Analysis: skeleton notes 2: sequences

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Informally a *sequence* (a_n) is a list $a_1, a_2, \ldots$ of real numbers. More formally it's a map $f : \mathbf{N} \to \mathbf{R}$ where we write a_n instead of $f(n)$. We call a_n the *n-th term* of the sequence (a_n). (Sometimes we start the indexing of a sequence at 0, or occasionally at some other starting value, e.g. $a_0, a_1, \ldots$ or $a_4, a_5, \ldots$).

The sequence (a_n) is *increasing* if $a_n \le a_{n+1}$ for all n and *decreasing* if $a_n \ge a_{n+1}$ for all n. It is *strictly increasing* if $a_n < a_{n+1}$ for all n and *strictly decreasing* if $a_n > a_{n+1}$ for all n. A *monotone* sequence is one which is either increasing or decreasing.

A sequence (a_n) is *bounded above*, if it has an *upper bound*, that is a number A with $a_n \le A$ for all n. Similarly it is *bounded below*, if it has a *lower bound*, that is a number B with $a_n \ge A$ for all n. Also it is *bounded* if it is both bounded above and bounded below, equivalently if there is a number C with $|a_n| \le C$ for all n. An increasing sequence is always bounded below and a decreasing sequence is always bounded above.

Often a sequence may fail to have one of these properties, but only due to a finite number of exceptional terms. So we say that a sequence (a_n) *eventually* has a certain property P if there is some positive integer N such that the sequence $a_N, a_{N+1}, \ldots$ has property P. For example the sequence (a_n) given by $a_n = (n-4)^2$ is not increasing (as $a_1 > a_2$) but it is eventually increasing (as $a_n \le a_{n+1}$ whenever $n \ge 4$).

We now define the notion of convergence. The idea is that a sequence (a_n) converges to a real number a if the sequence is eventually "close" to a. However, this term "close" is vague. We can make it precise by choosing a number $\varepsilon > 0$ and saying that a number x is *ε-close* to a if $|x - a| < \varepsilon$, equivalently $x \in (a - \varepsilon, a + \varepsilon)$. A sequence is *$\varepsilon$-close to a* if all its terms are ε-close to a and is *eventually ε-close to a* if all but finitely many of its terms are ε-close to a. When the sequence (a_n) converges to a we call a the *limit* of the sequence (a_n) and write $a = \lim_{n\to\infty} a_n$. (As an exercise you should prove that a convergent sequence cannot have to different limits).

A *null sequence* is a sequence that converges to 0. It follows straight from the definition that a sequence (a_n) converges to a if and only if the sequence $(a_n - a)$ is a null sequence.

A sequence which converges to a limit is *convergent*, and a sequence which is not convergent is *divergent*. A sequence (a_n) *diverges* to ∞ if for each number A, all but finitely many terms of the sequence satisfy $a_n > A$. I'll leave you to write down the corresponding definition of a sequence diverging to $-\infty$. Please note that there are many divergent sequences which neither diverge to ∞ nor to $-\infty$, for example taking $a_n = (-1)^n$.

One very useful lemma concerning limits is that they preserve "weak" inequalities. More precisely, if (a_n) converges to a and $a_n \geq b$ for all n then $a \geq b$. (We could replace $\geq$ by $\leq$). To prove this suppose that $a < b$ and let $\varepsilon = b - a$. Then $\varepsilon > 0$ and the sequence (a_n) must be ε-close to a, that is the terms a_n satisfy $a - \varepsilon < a_n < a + \varepsilon$ for all but finitely many n. But $a + \varepsilon = b$ so this means that $a_n < b$ which is contrary to the hypothesis that always $a_n \geq b$. By contradiction, then $a \geq b$.

An important example of a null sequence is $(1/n)$. It is perhaps less obvious than it looks that $(1/n)$ converges to zero. To prove this we need to prove that for each $\varepsilon > 0$ the sequence $(1/n)$ is ε-close to zero. Consider $1/\varepsilon$. Then $1/\varepsilon > 0$ and by the Archimedean property there is $N \in \mathbf{N}$ with $N > 1/\varepsilon$. If $n \in \mathbf{N}$ and $n \geq N$ then $n \geq N > 1/\varepsilon$ so that $1/n \leq 1/N < \varepsilon$. As $1/n > 0$ then certainly $|1/n| < \varepsilon$ for $n \geq N$ and so the sequence $(1/n)$is ε-close to 0. As this is true for all positive $\varepsilon > 0$ then $(1/n)$ is a null sequence. Note how essential the Archimedean property is here; in some sense it is equivalent to the assertion that $\lim_{n\to\infty} 1/n = 0$.

While one can give arguments from first principles like the above to prove what the limit of each convergent sequence is, it is good to have more systematic ways of proceeding in practice. The main method is the so-called "algebra of limits", which is the theorem that the arithmetic operations of addition, subtraction, multiplication and division preserve convergence and limits. To summarize: let (a_n) and (b_n) be sequences with limits a and b respectively. Then

- $(a_n + b_n)$ has limit $a + b$,
- $(a_n - b_n)$ has limit $a - b$,
- $(a_n b_n)$ has limit ab and
- provided that $b_n \neq 0$ for all n and also $b \neq 0$ then (a_n/b_n) has limit a/b.

In the last case, remember the maxim "thou shalt not divide by zero"! I will prove these in the lectures.

One big result on sequences is that each bounded monotone sequence is convergent. I'll prove this for increasing sequences that are bounded above, so let (a_n) be such a sequence. Define $A = \{a_n : n \in \mathbf{N}\}$, the set of all terms of the sequence. Then A has an upper bound since the sequence is bounded above. By the Completeness Axiom, it has a least upper bound α say. I claim that (a_n) converges to α. To prove this, let ε be any positive real number, and let us aim to show that the sequence (a_n) is ε-close to α. As $\alpha - \varepsilon < \alpha$ then $\alpha - \varepsilon$ is **not** an upper bound of A (since α is the **least** upper bound of A). So there is an element of A, say a_N, with $a_N > \alpha - \varepsilon$. For $n \geq N$, $a_n \geq a_N$ as (a_n) is increasing, so also $a_n > \alpha - \varepsilon$. Also $a_n \in A$ so $a_n \leq \alpha$ as α is an upper bound for A. So for $n \geq N$, $\alpha - \varepsilon < a_n \leq \alpha$ so that a_n is ε-close to α and the sequence (a_n) is eventually ε-close to α. The conclusion is that (a_n) converges to α.

A *subsequence* of a sequence (a_n) is roughly speaking a sequence formed by pulling out selected terms from (a_n) in the order they appear in the original sequence. More formally let (k_n) be a **strictly** increasing sequence of positive integers. Then (a_{k_n}) is a subsequence of (a_n). It's an easy exercise to prove that if a sequence converges to a limit, then each of its subsequences converges to the same limit. So a sequence having subsequences converging to different limits must be divergent. Thus there are two strategies for proving that a sequence is divergent:

- prove that it has two subsequences converging to different limits,
- prove that it has a subsequence diverging to ∞ or to $-\infty$.

Indeed one of these strategies will work on any divergent sequence.

A big result on subsequences is the Bolzano-Weierstrass theorem, which states that each bounded sequence has a convergent subsequence. There's a proof in my 2010 handout "proofs of some major theorems" but I'll give a different proof in this year's course. The Bolzano-Weierstrass theorem is one of the most useful in analysis, and I'll use it in proving some key theorems on real functions.