Analysis: skeleton notes 3: sequences

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Informally a *series* is an "infinite sum" $\sum_{n=1}^{\infty} a_n$ of real numbers a_n. What what does such an expression mean? The axioms for the reals allow us to add pairs of numbers, and a finite sum such as $\sum_{n=1}^{N} a_n$ makes sense as $(\cdots((a_1 + a_2) + a_3) + \cdots + a_{N-1}) + a_N$. Of course using the associative and commutative laws for addition it matters not how we bracket the terms, nor even in which order we take them, so we write $\sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N$.

But what about an infinite sum $\sum_{n=1}^{\infty} a_n$? We divide such series into convergent and divergent series by considering an associated sequence. Let $s_N = \sum_{n=1}^{N} a_n$. We call s_N the *N-th partial sum* of the series. Consider the sequence of partial sums (s_N). We say that the series $\sum_{n=1}^{\infty} a_n$ is *convergent* or *divergent* according to whether the sequence (s_n) of partial sums is convergent or divergent. If $\sum_{n=1}^{\infty} a_n$ is convergent its sum is $\lim_{N \to \infty} s_n = \lim_{N \to \infty} \sum_{n=1}^{N} a_n$. So for a convergent series $\sum_{n=1}^{\infty} a_n$ the sum is the limit of the sequence

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \ldots.$$

As with sequences we can start the indexing from a point other than $n = 1$, so we see series like $\sum_{n=0}^{\infty} b_n$ or $\sum_{n=5}^{\infty} c_n$ etc.

One important and basic theorem on series is that if a series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$. To see this note that $a_n = s_n - s_{n-1}$ where s_n is the n-th partial sum of the series. As the series is convergent then the sequence (s_n) is convergent to s say. Then also (s_{n-1}) converges to the same limit s, and by the algebra of limits $(a_n) = (s_n - s_{n-1})$ converges to $s - s = 0$.

However the converse of this theorem is **false**. There are divergent series $\sum_{n=1}^{\infty} a_n$ where $a_n \to 0$. The classic example is the *harmonic series* $\sum_{n=1}^{\infty} 1/n$. I will prove this series is divergent in the lectures.

A very important class of series is the geometric series $\sum_{n=0}^{\infty} r^n$ where r is a fixed real number. If $|r| \geq 1$ then it's easy to see the series diverges.

Suppose that $|r| < 1$. The N-th partial sum is

$$s_N = \sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$

and as $|r| < 1$, $\lim_{N \to \infty} r^{N+1} = 0$ leading to

$$\sum_{n=0}^{\infty} r^n = \lim_{N \to \infty} s_N = \frac{1}{1 - r}.$$

This is perhaps the most crucial series evaluation in analysis; you **must** know this and how to prove it.

If a series $\sum_{n=1}^{\infty} a_n$ has nonnegative terms (all $a_n \geq 0$) then its partial sums $s_N = a_1 + \cdots + a_N$ form an increasing sequence; if the sequence of partial sums is bounded above then the series converges, otherwise it diverges (to ∞). From this observation we deduce the *comparison test* for nonnegative series:

- Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with $0 \leq a_n \leq b_n$ for all n. If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

The proof is the observation that $\sum_{n=1}^{N} a_n \leq \sum_{n=1}^{\infty} b_n$ so if the second series converges, the partial sums of the first series are convergent, and as the first series has nonnegative terms, the first series is convergent.

In practice, a "souped-up" version of the comparison test, the *limit-comparison* test is more useful.

- Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with $a_n > 0$ and $b_n > 0$ for all n. Suppose also that the sequence b_n/a_n converges to a (finite) nonzero limit L. Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\sum_{n=1}^{\infty} b_n$ is convergent.

To prove this, suppose that $\sum_{n=1}^{\infty} b_n$ converges. Then $a_n < (L+1)b_n$ with only finitely many exceptions. We then apply the comparison test to $\sum_{n=1}^{\infty} a_n$ and the convergent series $\sum_{n=1}^{\infty}(L+1)b_n$ (if there are only finitely many n with $a_n \geq (L+1)b_n$ that doesn't affect convergence). The proof that $\sum_{n=1}^{\infty} a_n$ converges implies $\sum_{n=1}^{\infty} b_n$ converges is similar.

Applying the limit test with the geometric series, gives in essence the *ratio test*.

- Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. Suppose that the sequence (a_{n+1}/a_n) converges to a limit L. If $L < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent. If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

Suppose first that $L < 1$. Choose a number r with $L < r < 1$. Then for large enough n, say for $n \geq N$, $a_{n+1}/a_n < r$ and so for $n \geq N$, $a_n \leq a_N r^{n-N}$. For all but finitely many n, then $a_n \leq (a_N r^{-N}) r^n$ (note that $a_N r^{-N}$ is independent of n) and so we can use the fact that $\sum_{n=1}^{\infty} r^n$ converges to see that $\sum_{n=1}^{\infty} a_n$ converges.

If $L > 1$ then for large enough n, $a_{n+1} > a_n$, so the sequence (a_n) cannot converge to 0 and $\sum_{n=1}^{\infty} a_n$ must be divergent.

A series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ is convergent. As the name suggests, an absolutely convergent series is always convergent, but the proof isn't so obvious: see my 2010 handout on "proofs of major theorems". One can deduce stronger versions of the comparison, limit-comparison and ratio tests.

- If $\sum_{n=1}^{\infty} b_n$ is convergent and $|a_n| \leq b_n$ then $\sum_{n=1}^{\infty} a_n$ is (absolutely) convergent.

- If $\sum_{n=1}^{\infty} b_n$ is absolutely convergent, and the sequence $(|a_n/b_n|)$ converges to a (finite) limit L then $\sum_{n=1}^{\infty} a_n$ is (absolutely) convergent.

- Suppose that the sequence $(|a_{n+1}/a_n|)$ converges to a limit L. If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is (absolutely) convergent If $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

A convergent sequence that is not absolutely convergent is *conditionally convergent*. A nice theorem, which I won't prove, states that re-arranging the terms of an absolutely convergent series always gives an absolutely convergent series with the same sum, but if you re-arrange the terms of a conditionally convergent series, you can make the new series converge to any real number, or make it diverge.

One convergence test that works even when the series is conditionally convergent is *Leibniz's test* or the *alternating series test*. An alternating series is a series like $1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots$ where the terms are alternately positive and negative. More formally $\sum_{n=1}^{\infty} a_n$ is alternating if $a_n = (-1)^{n-1} b_n$ with each $b_n > 0$. Leibniz's test states that an alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$ is convergent if both

- $b_n \geq b_{n+1}$ for all n, and

- $\lim_{n \to \infty} b_n = 0$.

I'll sketch the proof, omitting some details. Let $a_N = \sum_{n=1}^{N} (-1)^n b_n$ be the N-th partial sum. Then $s_1 \geq s_3 \geq s_5 \geq \cdots$ and $s_2 \leq s_4 \leq s_6 \leq \cdots$ (proving this uses $b_n \geq b_{n+1}$). Both the sequences (s_{2N-1}) and (s_{2N}) are monotone

and bounded (why)? so they converge to limits L_1 and L_2 respectively. But $L_1 - L_2 = \lim_{N \to \infty} b_{2N} = 0$ so $L_1 = L_2$. As the "odd" partial sums and "even" partial sums converge to the same limit, the whole sequence of partial sums converges to that limit, so the series is convergent.