Previous exams: hints and outline solutions

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Throughout "bookwork" refers to standard theory, convered in lectures (and also in standard texts).

2009/2010

1(a) Show that if both x and $x + y$ are rational then so is y.

1(b) Convergent: ratio test.

1(c) This is basically 1(d) on sheet 2.

1(d) I am not covering closed sets this year, but you might instead try to decide whether $\{z \in \mathbf{C} : |z| > 1/3\}$ is an open set.

1(e) This is a standard type of example, question 16 of sheet 2 has three examples like this. It's an excellent follow-up to give the solutions in $x + iy$ form, but this will need finding sines/cosines of some "non-standard" angles.

1(f) The Cauchy-Riemann equations are bookwork. For the example expand out $(x + yi)^2 - 2i(x + yi)$ into real and imaginary parts, and compute their partial derivatives.

2(a) Prove the bound $1 < a_n < 3$ by induction: it's easy to show that $1 < a < 3$ implies $1 < \frac{1}{4}(a^2 + 3) < 3$. To show the sequence is decreasing, express $a_n - a_{n+1}$ in terms of a_n; you get a quadratic in a_n which factorizes and so should clearly be positive when $1 < a_n < 3$. Bounded monotone sequences converge, and the limit A satisfies $A = \frac{1}{4}(A^2 + 3)$ — this gives a quadratic equation for A only one of whose roots is a possible limit.

2(b) For existence apply the intermediate value theorem to $f(x) = 2\sin x - (x^2 - 1)$. For uniqueness, consider $f'(x)$ and show it's negative between $x = 1$ and $x = 2$ (so that f is strictly decreasing there).

2(c) Radius of convergence 1/2 by the ratio test.

3(a) Ugh! the plural of the Latin word supremum is suprema. It's what I call the least upper bound. Anyway this is question 4 of sheet 1 and the answer is of course $\alpha + \beta$.

3(b) The intermediate value theorem is bookwork. Apply it to $h(x) = f(x) - g(x)$.

3(c) This is A-level calculus (chain rule), not really second year University analysis.

4(a) Best done via fundamental theorem of calculus: $\sin 2z = f'(z)$ where $f(z) = -\frac{1}{2}\cos 2z$. Then the integral equals $f(-i) - f(1+i)$. This can be expanded out using $\cos(2+2i) = \frac{1}{2}(e^{2+2i} + e^{-2-2i})$ and $\cos(-2i) = \cos(2i) = \cosh(2)$ etc.

4(b) I didn't do any examples like this, but it follows from the facts that conjugation respects addition and multiplication (so that $\overline{z^n} = \overline{z}^n$ etc.). Also $\overline{a_j} = a_j$ as a_j is real (I can't say I like the use of i as a subscript when dealing with complex numbers). If you are particularly fastidious, write the proof as an induction on n.

4(c) Prove that $|1/(z^2+1)| \leq 1/3$ for z on γ (using $|z^2+1| \geq |z^2| - 1 = 3$) and that γ has length π.

2008/2009

1(a) Compare sheet 1, question 1(b). The results here are 1 more than those in the example, by the same method.

1(b) Yes, by the ratio test. The limit of a_{n+1}/a_n is 1/4.

1(c) 1: use the power series for the sine and exponential functions (or L'Hôpital if you really must).

1(d) This is an open disc, which I do in general in lectures.

1(e) This is very similar to question 16 of problem sheet 2, especially part (b). For that reason, I won't say more.

1(f) This is virtually identical to 1(f) of 2009/2010.

2(a) Compare question 21 of sheet 1. The method is the same. The limit satisfies $L = (3L+1)/(L+3)$ which is a quadratic equation.

2(b) This is a question which is I must admit is easier with the "epsilon-delta" definition of continuity rather than the "sequential" definition I useed.

Were there no such δ then for each n there would be $x_n \in [a,b]$ such that $f(x_n) = 0$ and $|x_n - c| < 1/n$. Then $(x_n) \to c$ and $f(x_n) = 0 \to 0 \neq f(c)$ contradicting continuity.

2(c) This is A-level calculus (chain rule).

3(a) Compare question 5 of sheet 1. This is the same with lub and glb interchanged, but the argument is eassentially the same.

3(b) The mean value theorem is bookwork. Apply it to f on the interval $[c, d]$ where $a \leq c < d \leq b$ to prove that $f(c) - f(d) < 0$.

3(c) Another A-level calculus question!

4(a) This is very similar to sheet 2, question 21. The same method works (with appropriate changes).

4(b) Identical to 4(a) of 2009/2010!

4(c) Identical to 4(c) of 2009/2010!

2007/2008

1(a) Identical to 1(a) of 2009/2010!

1(b) Identical to 1(b) of 2009/2010!

1(c) 0: note that $\cos x$ is bounded, and show that $x/(x^3+1) \to 0$ as $x \to \infty$.

1(d) This is bookwork: part of the proof of Rolle's theorem.

1(e) 1/3: by the ratio test.

1(f) This is an open disc, which I do in lectures.

1(g) Identical to 4(b) of 2009/2010!

1(h) Virtually identical to 1(f) of 2009/2010.

2(a) Identical to 2(a) of 2009/2010!

2(b) Identical to 2(b) of 2009/2010!

2(b) Identical to 2(c) of 2008/2009!

3(a) 0: by the substitution $x = 1 + y$ we get

$$\lim_{x \to 1} \frac{(x-1)^3}{\log x} = \lim_{y \to 0} \frac{y^3}{\log(1+y)}$$

and use the power series for $\log(1+y)$ or (if you must) L'Hôpital's rule.

3(b) Identical to 3(a) of 2009/2010!

3(c) Virtually identical to 3(b) of 2008/2009.

4(a) Virtually identical to 1(e) of 2008/2009.

4(b) Very similar to 2(b) of 2008/2009 with a complex function instead of a real function. The same proof works.

4(c) Use $1/|3 + z^3| \leq 1/2$ on γ and that γ has length π.