Analysis: skeleton notes 6: power series

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22 November 2011

A *power series* is a series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (*)$$

considered as a function of x. Some texts also call a series such as

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

a power series with *centre* c, but I shall largely stick to the definition $(*)$.

Natural questions concerning power series inculde: for which x does the series converge? what sort of function does it define? is it continuous? differentiable? etc.

Understanding convergence is relatively easy. Of course a power series always converges for $x = 0$. If $(*)$ converges for $x = x_0 \neq 0$ then it converges absolutely for all $x \in (-|x_0|, |x_0|)$ since then the sequence $(|a_n x_0^n|)$ is bounded above, and so then $\sum a_n x^n$ converges by comparison with the geometric series $\sum |x/x_0|^n$. It follows that if $(*)$ does converge for some nonzero x, it either converges absolutely for all x, or else there is $R > 0$ such that it converges absolutely for $|x| < R$ and diverges for $|x| > R$. This R is the *radius of convergence* and we write $R = 0$ if $f(x)$ only converges for $x = 0$ and $R = \infty$ if it converges for all x. To prove R exists, we consider the least upper bound of the set of r for which $f(x)$ converges on the interval $(-r, r)$.

An application of the ratio test proves that

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

provided this limit exists. (We allow ∞ as a “limit” in this context). This works for most of the standard power series.

If a power series has radius of convergence $R > 0$ it defines a function on $(-R, R)$ and we can ask whether this function is continuous, differentiable and so on. In fact it is, and its derivative is exactly what one would expect. However the proof is quite tricky if we start from first principles (see my "extra topics" document from last year). So to work up to it I start with what I call the "shifting theorem" (I don't think anyone else calls it that) which states that if $f(x)$ is given by a power series then for x_0 inside its radius of convergence then $g(t) = f(x_0 + t)$ is also given by a power series. More precisely

> Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and suppose that $|x_0| < R$. Then for $|t| < R - |x_0|$,
>
> $$f(x_0 + t) = \sum_{k=0}^{\infty} f_k(x_0) t^k$$
>
> where
>
> $$f_k(x_0) = \sum_{n=k}^{\infty} \binom{n}{k} a_n x_0^{n-k}.$$

To prove this just write

$$f(x_0 + t) = \sum_{n=0}^{\infty} a_n (x_0 + t)^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} a_n x_0^{n-k} t^k.$$

We would like to reverse the summation here and get

$$f(x_0 + t) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} a_n x_0^{n-k} t^k = \sum_{k=0}^{\infty} t^k \sum_{n=k}^{\infty} \binom{n}{k} a_n x_0^{n-k} = \sum_{k=0}^{\infty} f_k(x_0) t^k.$$

However, this step is highly dubious — in general even when a double series $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n}$ converges, the reversed double series $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_{m,n}$ may not converge, and even if it does, may converge to a **different** value to $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n}$. But there is a theorem that if the double series is absolutely convergent, that is $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |b_{m,n}|$ converges to (a finite) limit, then indeed $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n}$ and $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_{m,n}$ converge to the same value. Fortunately, that is the case here, as

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \left| \binom{n}{k} a_n x_0^{n-k} t^k \right| = \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} |x_0|^{n-k} |t|^k = \sum_{n=0}^{\infty} |a_n| (|x_0| + |t|)^n$$

which converges, since $|x_0| + |t| < R$.

We now show that power series define differentiable (and so continuous) functions. First we show that $f(x)$ given by $(*)$ is differenitiable at 0, provided that $R > 0$, and has derivative a_1. For $x \neq 0$ then

$$\frac{f(x) - f(0)}{x} - a_1 = \frac{f(x) - a_0}{x} - a_1 = x \sum_{n=2}^{\infty} a_n x^{n-2} = x \sum_{m=0}^{\infty} a_{m+2} x^m = x g(x).$$

say. The power series $g(x)$ has the same radius of convergence as $f(x)$ (why?) so we can pick a number r with $0 < r < R$. Then $\sum_{m=0}^{\infty} a_{m+2} r^m$ converges absolutely so let $A = \sum_{m=0}^{\infty} |a_{m+2}| r^m$. It's clear that $|g(x)| \leq A$ for $|x| < r$. If $x_n \to 0$ and $x_n \neq 0$ then eventually $0 < |x_n| < r$ so that eventually

$$\left| \frac{f(x_n) - f(0)}{x_n} - a_1 \right| = |x_n||g(x_n)| \leq A|x_n|.$$

By the squeeze principle $(f(x_n) - f(0))/x_n \to a_1$ and so $f'(0) = a_1$.

To prove that $f'(x_0)$ exists whenever $|x_0| < R$ we use the "shifting theorem". Then $h(t) = f(x_0 + t) = \sum_{k=0}^{\infty} f_k(x0) t^k$ and so applying the above case to this power series in t gives $f'(x_0) = h'(0) = f_1(x_0)$. So $f'(x)$ exists whenever $|x| < R$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

as one would expect. From this proof it follows that if the series for f' has radius of convergence R', then $R' \geq R$, but this can also be proved directly. Also using the comparison test it follows that $R' \leq R$ so actually $R' = R$. Of course one can now iterate the argument: the m-th derivative $f^{(m)}$ exists on $(-R, R)$ for all $m \in \mathbf{N}$.

One classic example of a power series is the exponential series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

which converges for all x. From this series, it's plain that $\exp' = \exp$. For $a \in \mathbf{R}$ is we consider the function $f_a(x) = \exp(x)\exp(a-x)$ we find that $f_a'(x) = 0$ so that f_a is constant: $f_a(x) = f_a(0) = \exp(a)$, that is $\exp(x)\exp(a - x) = \exp(a)$. Putting $a = x + y$ gives $\exp(x)\exp(y) = \exp(x + y)$ and so in particular $1 = \exp(x)\exp(-x)$. Clearly $\exp(x) > 0$ for $x \geq 0$ so it follows that $\exp(x) > 0$ for all x. As $\exp'(x) = \exp(x) > 0$ then exp is strictly increasing.

For $x > 0$, $\exp(x) > x$ so $\exp(x) \to \infty$ as $x \to \infty$ and consequently $\exp(x) \to 0$ as $x \to -\infty$. Indeed given any real α, there's $n \in \mathbf{N}$ with $n > \alpha$ and so if $x > 1$, $\exp(x) > x^n/n!$ and $\exp(n)/x^\alpha > x^{n-\alpha}/n!$ so that

$\exp(x)/x^\alpha \to \infty$ as $x \to \infty$. This means that $\exp(x)$ "grows faster" than any power of x.

The more fastidious texts on analysis treat sine and cosine in the same way. If we define

$$S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{and} \qquad C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

then these series converge for all real x. If we let $F(x) = S(x)^2 + C(x)^2$ then $F'(x) = 0$ so that F is constant: $S(x)^2 + C(x)^2 = F(0) = 1$. With much more effort one shows that S and C are periodic and they have all the properties of the sine and cosine functions: they indeed **are** the sine and cosine functions.

One useful application of power series is the evaluation of limits of the form $\lim_{c\to 0} f(x)/g(x)$. If one knows, or can find out, the power series for f and g, or indeed even the first nonzero term of that power series, the limit just drops out. As an example, consider

$$\lim_{x\to 0} \frac{\cos x - \cosh x}{\sin x \sinh x}.$$

The numerator is

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) - \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) = -x^2 + \frac{x^2}{12} + \cdots$$

and the denominator is

$$\left(x - \frac{x^3}{6} + \cdots\right)\left(x + \frac{x^3}{6} + \cdots\right) = x^2\left(1 - \frac{x^2}{6} + \cdots\right)\left(1 + \frac{x^2}{6} + \cdots\right).$$

so that

$$\frac{\cos x - \cosh x}{\sin x \sinh x} = \frac{-1 + x^2/12 + \cdots}{(1 - x^2/6 + \cdots)(1 + x^2/6 + \cdots)} \to -1$$

as $x \to 0$. One can adapt this to more general limits by writing $\lim_{x\to a} h(x) = \lim_{y\to 0} h(a+y)$.

A Appendix: double series

In this section, I'll use the usual notation for sequences starting the indexing at 1. A *double sequence* $(b_{m,n})$ is basically a map from $\mathbf{N} \times \mathbf{N}$ to $\mathbf{R}$, that is there is a number $b_{m,n}$ defined for all positive integers m and n. It can be visualized as an infinite 2-dimensional array:

$$\begin{array}{ccccc} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & \cdots \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & \cdots \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & \cdots \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}.$$

From this we can form the *double series* $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n}$ and $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{m,n}$. For the first double series to converge, we need each series $\sum_{m=1}^{\infty} b_{m,n}$ to converge (these are the "row sums" in the above array) to c_m say, and then the series $\sum_{m=1}^{\infty} c_m$ to converge. Likewise for $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{m,n}$ to converge, we need every "column sum" to converge and also the sum of the column sums to converge.

The problem with double series is that even if the first sum $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n}$ converges, the second sum $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{m,n}$ may not converge and even if it does it may converge to a different value to the first sum. As a simple example, define $b_{m,m} = 1$, $b_{m,m+1} = -1$ and $b_{m,n} = 0$ otherwise. This double sequence looks like

$$\begin{array}{rrrrr} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}.$$

The row sums are all zero, so that $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n} = \sum_{m=1}^{\infty} 0 = 0$. But the first column sum is 1 and the remaining column sums are zero so that $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{m,n} = 1 + \sum_{n=2}^{\infty} 0 = 1$. As a nice exercise you might prove that given any two sequences, there's a double sequence whose row sums form your first sequence and whose column sums form your second sequence.

The big theorem is that double sums can be reversed as long as we have absolute convergence. We say that $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n}$ is absolutely convergent if $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |b_{m,n}|$ is convergent. (Our previous example was certainly not absolutely convergent. The main result is that if $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n}$ is absolutely convergent, then so is $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{m,n}$ and they converge to the same value.

To prove this we first assume that each $b_{m,n} \geq 0$. Let $c_m = \sum_{n=1}^{\infty} b_{m,n}$ be the m-th row sum. Then the series $\sum_{m=1}^{\infty} c_m$ is convergent. For each m and n, $0 \leq b_{m,n} \leq c_m$, so for each n the column sum $d_n = \sum_{m=1}^{\infty} b_{m,n}$ is convergent by comparison with $\sum_{m=1}^{\infty} c_m$. For $N \in \mathbf{N}$ we have

$$\sum_{n=1}^{N} d_N = \sum_{n=1}^{N} \sum_{m=1}^{\infty} b_{m,n} = \sum_{m=1}^{\infty} \sum_{n=1}^{N} b_{m,n}.$$

We have reversed a summation here, but this is OK since one summation is finite, and we can prove this by induction on N. For each m and N, $\sum_{n=1}^{N} b_{m,n} \leq c_m$ and so $\sum_{n=1}^{N} \sum_{m=1}^{\infty} b_{m,n} \leq \sum_{m=1}^{\infty} c_m$. The sequence $\left(\sum_{n=1}^{N} \sum_{m=1}^{\infty} b_{m,n}\right)_{N=1}^{\infty}$ is bounded and increasing, so convergent. Hence $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{m,n}$ converges to a limit; moreover that limit is at most $\sum_{m=1}^{\infty} c_m = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n}$. We have show that $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{m,n}$ converges and that $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{m,n} \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n}$. Interchanging m and n in the above argument gives $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n} \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{m,n}$ so that $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{m,n}$.

To extend the result to general absolutely convergent series, we use the same trick as in the proof that absolutely convergent series are convergent. Write each $b_{m,n} = b_{m,n}^{+} - b_{m,n}^{-}$. Then $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n}^{+}$ and $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n}^{-}$ are absolutely convergent so we may reverse the summation in both, and so on.