a_n) be a decreasing sequence of positive numbers. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{m=1}^{\infty} 2^m a_{2^m}$ converges.

   Prove it.

6. Let (a_n) be a divergent series of real numbers. Prove that (a_n) has either

   - a subsequence diverging to ∞, or
   - a subsequence diverging to $-\infty$, or
   - two subsequences converging to different limits.

7. Consider a sequence (P_n) not of numbers, but of points in the plane (you can write $P_n = (x_n, y_n)$ where x_n and y_n are real numbers). Give suitable definitions for *convergence* and *boundedness* for such a sequence, and prove that a bounded sequence (P_n) of points in the plane has a convergent subsequence.

8. Let (a_n) be a convergent sequence with limit L. We define a new sequence (b_n) by
$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$
Prove that (b_n) also converges to L.

9. Find a continuous function on the interval $[0, \infty)$ with range $(-1, 1)$.

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