Proofs of some major theorems

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I give proofs for some of the more important results in the course. My thanks to Andrew Barratt and Ryan Stanley for pointing out errors in earlier versions.

Absolutely convergent series are convergent

A series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent. The terminology suggests that absolutely convergent series are convergent, but this isn't quite immediate.

In the proof I employ some useful but non-standard notation. For $x \in \mathbf{R}$ define

$$x^+ = \max(x, 0) = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases}$$

and

$$x^- = \max(-x, 0) = \begin{cases} 0 & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

For instance $2^+ = 2$ and $2^- = 0$. Also $(-3)^+ = 0$ and $(-3)^- = 3$.

In all cases $0 \leq x^+ \leq |x|$, $0 \leq x^- \leq |x|$ and $x = x^+ - x^-$.

Theorem. *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Then $\sum_{n=1}^{\infty} a_n$ is a convergent series.*

Proof As $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} |a_n|$ is convergent. As $0 \leq a_n^+ \leq |a_n|$ then $\sum_{n=1}^{\infty} a_n^+$ is convergent by the comparison test. Similarly $\sum_{n=1}^{\infty} a_n^-$ is convergent. Therefore

$$\sum_{n=1}^{\infty} (a_n^+ - a_n^-) = \sum_{n=1}^{\infty} a_n$$

is convergent (essentially by the difference rule for convergence of sequences). $\square$

The Bolzano-Weierstrass theorem

This states that a bounded sequence always has a convergent subsequence. Before proving this, we prove two preliminary results, each of interest in its own right.

Lemma. *Let (a_n) be a monotone bounded sequence. Then (a_n) is convergent.*

Proof First suppose that (a_n) is increasing. As (a_n) is bounded, the set $A = \{a_n : n \in \mathbf{N}\}$ is nonempty and bounded, so has a least upper bound α by the completeness axiom. We claim that $a_n \to \alpha$ as $n \to \infty$. Given any $\varepsilon > 0$, then $\alpha - \varepsilon < \alpha$ so that $\alpha - \varepsilon$ is not an upper bound of A (as α is the **least** upper bound of A). Therefore there is $N \in \mathbf{N}$ with $a_N > \alpha - \varepsilon$. As (a_n) is increasing, but is bounded above by α, then for each $n \geq N$,

$$\alpha - \varepsilon < a_N \leq a_n \leq \alpha$$

so that eventually $|a_n - \alpha| < \varepsilon$. Hence $a_n \to \alpha$ as $n \to \infty$.

If (a_n) is decreasing and bounded, then the sequence $(-a_n)$ is increasing and bounded. Hence by the foregoing $(-a_n)$ is convergent, and then so is (a_n). $\square$

Lemma. *Every sequence has a monotone subsequence.*

Proof Let (a_n) be a sequence. We call $n \in \mathbf{N}$ *special* if a_n is strictly larger than all subsequent terms of the sequence. That is, n is special if $a_n > a_m$ for all m with $m > n$. Let S be the set of all special numbers. Then S is a subset of $\mathbf{N}$. We divide into two cases.

Case (i): S is an infinite set. In this case let us write the elements of S in ascending order:

$$S = \{n_1, n_2, n_3, \ldots\}$$

where $n_k < n_{k+1}$ for all k. As each n_k is special, and $n_{k+1} > n_k$, then $a_{n_k} > a_{n_{k+1}}$. Therefore the sequence (a_{n_k}) is a decreasing subsequence of (a_n).

Case (ii): S is a finite set. In this case there is a number $M \in \mathbf{N}$ such that $M > n$ for all $n \in S$. Thus if $m \geq M$, m is not special, and there is $m' > m$ for which $a_{m'} \geq a_m$. Define recursively $m_1 = M$, and for each k, m_{k+1} is a number with $m_{k+1} > m_k$ and $a_{m_{k+1}} \geq a_{m_k}$. Then (a_{m_k}) is an increasing subsequence of (a_n). $\square$

Theorem (Bolzano-Weierstrass)**.** *Every bounded sequence has a convergent subsequence.*

Proof Let (a_n) be a bounded sequence. Then (a_n) has a monotone subsequence (a_{n_k}). The sequence (a_{n_k}) is *a fortiori* bounded. Hence (a_{n_k}) is convergent. $\square$

The intermediate value theorem

Theorem. *Let $f : [a, b] \to \mathbf{R}$ be a continuous function, and suppose that $f(a) < r < f(b)$. Then there is $t \in (a, b)$ with $f(t) = r$*

Proof We shall define two sequences of elements (a_n) and (b_n) of elements of $[a, b]$ with the following properties;

- $a_0 = a$ and $b_0 = b$,
- (a_n) is increasing and (b_n) is decreasing,
- $b_n - a_n = 2^{-n}(b - a)$,
- $f(a_n) \leq r \leq f(b_n)$.

We start with $a_0 = a$ and $b_0 = b$. When we have defined a_n and b_n, with the above properties, let $c_n = \frac{1}{2}(a_n + b_n)$. Then $c_n - a_n = b_n - c_n = \frac{1}{2}(b_n - a_n)$. If $f(c_n) \leq r$ let $a_{n+1} = c_n$ and $b_{n+1} = b$; if $f(c_n) > r$ let $a_{n+1} = a_n$ and $b_{n+1} = c_n$. Then $a_n \leq a_{n+1}$, $b_n \geq b_{n+1}$, $f(a_{n+1}) \leq r \leq f(b_{n+1})$ and $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) = 2^{-(n+1)}(b - a)$. Thus the sequences (a_n) and (b_n) have the stated properties.

The sequence (a_n) is increasing and bounded. It converges to its least upper bound t, and as $a \leq a_n \leq b$ for all n then $a \leq t \leq b$. As $b_n = a_n + 2^{-n}(b - a)$ it follows that $b_n \to t$ as $n \to \infty$ also. By the continuity of f, both $f(a_n) \to f(t)$ and $f(b_n) \to f(t)$ as $n \to \infty$. We cannot have $f(t) > r$ for each $f(a_n) \leq r$ and so $|f(a_n) - f(t)| = f(t) - f(a_n) \geq f(t) - r > 0$ for all n and so $(f(a_n))$ cannot converge to $f(t)$. Similarly, considering $f(b_n)$ we cannot have $f(t) < r$. We conclude that $f(t) = r$. $\square$

The boundedness theorem

Theorem. *Let $f : [a, b] \to \mathbf{R}$ be a continuous function. Then f is bounded on $[a, b]$ and attains its bounds. More precisely there are c, $d \in [a, b]$ with $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.*

Proof Suppose that f is not bounded above on $[a, b]$, that is there is no M such that $f(x) < M$ for all $x \in [a, b]$. Then for each $n \in \mathbf{N}$ there is $x_n \in [a, b]$ with $f(x_n) > n$. By the Bolzano-Weierstrass theorem, (x_n) has a convergent subsequence (x_{n_k}) converging to $r \in \mathbf{R}$. Then $r \in [a, b]$ (why?) and by the continuity of f, $f(x_{n_k}) \to f(r)$ as $k \to \infty$. But $f(x_{n_k}) > n_k$ and so $(f(x_{n_k}))$ is an unbounded, and so divergent, sequence. This gives a contradiction and shows that f is bounded above on $[a, b]$.

To show f is bounded below, either adapt the above argument or apply it to $-f$ rather than f.

To prove that f attains its bounds, we employ a cheap trick. Let M be the least upper bound of the values of f on $[a, b]$. If $f(x) < M$ for all $x \in [a, b]$ then the function $g : [a, b] \to \mathbf{R}$ defined by $g(x) = 1/(M - f(x))$ is well-defined, continuous and takes positive values. By the first part of the theorem, g has an upper bound C on $[a, b]$. Then

$$0 < \frac{1}{M - f(x)} = g(x) \leq C$$

for all $x \in [a, b]$. Thus

$$M - f(x) \geq \frac{1}{C}$$

and so

$$f(x) \leq M - \frac{1}{C} < M$$

for all $x \in [a, b]$, contradicting M being the least upper bound for f on $[a, b]$. This contradiction shows there is $d \in [a, b]$ with $d = M$ and so $f(x) \leq f(d)$ for all $x \in [a, b]$.

Adapting the above argument or applying it to $-f$ rather than f shows that there is $c \in [a, b]$ with $f(c) \leq f(x)$ for all $x \in [a, b]$. $\square$

Rolle's theorem

Theorem. *Let $a < b$, $f : [a, b] \to \mathbf{R}$ be a continuous function with $f(a) = f(b)$ and suppose that f is differentiable on (a, b). Then there is $t \in (a, b)$ with $f'(t) = 0$.*

Proof By the boundedness theorem, there are c, $d \in [a, b]$ with $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$. If $f(c) = f(d)$ then f is constant, and so $f'(x) = 0$ for all $x \in (a, b)$ so we can take t to be any element of (a, b) for instance $t = \frac{1}{2}(a + b)$.

In general,

$$f(c) \leq f(a) = f(b) \leq f(d).$$

If $f(d) > f(a)$ then $d \in (a, b)$. If $a \leq x < d$ then

$$\frac{f(x) - f(d)}{x - d} \geq 0$$

as $f(x) - f(d) \leq 0$ and $x - d < 0$. Taking a sequence (x_n) of elements of $[a, d)$ converging to d we find

$$f'(d) = \lim_{x \to d} \frac{f(x) - f(d)}{x - d} = \lim_{n \to \infty} \frac{f(x_n) - f(d)}{x_n - d} \geq 0.$$

On the other hand if $d < x \le b$ then

$$\frac{f(x) - f(d)}{x - d} \le 0$$

as $f(x) - f(d) \le 0$ and $x - d > 0$. Taking a sequence (y_n) of elements of $(d, b]$ converging to d we find

$$f'(d) = \lim_{x \to d} \frac{f(x) - f(d)}{x - d} = \lim_{n \to \infty} \frac{f(y_n) - f(d)}{y_n - d} \le 0.$$

We conclude that $f'(d) = 0$

If $f(c) < f(a)$ then $c \in (a, b)$ and we can adapt the above argument to show that $f'(c) = 0$. The only remaining possibility is when $f(c) = f(a) = f(d)$ which we have already dealt with. $\square$

The mean value theorem

Theorem. *Let $a < b$, $f : [a, b] \to \mathbf{R}$ be a continuous function and suppose that f is differentiable on (a, b). Then there is $t \in (a, b)$ with*

$$f'(t) = \frac{f(b) - f(a)}{b - a}.$$

Proof Define

$$g(x) = f(x) - (x - a)\frac{f(b) - f(a)}{b - a}.$$

Then g is continuous on $[a, b]$, differentiable on (a, b),

$$g(b) = f(b) - (b - a)\frac{f(b) - f(a)}{b - a} = f(b) - (f(b) - f(a)) = f(a) = g(a)$$

and

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Thus we may apply Rolle's theorem to g and conclude there is $t \in (a, b)$ with $g'(t) = 0$. This means that

$$0 = f'(t) - \frac{f(b) - f(a)}{b - a}$$

that is

$$f'(t) = \frac{f(b) - f(a)}{b - a}.$$

$\square$

The Cauchy-Riemann equations

Recall that a function f is analytic on an open set $U \subseteq \mathbf{C}$ is it is differentiable if for each $a \in U$, the complex derivative $f'(a)$ exists, and the definition of $f'(a)$ is

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

if, of course, this limit exists. We can also regard f as a pair of two real-valued functions of two variables: precisely

$$f(x + iy) = u(x, y) + iv(x, y)$$

where x, y, $u(x, y)$, $v(x, y) \in \mathbf{R}$. We can now state and prove the Cauchy-Riemann equations.

Theorem. *Let f, U, u and v be as defined above. If f is analytic in U then the partial derivatives of u and v exist in U and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proof Suppose that f is analytic in U. Let $a \in U$. Then $f'(a)$ exists. Write $a = x_0 + iy_0$. Let (h_n) be any null sequence of nonzero reals. Then $a + h_n \to a$ as $n \to \infty$. Therefore

$$\begin{aligned} f'(a) &= \lim_{n \to \infty} \frac{f(a + h_n) - f(a)}{h_n} \\ &= \lim_{n \to \infty} \left(\frac{u(x_0 + h_n, y_0) - u(x_0, y_0)}{h_n} + i \frac{v(x_0 + h_n, y_0) - v(x_0, y_0)}{h_n} \right). \end{aligned}$$

It follows that

$$\lim_{n \to \infty} \frac{u(x_0 + h_n, y_0) - u(x_0, y_0)}{h_n} = \operatorname{Re} f'(a)$$

and as this limit is independent of the sequence (h_n) then

$$\lim_{x \to \infty} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0}$$

exists and equals $\operatorname{Re} f'(a)$. But this limit is, by definition, the partial derivative $\partial u / \partial x$ at the point (x_0, y_0). Therefore

$$\frac{\partial u}{\partial x}(x_0, y_0) = \operatorname{Re} f'(a).$$

Applying this argument to the imaginary part gives

$$\frac{\partial v}{\partial x}(x_0, y_0) = \operatorname{Im} f'(a).$$

Again let (h_n) be a null sequence of nonzero reals. Then $a + ih_n \to a$ as $n \to \infty$. Therefore

$$\begin{aligned} f'(a) &= \lim_{n\to\infty} \frac{f(a+ih_n) - f(a)}{ih_n} \\ &= \lim_{n\to\infty} \left(-i\frac{u(x_0, y_0 + h_n) - u(x_0, y_0)}{h_n} + \frac{v(x_0, y_0 + h_n) - v(x_0, y_0)}{h_n} \right). \end{aligned}$$

By a similar argument to above, we get that the partial derivatives in the y-direction of u and v exist, and that

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\operatorname{Im} f'(a)$$

and

$$\frac{\partial v}{\partial y}(x_0, y_0) = \operatorname{Re} f'(a).$$

We conclude that

$$\frac{\partial u}{\partial x} = \operatorname{Re} f' = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\operatorname{Im} f' = -\frac{\partial v}{\partial x}.$$

□