ECM3703: Complex Analysis

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Contents

Introduction to Complex Analysis Complex analysis is the study of functions involving complex numbers. It has "real" applications, for example, evaluating integrals like

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} = \pi$$

but contour integration easily gives us

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\mathrm{d}x = \frac{\pi}{\mathrm{e}}.$$

Also we can derive results such as

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$
$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$
etc.

Also complex analysis has applications to many other branches of mathematics from number theory to fluid mechanics. For example, the Riemann ζ-function explains the distribution of prime numbers.

1 Review of Complex Numbers

1.1 Basics

The set of complex numbers, $\mathbb{C}$ is defined by

$$\mathbb{C} = \left\{ x + \mathrm{i}y |\, x, y \in \mathbb{R}, \mathrm{i}^2 = -1 \right\}.$$

Addition and subtraction are all easy, for example, if $z = 3 - \mathrm{i}$ and $w = 1 + 4\mathrm{i}$ then

$$\begin{aligned} z + w &= (3 - \mathrm{i}) + (1 + 4\mathrm{i}) = 4 + 3\mathrm{i} \\ z - w &= (3 - \mathrm{i}) - (1 + 4\mathrm{i}) = 2 - 5\mathrm{i} \\ zw &= 3 - \mathrm{i} + 12\mathrm{i} - 4\mathrm{i}^2 \\ &= 7 + 11\mathrm{i} \end{aligned}$$

Division is a bit trickier, but $\frac{z}{w} = z\left(\frac{1}{w}\right)$ so division boils down to evaluating $\frac{1}{x+\mathrm{i}y}$.

The trick is similar to "rationalising the denominator":

$$\begin{aligned} \frac{1}{x + \mathrm{i}y} &= \frac{x - \mathrm{i}y}{(x + \mathrm{i}y)(x - \mathrm{i}y)} \\ &= \frac{x - \mathrm{i}y}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}\mathrm{i}. \end{aligned}$$

Consider $x^2 + y^2$ for $x, y \in \mathbb{R}$. Since $x^2 \geq 0$ and $y^2 \geq 0$, $x^2 + y^2 \geq 0$ also, and $x^2 + y^2 > 0$ unless $x^2 = 0$ and , which can only happen if $x = 0$ and $y = 0$. So we can always divide by a complex number as long as it's not zero.

Example

$$z = 3 - \mathrm{i}, w = 1 + 4\mathrm{i}.$$

$$\begin{aligned}
\frac{3-\mathrm{i}}{1+4\mathrm{i}} &= \frac{(3-\mathrm{i})(1-4\mathrm{i})}{(1+4\mathrm{i})(1-4\mathrm{i})} \\
&= \frac{3-\mathrm{i}-12\mathrm{i}-4\mathrm{i}^2}{1^2+4^2} \\
&= \frac{-1-13\mathrm{i}}{17} \\
&= -\frac{1}{17} - \frac{13}{17}\mathrm{i}.
\end{aligned}$$

Jargon For $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$,

- $x = \Re(z)$ - the **real part** of z
- $y = \Im(z)$ - the **imaginary part** of z
- $x - \mathrm{i}y = \overline{z} = z^\star$ - the **complex conjugate** of z.

Also,

$$\begin{aligned}
x &= \frac{(x+\mathrm{i}y)+(x-\mathrm{i}y)}{2} \\
\implies \Re(z) &= \frac{z+z^\star}{2} \\
\text{and } \Im(z) &= \frac{z-z^\star}{2}
\end{aligned}$$

and

$$\begin{aligned}
zz^\star &= (x+\mathrm{i}y)(x-\mathrm{i}y) \\
&= x^2+y^2 \\
&\text{which is a non-negative real number.}
\end{aligned}$$

Let $|z| = \sqrt{x^2+y^2} = \sqrt{zz^\star}$ be the **absolute value** (or **modulus**) of z.

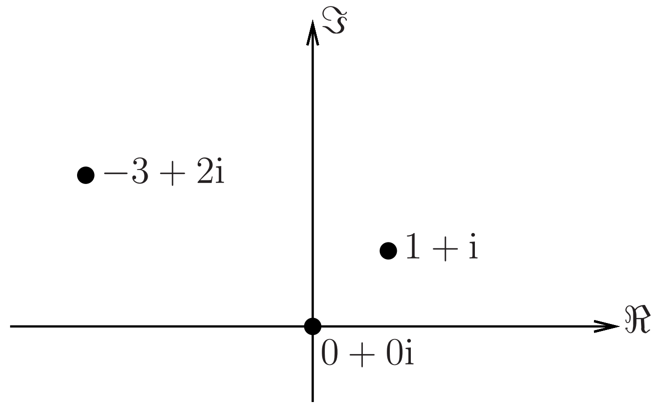
Note that

$$\begin{aligned}
x^2 &\le x^2+y^2 \\
\implies \sqrt{x^2} &\le \sqrt{x^2+y^2} \\
\implies |x| &\le \sqrt{x^2+y^2}
\end{aligned}$$

i.e. $|\Re(z)| \le |z|$ and $|\Im(z)| \le |z|$.

1.2 Argand diagram

Regard $x + \mathrm{i}y$ as the point (x, y) in the Cartesian plane.



Addition and subtraction in $\mathbb{C}$ correspond to addition and subtraction of position vectors.

By Pythagoras' theorem, $|z| = \sqrt{x^2 + y^2}$ is the distance from 0 to z in the Argand diagram!

Lemma If $z, w \in \mathbb{C}$ then

- $(z \pm w)^\star = z^\star \pm w^\star$
- $(zw)^\star = z^\star w^\star$
- $(z/w)^\star = z^\star / w^\star$ if $w \neq 0$.

Idea of Proof Write $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

Note that $(-\mathrm{i})^2 = \mathrm{i}^2 = -1$ in a formula is valid as long as all variables are real.

Corollary If $z, w \in \mathbb{C}$ then $|zw| = |z|\,|w|$.

Proof

$$
\begin{aligned}
|zw|^2 &= (zw)(zw)^\star \\
&= zw \cdot z^\star w^\star \\
&= zz^\star ww^\star \\
&= |z|^2\,|w|^2 \\
\implies |zw| &= |z|\,|w| \text{ (taking square roots)}
\end{aligned}
$$

$\square$

Corollary (Triangle Inequality) If $z, w \in \mathbb{C}$ then

$$|z+w| \leq |z| + |w|$$

Proof (Estermann) If $z + w = 0$ then this is definitely true!

Otherwise assume $z + w \neq 0$. Then

$$\frac{z}{z+w} + \frac{w}{z+w} = 1$$
$$\implies \Re\left(\frac{z}{z+w}\right) + \Re\left(\frac{w}{z+w}\right) = 1$$

Also note that

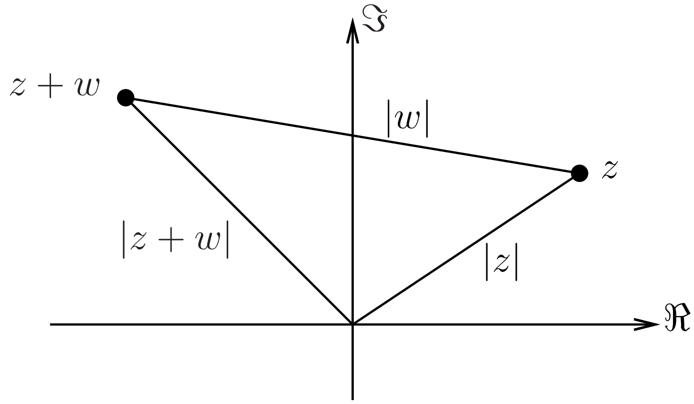
$$\Re\left(\frac{z}{z+w}\right) \leq \left|\frac{z}{z+w}\right| \text{ and } \Re\left(\frac{w}{z+w}\right) \leq \left|\frac{w}{z+w}\right|$$

so

$$\left|\frac{z}{z+w}\right| + \left|\frac{w}{z+w}\right| \geq 1$$
$$\implies |z| + |w| \geq |z+w| \text{ (multiplying both sides by } |z+w|\text{).}$$

□

The triangle inequality says that the length of a side of a triangle is at most the sum of the lengths of the other two sides. Remember it, as it will be used a lot in this module!



Also, another very useful inequality is

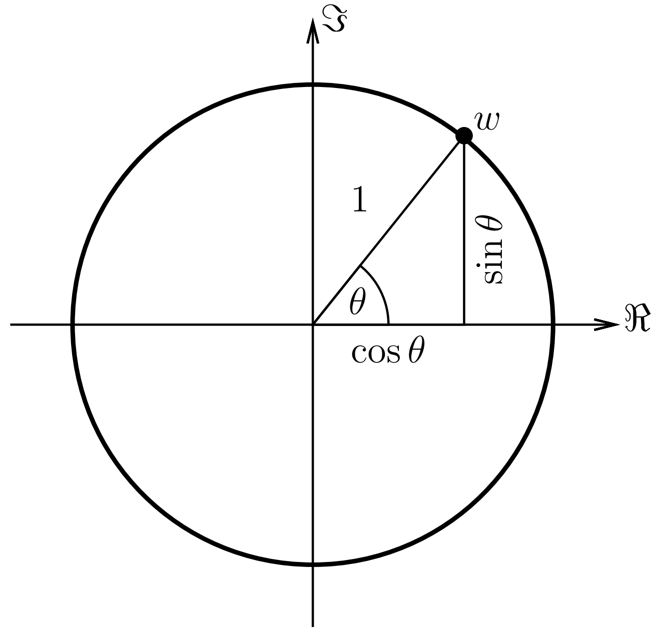
$$|z+w| \geq |z| - |w|$$

whose proof is a question on the problem sheet.

Let $z \neq 0$ and $z \in \mathbb{C}$. Consider $w = z/|z|$. then $|w| = 1$.

We claim there is some $\theta \in \mathbb{R}$ such that $w = \cos\theta + \mathrm{i}\sin\theta$.

If w is in the first quadrant:



In general to get to the other 3 quadrants, we need θ outside the interval $\left[0, \frac{\pi}{2}\right]$.

For example, if $\theta = \frac{3\pi}{4}$ then $\cos\theta + \mathrm{i}\sin\theta = -\frac{1}{\sqrt{2}} + \frac{\mathrm{i}}{\sqrt{2}}$, and if $\theta = -\frac{\pi}{4}$ say, then $\cos\theta + \mathrm{i}\sin\theta = \frac{1}{\sqrt{2}} - \frac{\mathrm{i}}{\sqrt{2}}$.

If $\theta > 0$, $\cos\theta + \mathrm{i}\sin\theta$ is obtained by starting at the point 1 and rotating an angle θ anticlockwise. If $\theta < 0$ we get $\cos\theta + \mathrm{i}\sin\theta$ by starting at 1 and rotating clockwise through an angle $|\theta|$.

A corollary to this is that each nonzero complex number has the form $z = r\mathrm{e}^{\mathrm{i}\theta}$ where $r = |z|$ and $\mathrm{e}^{\mathrm{i}\theta}$ is defined to be $\cos\theta + \mathrm{i}\sin\theta$. This is called the **polar form** of z.

Lemma For $\theta, \phi \in \mathbb{R}$,

$$\mathrm{e}^{\mathrm{i}(\theta+\phi)} = \mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\phi}.$$

Proof The LHS is $\cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi)$.

The RHS is

$$(\cos\theta + \mathrm{i}\sin\theta)(\cos\phi + \mathrm{i}\sin\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi + \mathrm{i}(\cos\theta\sin\phi + \sin\theta\cos\phi).$$

These are the same (they are addition formulae for trigonometric functions!)

$\square$

Corollary

$$\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\left(s\mathrm{e}^{\mathrm{i}\phi}\right) = rs\mathrm{e}^{\mathrm{i}(\theta+\phi)}.$$

We call θ in $z = r\mathrm{e}^{\mathrm{i}\theta}$ an **argument** of z.

Note that arguments are not unique, as $\mathrm{e}^{2\pi\mathrm{i}} = \cos 2\pi + \mathrm{i}\sin 2\pi = 1 + \mathrm{i}\cdot 0 = 1 = \mathrm{e}^{\mathrm{i}\cdot 0}$.

Similarly, $\mathrm{e}^{\mathrm{i}(\theta+2\pi)} = \mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{2\pi\mathrm{i}} = \mathrm{e}^{\mathrm{i}\theta}$. This can also be extended to $\mathrm{e}^{\mathrm{i}(\theta+2n\pi)} = \mathrm{e}^{\mathrm{i}\theta}$ if $n \in \mathbb{Z}$.

In general, $\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}\phi} \iff \theta - \phi = 2n\pi$ with $n \in \mathbb{Z}$. Arguments are only defined "modulo 2π".

Define the **principal argument** of $z \neq 0$ to be the unique θ satisfying $-\pi < \theta \leq \pi$ with $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$.

We write the principal argument as $\mathrm{Arg}(z)$.

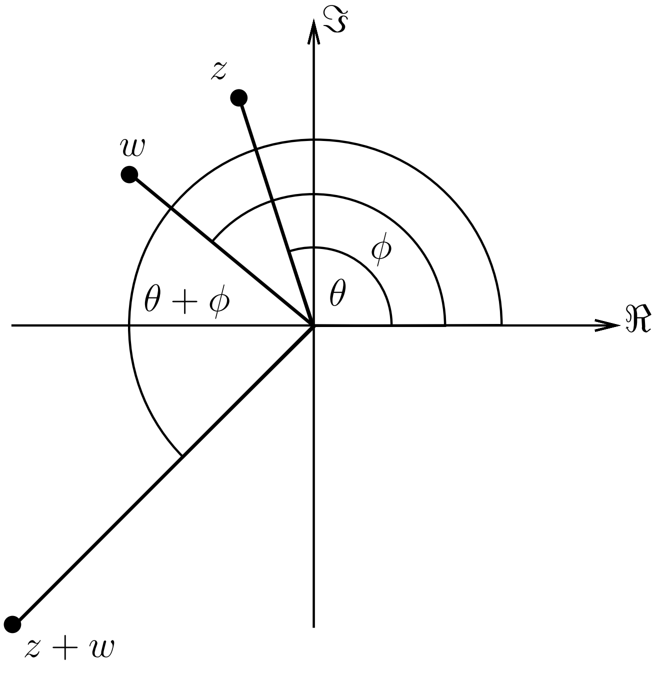
For example, we have

$$\begin{aligned}
\mathrm{Arg}(1) &= 0\\
\mathrm{Arg}(\mathrm{i}) &= \tfrac{\pi}{2}\\
\mathrm{Arg}(-1) &= \pi\\
\mathrm{Arg}\left(-\tfrac{1}{\sqrt{2}} + \tfrac{\mathrm{i}}{\sqrt{2}}\right) &= \tfrac{3\pi}{4}\\
\mathrm{Arg}\left(\tfrac{1}{\sqrt{2}} - \tfrac{\mathrm{i}}{\sqrt{2}}\right) &= -\tfrac{\pi}{4}.
\end{aligned}$$

Also we have $\mathrm{Arg}(z+w) = \mathrm{Arg}(z) + \mathrm{Arg}(w) + 2m\pi$ where $m \in \mathbb{Z}$, and m depends on z and w.

In the product $\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\left(s\mathrm{e}^{\mathrm{i}\phi}\right) = rs\mathrm{e}^{\mathrm{i}(\theta+\phi)}$, although θ and ϕ may be the principal arguments of z and w, $\theta + \phi$ may not be the principal argument of zw.

But $\theta+\phi$ differs from $\mathrm{Arg}(zw)$ by a multiple of 2π, e.g. $(-1)(-1) = 1$ and $\mathrm{Arg}(-1)+\mathrm{Arg}(-1) = 2\pi \neq 0 = \mathrm{Arg}(1)$.



1.3 Complex Functions

Complex analysis deals with functions of complex numbers.

We consider functions of the form $f : D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$.

Example

- $f(z) = z^2 + 1$. Note that $f(\pm \mathrm{i}) = 0$ since f can take the value 0 over $\mathbb{C}$.
- **Polynomials:** $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ are all functions from $\mathbb{C}$ to $\mathbb{C}$.
- **Rational functions:** $f(z) = \dfrac{1}{z}$ or $f(z) = \dfrac{1}{z^2+1}$ etc.

  In general, $f(z) = \dfrac{g(z)}{h(z)}$ is a **rational function** when g and h are polynomials.

  - $f(z) = \dfrac{1}{z}$ is a function from $\mathbb{C}\backslash\{0\}$ to $\mathbb{C}$.
  - $f(z) = \dfrac{z}{z^2+1}$ is defined on $\mathbb{C}\backslash\{\mathrm{i}, -\mathrm{i}\}$ and is a function from $\mathbb{C}\backslash\{\mathrm{i}, -\mathrm{i}\}$ to $\mathbb{C}$.

Theorem If $g(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ with $a_n \neq 0$, and $a_1, a_2, \ldots, a_n \in \mathbb{C}$, then the equation $g(z) = 0$ has at most n complex roots.

Proof We need to use the following result.

Lemma (Remainder Theorem) If $g(z)$ is a polynomial over $\mathbb{C}$ and $b \in \mathbb{C}$, then

$$g(z) = (z-b)h(z) + g(b)$$

where $h(z)$ is a polynomial.
Proof By long division of polynomials, $g(z) = (z-b)h(z) + c$ where $c \in \mathbb{C}$.
Substitute $z = b$ to get $g(b) = 0 \cdot h(b) + c$, i.e. $c = g(b)$.

□

If $g(z) = 0$ had no roots then we win.

Suppose z_1 is a root. Then $g(z_1) = 0$ and so by the remainder theorem, $g(z) = (z - z_1)h(z)$ where h is a polynomial.

As $g(z) = a_n z^n + \ldots$ then $h(z) = a_n z^{n-1} + \ldots$.

If $h(z_2) = 0$ then we can remove another factor $(z - z_2)$ and so on.

Eventually

$$g(z) = (z - z_1)(z - z_2) \ldots (z - z_n)\, p(z)$$

where $p(z)$ is a polynomial and $p(z) = 0$ has no solutions. $p(z) = a_n z^{n-k} + \ldots$ (with $k \leq n$).

$$
\begin{aligned}
g(z_1) = 0 \iff & z - z_1 = 0 \\
& \text{or } z - z_2 = 0 \\
& \vdots \\
& \text{or } z - z_k = 0 \\
& \text{or } p(z) = 0 \\
\iff & z \in \{z_1, \ldots, z_k\}.
\end{aligned}
$$

Therefore $g(z) = 0$ has $k \leq n$ distinct solutions.

$\square$

The only polynomials $p(z)$ over $\mathbb{C}$ with no solution to $p(z) = 0$ are constant polynomials (*Fundamental Theorem of Algebra*).

In general, every polynomial over $\mathbb{C}$ does factorise as

$$
a\left(z - z_1\right)\left(z - z_2\right)\ldots\left(z - z_n\right).
$$

Given a rational function $f(z)$, $f(z)$ is defined for all but finitely many complex values. If $f(z) = \frac{g(z)}{h(z)}$ then $h(z) \neq 0$ except for finitely many values of z.

1.3.1 The Exponential Function

Let $z = x + \mathrm{i}y$, with $x, y \in \mathbb{R}$.

Define

$$
\begin{aligned}
\exp(z) &= \mathrm{e}^z \\
&= \mathrm{e}^{x + \mathrm{i}y} \\
&= \mathrm{e}^x \left(\cos y + \mathrm{i} \sin y\right).
\end{aligned}
$$

The formula $\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ should generalise to $\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

For this to be the case,

$$
\begin{aligned}
\mathrm{e}^{y\mathrm{i}} &= \sum_{n=0}^{\infty} \frac{(\mathrm{i}y)^n}{n!} \\
&= 1 + \mathrm{i}y - \frac{y^2}{2!} - \frac{\mathrm{i}y^3}{3!} + \frac{y^4}{4!} + \ldots \\
&= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \ldots\right) + \left(\mathrm{i}y - \frac{\mathrm{i}y^3}{3!} + \ldots\right) \\
&= \cos y + \mathrm{i} \sin y.
\end{aligned}
$$

Properties

- $e^{z+w} = e^z e^w$ follows from $e^{x+y} = e^x e^y \, \forall x, y \in \mathbb{R}$. (This is a homomorphism from $(\mathbb{C}, +)$ to $(\mathbb{C}\backslash\{0\}, \times)$.)
- $e^{z+2\pi i} = e^z$ (as $e^{z+2\pi i} = e^z e^{2\pi i} = e^z \cdot 1 = e^z$), so $\exp$ is a **periodic** function on $\mathbb{C}$ with period $2\pi i$.
- $\left|e^{x+iy}\right| = |e^x| \left|e^{iy}\right| = e^x \cdot |\cos y + i \sin y| = e^x \sqrt{\cos^2 y + \sin^2 y} = e^x \cdot 1 = e^x$. So $|e^z| = e^{\Re(z)}$.
- For $y \in \mathbb{R}$, $e^{iy} = \cos y + i \sin y$ and $e^{-iy} = \cos(-y) + i \sin(-y) = \cos y - i \sin y$. So

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and } \sin y = \frac{e^{iy} - e^{-iy}}{2i}$$

Now define, for $z \in \mathbb{C}$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Then

$$\begin{aligned}
\cos ix &= \frac{e^{-x} + e^x}{2} \\
&= \cosh x \\
\sin ix &= \frac{e^{-x} - e^x}{2i} \\
&= i \left(\frac{e^x + e^{-x}}{2} \right) \\
&= i \sinh x.
\end{aligned}$$

1.3.2 Complex Logarithms

Given $w \in \mathbb{C}$, what are the solutions of $\exp z = w$?

We need $w \neq 0$ as $\exp(z) \neq 0$ (because $\exp(z)\exp(-z) = 1$).

For any nonzero w, $\exp(z) = w$ is soluble but with infinitely many solutions!

Write $w = re^{i\theta}$ in polar form for $y \in \mathbb{R}$. Set $z = x + iy$, with $x, y \in \mathbb{R}$.

Then $\exp(z) = e^x e^{iy}$ should equal $re^{i\theta}$.

$e^x = \left|e^x e^{iy}\right| = \left|re^{i\theta}\right| = r > 0$. So x has to equal $\log r$.

Then we get $e^{iy} = e^{i\theta}$, i.e. $y = \theta + 2n\pi$ where $n \in \mathbb{Z}$.

So $e^z = re^{i\theta}$ has solutions

$$\boxed{z = \log r + i(\theta + 2n\pi)}$$

where $n \in \mathbb{Z}$.

Any of these solutions is a **logarithm** of $re^{i\theta} = w$, that is, if $w \in \mathbb{C}\backslash\{0\}$ then a **logarithm** of w is a solution to $\exp z = w$.

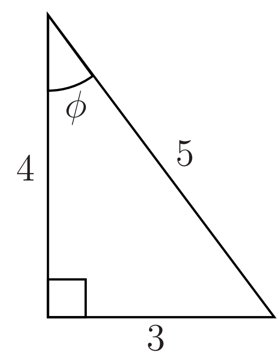
One logarithm of w is $\log|w| + i\mathrm{Arg}(w)$. We call this the **principal logarithm** and write it as $\mathrm{Log}(w)$.

As before we can deduce that

$$
\begin{aligned}
\log(zw) = \mathrm{Log}(z) + \mathrm{Log}(w) \\
+ 2\pi n\mathrm{i} \\
(n \in \mathbb{N}).
\end{aligned}
$$

Example

$$\mathrm{Log}(-3-4\mathrm{i})$$



Here,

$$
\begin{aligned}
\mathrm{Log}(-3-4\mathrm{i}) &= \log|-3-4\mathrm{i}| - \theta\mathrm{i} \\
&= \log 5 - \mathrm{i}\left(\frac{\pi}{2} + \arctan\frac{3}{4}\right).
\end{aligned}
$$

2 Main Course

We want to be able to study differentiable functions on subsets of $\mathbb{C}$.

2.1 Open and closed sets

An **open disc** is a set of the form

$$\mathcal{D}(a,r) = \{\, z \in \mathbb{C} |\, |z-a| < r\}$$

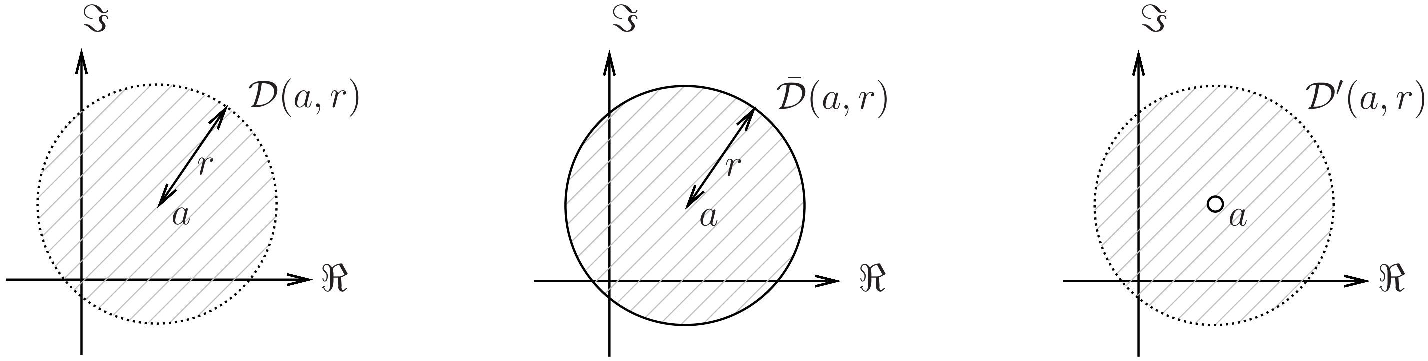
where $a \in \mathbb{C}$ and $r \in \mathbb{R}^+$.

A **closed disc** is one of the form

$$\overline{\mathcal{D}}(a,r) = \{\, z \in \mathbb{C} |\, |z-a| \le r\}$$

and a **punctured open disc** is one of the form

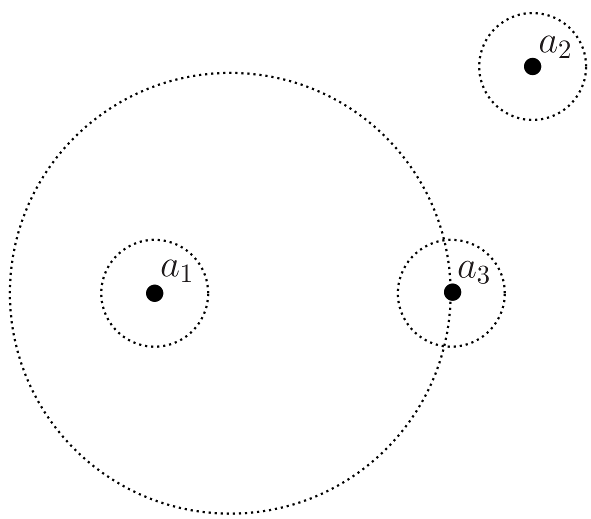
$$\mathcal{D}'(a,r) = \{\, z \in \mathbb{C} |\, 0 < |z-a| < r\}\,.$$



Let $A \subseteq \mathbb{C}$. Each point $a \in \mathbb{C}$ is either an **interior point** of A or an **exterior point** of A, or a **boundary point** of A.

a is:

- an **interior point** of A is there is no $r > 0$ with $\mathcal{D}(a, r) \subseteq A$ (for example, a_1)
- an **exterior point** of A if there is $r > 0$ with $\mathcal{D}(a, r) \cap A = \emptyset$ (for example a_2)
- a **boundary point** of A if a is neither an interior nor exterior point of A (for example, a_3)



A subset $A \subseteq \mathbb{C}$ is **open** if all $a \in A$ are interior points of A, i.e. no element of A is a boundary point of A.

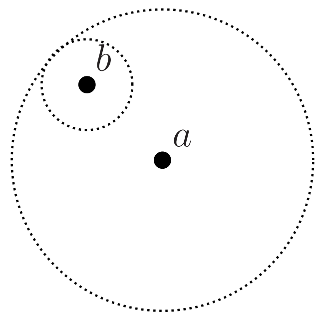
A subset $A \subseteq \mathbb{C}$ is **closed** if every $a \in \mathbb{C}$ which is a boundary point of A is an element of A, i.e. A contains **all** its boundary points.

Claim

1. $\mathcal{D}(a, r)$ is open
2. $\overline{\mathcal{D}}(a, r)$ is closed.

Proof

1. Let $b \in \mathcal{D}(a, r)$. We need to prove b is an interior point of $\mathcal{D}(a, r)$, i.e. we need to find $s > 0$ with $\mathcal{D}(b, s) \subseteq \mathcal{D}(a, r)$.



Try $s = r - |b - a|$. Let $z \in \mathcal{D}(b, s)$. Then $|z - b| < s$.

We need to prove that $z \in \mathcal{D}(a, r)$, i.e. that $|z - a| < r$. then

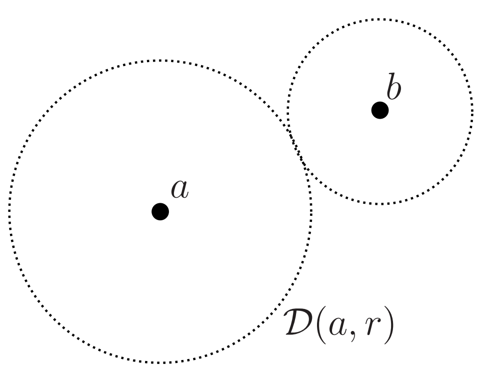
$$\begin{aligned} |z-a| &= |(z-b)+(b-a)| \\ &\leq |z-b|+|b-a| \text{ (by the \textit{triangle inequality})} \\ &< s+|b-a| \\ &= r \end{aligned}$$

So $z \in \mathcal{D}(a, r)$ and $\mathcal{D}(b, s) \subseteq \mathcal{D}(a, r)$.

We should have proved that $s > 0$ but as $b \in \mathcal{D}(a, r)$ then $|b - a| < r$ and so $s = r - |b - a| > 0$.

2. To show $\overline{\mathcal{D}}(a, r)$ is closed, we need to show every boundary point of $\overline{\mathcal{D}}(a, r)$ is an element of $\overline{\mathcal{D}}(a, r)$. It suffices to prove that if $b \notin \overline{\mathcal{D}}(a, r)$ then b is not a boundary point of $\overline{\mathcal{D}}(a, r)$ (which is the contrapositive), so that $|b - a| \neq r$.

So we shall show that each $b \notin \overline{\mathcal{D}}(a, r)$ is an exterior point.



We claim there is $s > 0$ with $\mathcal{D}(b, s) \cap \overline{\mathcal{D}}(a, r) = \emptyset$. Take $s = |b - a| - r$.

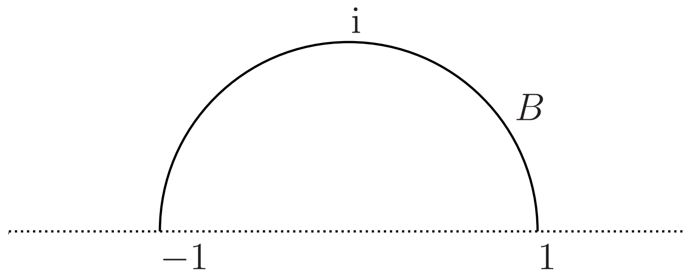
Note that as $b \notin \overline{\mathcal{D}}(a, r)$ then $|b - a| > r$.

Skipping the details, we again use the triangle inequality.

□

Examples

- $\mathbb{C}$ is both open and closed.
- $B = \{z \in \mathbb{C} | \, |z| \leq 1$ and $\Im(z) > 0\}$ is neither open nor closed.
  - For example, 0 and i are boundary points of B. $0 \in B$ so B isn't closed, but $\mathrm{i} \in B$ so B isn't open.



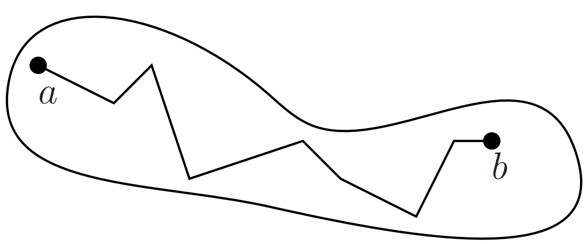
2.2 Domains

A **domain** is a **connected** open set.

A subset $A \subseteq \mathbb{C}$ is **bounded** if $A \subseteq \mathcal{D}(a, r)$ for some $a \in \mathbb{C}$, $r > 0$.

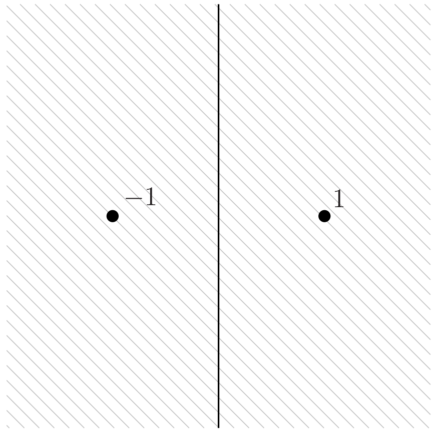
$\mathbb{C}$ is unbounded, as we can't have $\mathbb{C} \subseteq \mathcal{D}(a, r)$ as $a + 2r \notin \mathcal{D}(a, r)$.

Definition An open set $\mathcal{U} \subseteq \mathbb{C}$ is **connected** if for any 2 points a and b in $\mathcal{U}$, I can join a to b in a finite sequence of straight line segments contained within $\mathcal{U}$. (This is only appropriate for open sets.)

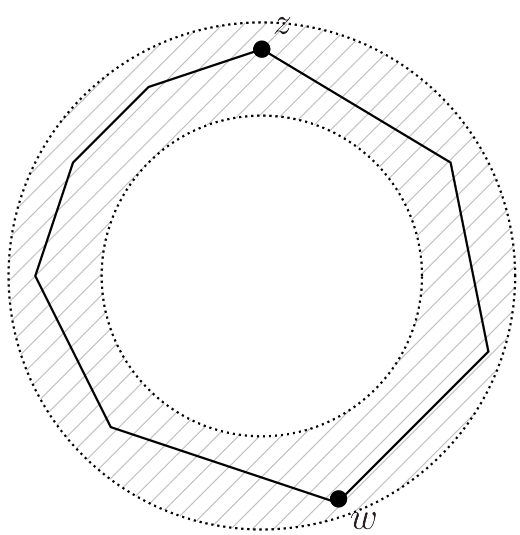


Examples

- $\mathcal{D}(a, r)$ is connected.
  - Let $b, c \in \mathcal{D}(a, r)$. One can join b and c directly by a straight line segment which itself lies within $\mathcal{D}(a, r)$.
- $\mathcal{U} = \{z \in \mathbb{C} | \, \Re(z) \neq 0\}$ is not connected.
  - $\mathcal{U}$ consists of points off the imaginary axis. Any sort of path made up of line segments from -1 to 1 must cross the y-axis so can't lie entirely inside $\mathcal{U}$.



- Consider $A(a,r,s) = \{z \in \mathbb{C} | \, r < |z-a| < s, r, s \in \mathbb{R}, 0 < r < s\}$, the annulus with centre a and the region between the two circles $|z-a| = r$ and $|z-a| = s$. This is connected.
  - Draw a polygon inside $A(a,r,s)$ encircling the centre. For $z, w \in A(a,r,s)$ we can join z and w to the polygon by straight line segments, so to get z to w, we walk from z to the polygon, walk around the polygon, then walk to w. So the annulus is connected.



2.3 Convergence of sequences

Let (z_n) be a sequence of complex numbers. We say (z_n) converges to z or $z_n \to z$ or $\lim_{n\to\infty} z_n = z$ if either

- $\Re(z_n) \to \Re(z)$ and $\Im(z) \to \Im(z)$
- $|z_n - z| \to 0$.

(These are equivalent.)

$\sum_{n=0}^{\infty} z_n$ converges to z if $\sum_{n=0}^{\infty} z_n$ converges to z as $N \to \infty$ (same as in real analysis).

We claim that if $|z| < 1$ then $\sum_{n=0}^{\infty} z^n$ converges to $\frac{1}{1-z}$.

$$\sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z}$$

so

$$\left| \frac{1}{1-z} - \sum_{n=0}^{N} z^n \right| = \left| \frac{z^{N+1}}{1-z} \right| \\ \to 0$$

as $|z| < 1$.

As with real series, $\sum_{n=0}^{\infty} z_n$ is divergent when $z_n \not\to 0$. If $|z| > 1$ then $z^n \not\to 0$.

The "calculus of limits" works over $\mathbb{C}$ in exactly the same way as over $\mathbb{R}$.

Lemma (Absolute Convergence) If $\sum_{n=0}^{\infty} |z_n|$ converges then so does $\sum_{n=0}^{\infty} z_n$.

Outline of Proof Apply the real version to $\sum \Re(z_n)$ and $\sum \Im(z_n)$, etc.

$\square$

We say $\sum_{n=0}^{\infty} z_n$ is **absolutely convergent** if $\sum_{n=0}^{\infty} |z_n|$ converges.

Example For all $z \in \mathbb{C}$, $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is absolutely convergent since

$$\sum_{n=0}^{\infty} \left| \frac{z^n}{n!} \right| = \sum_{n=0}^{\infty} \frac{|z|^n}{n!}$$

is a convergent series over $\mathbb{R}$.

"Proof" that $\sum_{n=0}^{\infty} \frac{z^n}{n!} = \mathrm{e}^z$ Note that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^{n-k} w^k \text{ (by the binomial theorem)} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k} w^k}{(n-k)!k!} (*) \\ &= \sum_{k=0}^{\infty} \frac{w^k}{k!} \sum_{m=0}^{\infty} \frac{z^m}{m!} \text{ (letting } m = n-k). \end{aligned}$$

$\square$

At $(*)$, we interchanged the order of summation. This is legitimate as the "double" series is absolutely convergent.

In particular, if $x, y \in \mathbb{R}$, then

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(x+\mathrm{i}y)^n}{n!} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{n=0}^{\infty} \frac{(\mathrm{i}y)^n}{n!} \\ &= \mathrm{e}^x(\cos y + \mathrm{i}\sin y) \\ &= \mathrm{e}^{x+\mathrm{i}y}.\end{aligned}$$

2.4 Continuity

Definition Let $f : A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$. We say that f is **continuous at** $a \in A$ if all sequences $(a_n) \in A$ with $a_n \to a$ satisfy $f(a_n) \to f(a)$.

We say f is **continuous on** A if f is continuous at all $a \in A$.

Examples Obviously constant functions are continuous and $f(z) = z$ is continuous.

Theorem Continuity holds under addition, subtraction, multiplication, division and composition, subject to the usual caveats (dividing by zero/well-defined composition, etc.)

Proof Same as for real analysis.

□

Remark Polynomials are continuous on all of $\mathbb{C}$, rational functions are continuous where defined.

Important Example $f(z) = \exp z$ is continuous.

Proof Suppose $a_n \to a$. We will show that $\mathrm{e}^{a_n} \to \mathrm{e}^a$.

Note that

$$\begin{aligned}\mathrm{e}^{a_n} &= \mathrm{e}^a \mathrm{e}^{a_n - a} \\ &= \mathrm{e}^a \mathrm{e}^{b_n}\end{aligned}$$

where $b_n = a_n - a \to 0$. It suffices to show that $\mathrm{e}^{b_n} \to 1$.

$$\begin{aligned}\mathrm{e}^{b_n} - 1 &= \sum_{k=0}^{\infty} \frac{b_n^k}{k!} - 1 \\ &= \sum_{k=1}^{\infty} \frac{b_n^k}{k!}\end{aligned}$$

As $b_n \to 0$, $|b_n| \leq 1$ **eventually**. That is, there exists $N \in \mathbb{N}$ with $|b_n| \leq 1 \forall n \geq N$.

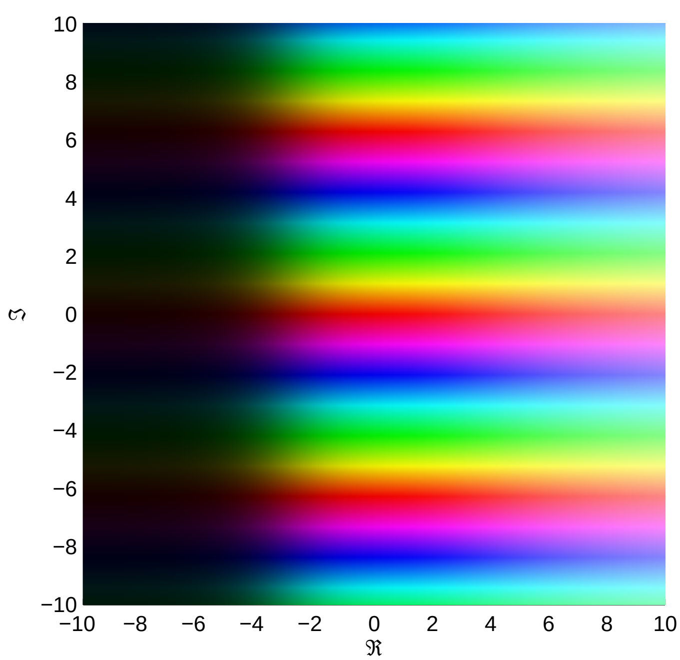
Hence

$$\begin{aligned} \left|\mathrm{e}^{b_n} - 1\right| &= |b_n| \left|\sum_{k=1}^{\infty} \frac{b_n^{k-1}}{k!}\right| \\ &\leq |b_n| \sum_{k=1}^{\infty} \frac{\left|b_n^{k-1}\right|}{k!} \\ &\leq |b_n| \sum_{k=1}^{\infty} \frac{1}{k!} \\ &= |b_n| (\mathrm{e} - 1) \\ &\text{as } |b_n| \leq 1 \text{for } n \geq N. \end{aligned}$$

As $b_n \to 0$, $\left|\mathrm{e}^{b_n} - 1\right| \to 0$ so that $\mathrm{e}^{b_n} \to 1$ as required.

Hence $\exp(z)$ is continuous on all of $\mathbb{C}$.

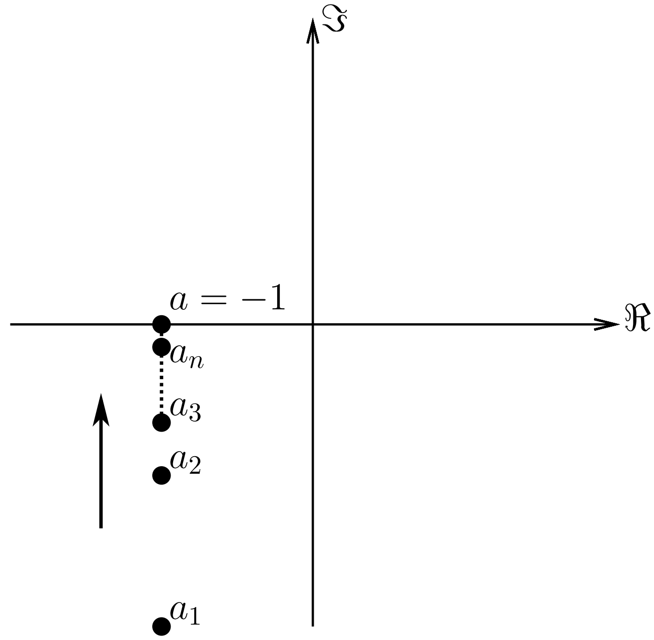
$\square$



Here is a complex plot, generated with MATLAB, to illustrate the continuity of $\exp z$. The brightness illustrates the absolute value (so $|z| = 0$ is black, and $|z| \to \infty$ leads to white), and the arguments are represented by hue (on a colour scale) where an argument of 0 is red, and an argument of π is cyan. Notice that there are no discontinuities (e.g. changing colours/shading along lines or points) so this is visual confirmation that $\exp z$ is continuous, and you can see that it is actually periodic (with period $2\pi\mathrm{i}$ - the colour of a point will the exactly the same as that $2\pi\mathrm{i}$ above it).

Important Example $g(z) = \mathrm{Log}(z)$ is not continuous.

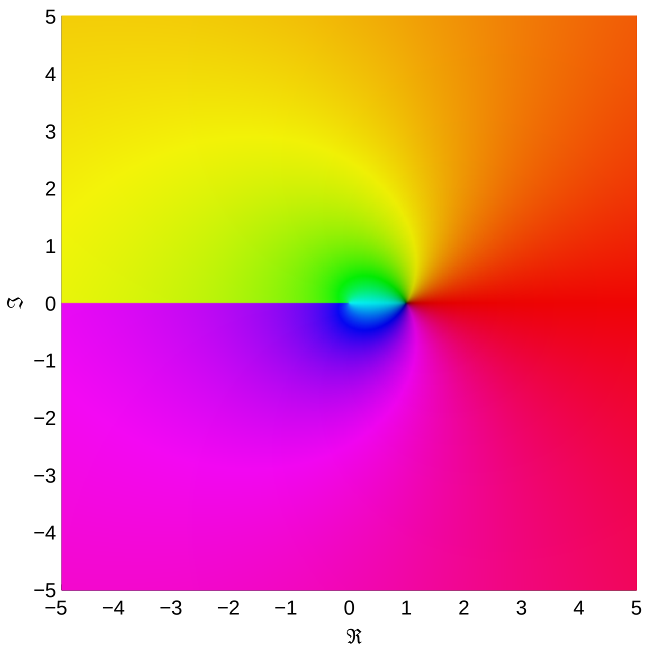
Let (a_n) be a sequence of points convergine to $a = -1$ from below the real axis (e.g. $a_n = -1 - \frac{i}{n}$).



Now $g(a) = \mathrm{Log}(-1) = \mathrm{i}\pi$ but the principal arguments of the a_n are negative and tend to $-\pi$ as $n \to \infty$ so that

$$\lim_{n\to\infty} \mathrm{Log}(a_n) = -\mathrm{i}\pi \neq \mathrm{Log}(a).$$

As in real analysis we can deal with limits of functions, i.e. $\lim_{z\to a} f(z) = b$ means that $f(a_n) \to b$ for all sequences (a_n) with $a_n \to a$ such that $a_n \neq a$.



There is a slit on the negative real axis; the colours are not the same on either side of the line, so the arguments on either side do not reach the same limit.

2.5 Holomorphic Functions

Definition Let $f : A \to \mathbb{C}$ (with $A \subseteq \mathbb{C}$). We say that f is **differentiable** at $a \in A$ if

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists. This limit is called $f'(a)$ provided it exists.

Definition A function on an open set $\mathcal{U}$ which is differentiable at every point of $\mathcal{U}$ is called a **holomorphic** function on $\mathcal{U}$.

Theorem As in real analysis, if f and g are differentiable then so are $f \pm g$, fg, f/g, $f \circ g$, $g \circ f$ subject to the usual caveats, and their derivatives are given by the standard formulae from calculus (product/quotient/chain rules).

The proof is the same as in real analysis.

Examples

- $\frac{1}{z^2+1}$ has derivative $\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z^2+1}\right) = -\frac{2z}{(z^2+1)^2}$.
- e^{z^2} has derivative $\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathrm{e}^{z^2}\right) = 2z\mathrm{e}^{z^2}$ etc.

Theorem $\exp(z)$ is differentiable and it is its own derivative.

Proof

$$\frac{\mathrm{e}^z - \mathrm{e}^a}{z-a} = \mathrm{e}^a \left(\frac{\mathrm{e}^{z-a} - 1}{z-a} \right)$$

so

$$\begin{aligned} \lim_{z \to a} \frac{\mathrm{e}^z - \mathrm{e}^a}{z-a} = \mathrm{e}^z &\iff \lim_{z \to a} \frac{\mathrm{e}^{z-a} - 1}{z-a} = 1 \\ &\iff \lim_{w \to 0} \frac{\mathrm{e}^w - 1}{w} = 1 \text{ (setting } w = z-a) \end{aligned}$$

But

$$\begin{aligned} \frac{\mathrm{e}^w - 1}{w} - 1 &= \sum_{n=1}^{\infty} \frac{w^{n-1}}{n!} \\ &= w \sum_{n=2}^{\infty} \frac{w^{n-2}}{n!} \end{aligned}$$

and by an argument similar to that which was used for the continuity of $\exp(z)$ we get that

$$\lim_{w \to 0} \left(\frac{\mathrm{e}^w - 1}{w} - 1 \right) = 0$$

whence $\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathrm{e}^z\right) = \mathrm{e}^z$ for all $z \in \mathbb{C}$.

$\square$

Corollary

$$\frac{\mathrm{d}}{\mathrm{d}z}(\cos z) = -\sin z, \text{ etc.}$$

Proof

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}(\cos z) &= \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}z} - \mathrm{i}\mathrm{e}^{-\mathrm{i}z}}{2} \\ &= \frac{-\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}} \\ &= -\sin z, \text{ etc.} \end{aligned}$$

$\square$

Example Let $g(z) = z^\star$.

Let $a \in \mathbb{C}$. Then

$$\begin{aligned} \frac{g(z)-g(a)}{z-a} &= \frac{z^\star - a^\star}{z-a} \\ &= \frac{(z-a)^\star}{(z-a)} \end{aligned}$$

so that

$$\begin{aligned} g'(z) &= \lim_{z\to a} \frac{(z-a)^\star}{(z-a)} \\ &= \lim_{w\to 0} \frac{z^\star}{z} \end{aligned}$$

provided the limit exists.

But this limit does **not** exist! Let $a_n = \frac{1}{n} \to 0$ as $n \to \infty$. Now

$$\begin{aligned} \frac{(a_n)^\star}{a_n} &= \frac{1/n}{1/n} \\ &= 1 \to 1. \end{aligned}$$

But if $b_n = \frac{\mathrm{i}}{n} \to 0$ as $n \to \infty$ we have

$$\begin{aligned} \frac{(b_n)^\star}{b_n} &= \frac{-\mathrm{i}/n}{\mathrm{i}/n} \\ &= -1 \to -1. \end{aligned}$$

So $g(z)$ is not differentiable anywhere!

2.5.1 Theorem (Cauchy-Riemann Equations)

Let

$$f(x+\mathrm{i}y) = u(x,y) + \mathrm{i}v(x,y)$$

where $x, y, u(x,y), v(x,y) \in \mathbb{R}$.
If f is holomorphic on an open set $\mathcal{U}$ then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

So the real and imaginary parts of a holomorphic function f almost detemine each other; $\Re(f) = u$ and $\Im(f) = v$ are deeply entwined with each other.

Proof Let $z_0 = x_0 + \mathrm{i}y_0 \in \mathcal{U}$. Then

$$\begin{aligned} f'(z_0) &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \\ &= \lim_{h\to 0} \frac{f(z+h)-f(z_0)}{h} \end{aligned}$$

1. Specialise $h \in \mathbb{R}$. Then

$$
\begin{aligned}
f'(z_0) &= \lim_{h\to 0} \frac{[u(x_0+h,y_0)+\mathrm{i}v(x_0+h,y_0)-u(x_0,y_0)-\mathrm{i}v(x_0,y_0)]}{h} \\
&= \lim_{h\to 0} \frac{u(x_0+h,y_0)+u(x_0,y_0)}{h} + \mathrm{i}\lim_{h\to 0}\frac{v(x_0+h,y_0)-v(x_0,y_0)}{h} \\
&= \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + \mathrm{i}\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}
\end{aligned}
$$

2. Specialise $h = \mathrm{i}k$ with $k \in \mathbb{R}$. Then

$$
\begin{aligned}
f'(z_0) &= \lim_{k\to 0} \frac{[u(x_0,y_0+k)+\mathrm{i}v(x_0,y_0+k)-u(x_0,y_0)-\mathrm{i}v(x_0,y_0)]}{\mathrm{i}k} \\
&= \lim_{k\to 0} \frac{v(x_0,y_0+k)-v(x_0,y_0)}{k} - \mathrm{i}\lim_{k\to 0}\frac{u(x_0,y_0+k)-u(x_0,y_0)}{k} \\
&= \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} - \mathrm{i}\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}.
\end{aligned}
$$

Compare both results (equating real and imaginary parts) to give

$$
\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} \quad \text{and} \quad \left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} = -\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}.
$$

□

A converse If $f(x+\mathrm{i}y) = u(x,y)+\mathrm{i}v(x,y)$ is a function on an open set $\mathcal{U}$, then if f satisfies the *Cauchy-Riemann equations* **and** $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous on $\mathcal{U}$ then f is holomorphic. The proof is omitted here, but see *Chen's notes* - to which there is a link on the website.

Example

$$f(z) = \mathrm{e}^z$$

In this case, $f(x+\mathrm{i}y) = \mathrm{e}^x(\cos y + \mathrm{i}\sin y)$ with $u = \mathrm{e}^x\cos y$ and $v = \mathrm{e}^x \sin y$. Then

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \mathrm{e}^x\cos y \qquad & \frac{\partial v}{\partial x} &= \mathrm{e}^x \sin y \\
\frac{\partial v}{\partial y} &= \mathrm{e}^x\cos y \qquad & -\frac{\partial u}{\partial y} &= \mathrm{e}^x \sin y
\end{aligned}
$$

confirming f to be holomorphic.

Example

$$f(z) = z^\star$$

In this case, $f(x+\mathrm{i}y) = x - \mathrm{i}y$ with $u = x$ and $v = -y$.

But $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial y} = -1 \neq \frac{\partial u}{\partial x}$ so f isn't holomorphic.

Example

$$f(z) = |z|$$

In this case, $f(x+\mathrm{i}y) = \sqrt{x^2+y^2}$ with $u = \sqrt{x^2+y^2}$ and $v = 0$. Therefore

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \frac{\partial u}{\partial y} = -\frac{y}{\sqrt{x^2+y^2}}$$

but also

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

The Cauchy-Riemann equations can't hold unless $x = y = 0$, but then $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ would not exist.

Therefore f isn't differentiable anywhere!

Application A holomorphic real-valued function has to be constant. Since $v = 0$, then by the *Cauchy-Riemann equations*,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

so u is constant.

Claim $\mathrm{Log}(z)$ is holomorphic on the set $\mathbb{C}\setminus\{x|\, x\in\mathbb{R}, x\leq 0\}$.

Proof For convenience, we'll prove this for $\{z = x+\mathrm{i}y|\, x>0\}$.

For $x > 0$,

$$\begin{aligned}\mathrm{Log}(x+\mathrm{i}y) &= \log|x+\mathrm{i}y| + \mathrm{i}\mathrm{Arg}(x+\mathrm{i}y)\\ &= \log\sqrt{x^2+y^2} + \mathrm{i}\arctan\frac{y}{x}.\end{aligned}$$

In this case, $u = \frac{1}{2}\log(x^2+y^2)$ and $v = \arctan\frac{y}{x}$.

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{2}\frac{2x}{x^2+y^2} \\
&= \frac{x}{x^2+y^2} \\
\frac{\partial u}{\partial y} &= \frac{y}{x^2+y^2} \\
\frac{\partial v}{\partial x} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right) \\
&= \frac{x^2}{x^2+y^2}\cdot\left(-\frac{y}{x^2}\right) \\
&= -\frac{y}{x^2+y^2} \\
\frac{\partial v}{\partial y} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right) \\
&= \frac{x^2}{x^2+y^2}\cdot\frac{1}{x} \\
&= \frac{x}{x^2+y^2}
\end{aligned}
$$

The derivatives all agree, so $\mathrm{Log}(z)$ is holomorphic for $\Re(z) = x > 0.$

Then for when $x < 0$ and $y > 0,$ u is the same as before and $v = \mathrm{i}\left[\pi - \arctan\frac{y}{x}\right]$. And for when when $x < 0$ and $y < 0,$ u is the same and v changes sign from the previous case.

□

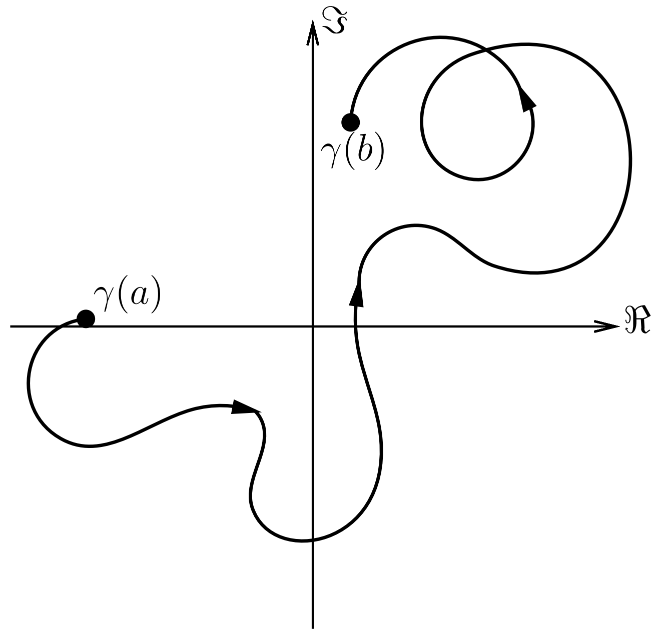
2.6 Paths and path integrals

These are analogous to "line integrals" in the plane.

A **path** is a continuous map $\gamma : [a, b] \to \mathbb{C}$. It is smooth if γ is differentiable and γ' is continuous.

We can write $\gamma(t) = x(t) + \mathrm{i}y(t)$, with $x, y : [a, b] \to \mathbb{R}$.

γ is continuous (and differentiable) if both x and y are.

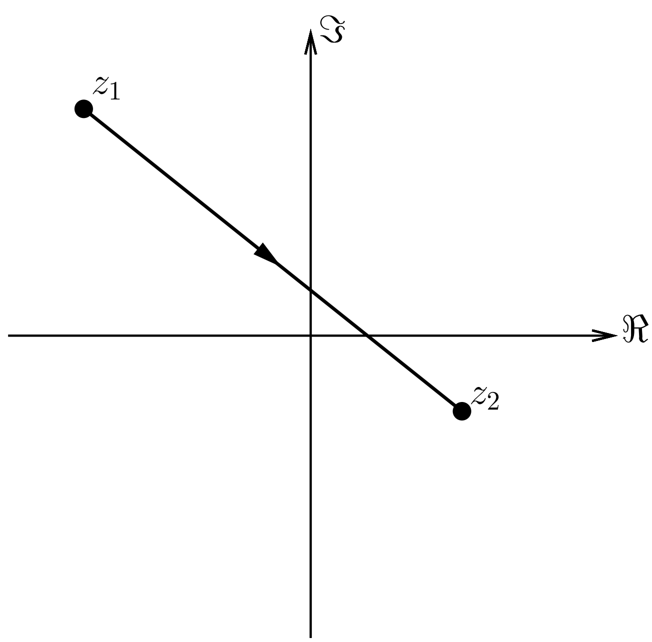


Examples

1. Let $z_1, z_2 \in \mathbb{C}$.
   The line segment from z_1 to z_2 is the path $\gamma : [0,1] \to \mathbb{C}$ defined by

$$\gamma(t) = z_1 + t(z_2 - z_1).$$

   Sometimes we denote this as $[z_1, z_2]$.

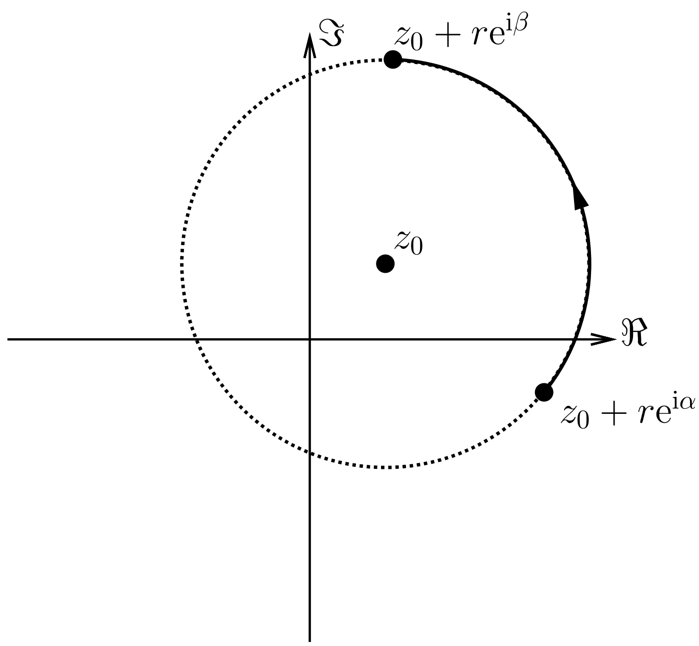


2. Let $z_0 \in \mathbb{C}$, $r > 0$, $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$.

Define $\gamma : [\alpha, \beta] \to \mathbb{C}$ by

$$\gamma(t) = z_0 + r\mathrm{e}^{\mathrm{i}t}.$$

γ is an arc of the circle centre z and radius r.



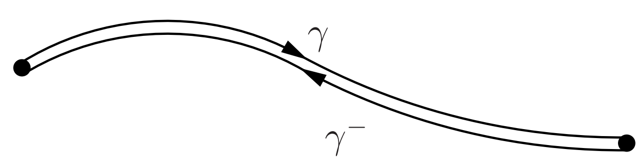
If $\alpha = 0$, $\beta = 2\pi$, one traverses the whole circle. If $\alpha = 0$ and $\beta = 4\pi$, then the circle is traversed twice, and so on.

Both of these paths are smooth, as $\gamma'(t) = z_2 - z_1$ and $\gamma(t) = \mathrm{i}r\mathrm{e}^{\mathrm{i}t}$ respectively (which are continuous).

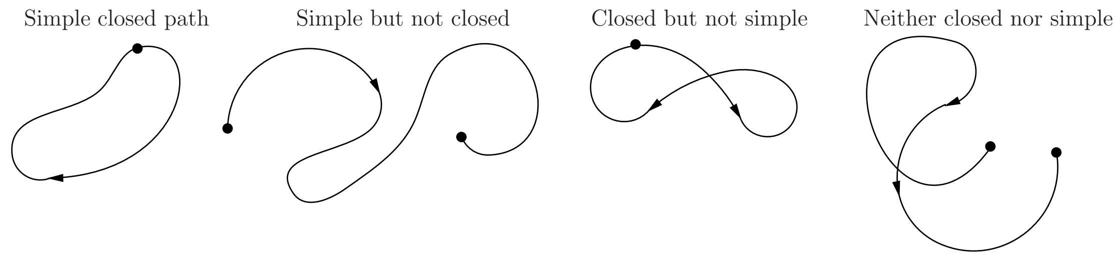
Path jargon

For the path $\gamma : [a, b] \to \mathbb{C}$:

- $\gamma(a)$ is the **start point**
- $\gamma(b)$ is the **end point**
- If $\gamma(a) = \gamma(b)$ then the path is **closed** (e.g. a full circle)
- If $\gamma(t) = \gamma(s) \implies t = s$ then the path is **simple** (i.e. injective). It doesn't intersect itself except it might close up at the end.
- $\gamma^- = \gamma(-t)$ is a map $\gamma^- : [-b, -a] \to \mathbb{C}$ is the **reversal** of γ.



Examples



Let f be continuous on an open set $\mathcal{U}$. Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path contained in $\mathcal{U}$.

Define

$$\boxed{\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t}$$

(as a mnemonic, $z = \gamma(t)$ so $\mathrm{d}z = \gamma'(t)\mathrm{d}t$),

In the last integral, we integrate a complex function over a real variable.

Naturally

$$\int_a^b (u(t) + \mathrm{i}v(t))\, \mathrm{d}t = \int_a^b u(t)\mathrm{d}t + \mathrm{i}\int_a^b v(t)\mathrm{d}t.$$

Example Integrating

$$f(z) = z$$

over γ, the line segment from 1 to $2 + 2\mathrm{i}$. So then

$$\begin{aligned} \gamma : [0, 1] &\to \mathbb{C} \\ t &\mapsto 1 + t(1 + 2\mathrm{i}) \end{aligned}$$

so $\gamma'(t) = 1 + 2\mathrm{i}$.

$$\begin{aligned} \int_\gamma f(z)\mathrm{d}z &= \int_0^1 (1 + (1 + 2\mathrm{i})t)\,(1 + 2\mathrm{i})\mathrm{d}t \\ &= \int_0^1 ((1 + 2\mathrm{i}) + (-3 + 4\mathrm{i})t)\,\mathrm{d}t \\ &= \left[(1 + 2\mathrm{i})t + \frac{1}{2}\,(-3 + 4\mathrm{i})\,t^2\right]_0^1 \\ &= 1 + 2\mathrm{i} + \frac{1}{2}(-3 + 4\mathrm{i}) \\ &= -\frac{1}{2} + 4\mathrm{i}. \end{aligned}$$

Remark

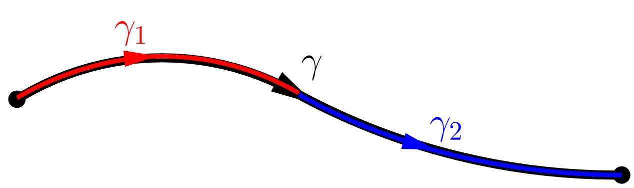
There is a connection with line integrals:

$$f(x+\mathrm{i}y) = u(x,y) + \mathrm{i}v(x,y)$$

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma \left(u(x,y) + \mathrm{i}v(x,y)\right)(\mathrm{d}x + \mathrm{i}\mathrm{d}y)$$

Some properties

- $\int_\gamma \left(f(z) + g(z)\right)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}z + \int_\gamma g(z)\mathrm{d}z$
- $\int_\gamma \alpha f(z)\mathrm{d}z = \alpha \int_\gamma f(z)\mathrm{d}z \ \forall \alpha \in \mathbb{C}$
- If γ^- is the reversal of γ, then $\int_{\gamma^-} f(z)\mathrm{d}z = -\int_\gamma f(z)\mathrm{d}z$.
- $\int_\gamma f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z$ where $\gamma : [a,b] \to \mathbb{C}$, $\gamma_1 : [a,c] \to \mathbb{C}$ and $\gamma_2 : [c,b] \to \mathbb{C}$, with $a < c < b$. Also $\gamma_1(t) = \gamma(t)$ and $\gamma_2(t) = \gamma(t)$.



- **Notation:** $\int_{w_1}^{w_2} f(z)\mathrm{d}z$ means $\int_\gamma f(z)\mathrm{d}z$ where γ is the line segment from w_1 to w_2 - this only makes sense if f is defined in a domain containing this segment.

2.6.1 Theorem (Fundamental Theorem of Calculus)

Assume $f : \mathcal{U} \to \mathbb{C}$ is holomorphic and $\mathcal{U}$ is open. Also assume that f' is continuous. Then

$$\boxed{\int_\gamma f'(z)\mathrm{d}z = f\left(\gamma(b)\right) - f\left(\gamma(a)\right)}$$

If γ is a closed path, then $\int_\gamma f'(z)\mathrm{d}z = 0$.

Proof

$$
\begin{aligned}
\int_\gamma f'(z)\mathrm{d}z &= \int_a^b f'\left(\gamma(t)\right)\gamma'(t)\mathrm{d}t \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f\left(\gamma(t)\right)\mathrm{d}t \text{ (applying the chain rule in reverse)} \\
&= \left[f\left(\gamma(t)\right)\right]_{t=a}^{t=b} \text{ (by the real version of the Fundamental Theorem of Calculus)} \\
&= f\left(\gamma(b)\right) - f\left(\gamma(a)\right)
\end{aligned}
$$

□

Example

$$\int_1^{2+2\mathrm{i}} z\mathrm{d}z$$

using the Fundamental Theorem of Calculus.

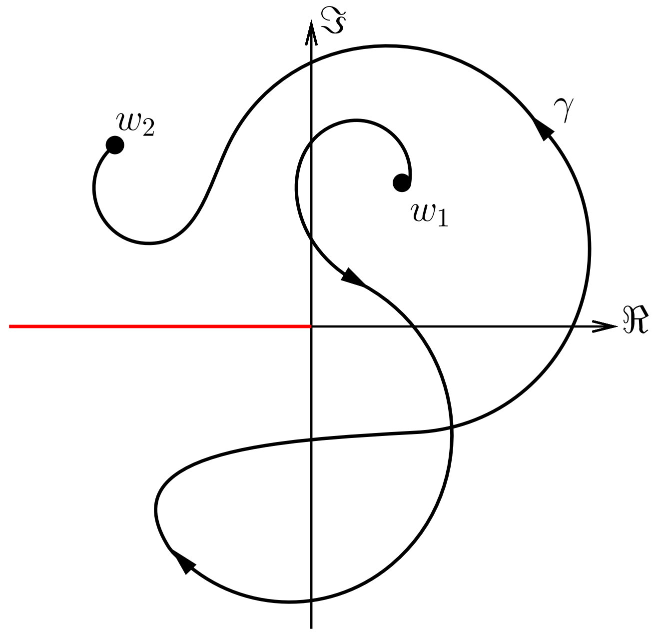
$$
\begin{aligned}
\int_1^{2+2\mathrm{i}} z\mathrm{d}z &= \int_1^{2+2\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^2}{2}\right)\mathrm{d}z \\
&= \frac{(2+2\mathrm{i})^2}{2} - \frac{1^2}{2} \\
&= \frac{4+8\mathrm{i}-4-1}{2} \\
&= -\frac{1}{2} + 4\mathrm{i}.
\end{aligned}
$$

Example

On $\mathcal{U} = \mathbb{C}\backslash(-\infty, 0]$, $\mathrm{Log}(z)$ has derivative $\frac{1}{z}$.

So if γ is a path from w_1 to w_2, which **does not** touch the negative real axis or 0, then

$$\int_\gamma \frac{\mathrm{d}z}{z} = \mathrm{Log}(w_2) - \mathrm{Log}(w_1).$$



However, if γ is the unit circle:

$$\begin{aligned} \gamma : [0, 2\pi] &\to \mathbb{C} \\ t &\mapsto \mathrm{e}^{\mathrm{i}t} \end{aligned}$$

then

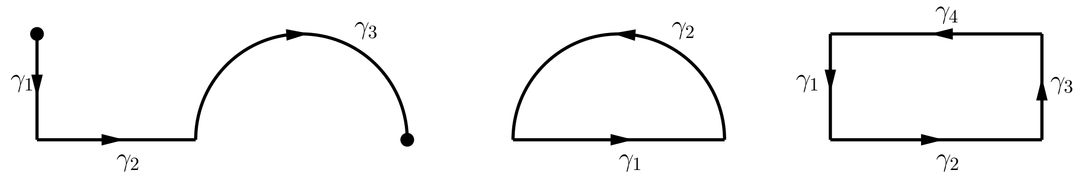
$$\begin{aligned} \int_\gamma \frac{\mathrm{d}z}{z} &= \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)} \mathrm{d}t \\ &= \int_0^{2\pi} \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}} \mathrm{d}t \\ &= \mathrm{i} \int_0^{2\pi} t \mathrm{d}t \\ &= 2\pi\mathrm{i} \neq 0. \end{aligned}$$

Corollary

$\frac{1}{z}$ is **not** the derivative of any holomorphic function defined on a region containing the unit circle.

2.7 Contours

A contour of $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a sequence of smooth paths "arranged end-to-end", i.e. the end-point of γ_j is the start point of γ_{j+1}. It is **closed** if the end-point of γ_n is the start point of γ_1 (The last two contours shown are closed.)



For a contour $\gamma = (\gamma_1, \ldots, \gamma_n)$ define

$$\int_\gamma f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \ldots \int_{\gamma_n} f(z)\mathrm{d}z.$$

The Fundamental Theorem of Calculus also works for contours. In particular, $\int_\gamma f'(z)\mathrm{d}z = 0$ if γ is a closed contour.

Estimations of contour integrals

If γ is a smooth path $\gamma : [a, b] \to \mathbb{C}$, the length of γ is

$$\ell(\gamma) = \int_a^b \left|\gamma'(t)\right| \mathrm{d}t$$

and the length of a contour $\gamma = (\gamma_1, \ldots, \gamma_n)$ is

$$\ell(\gamma) = \sum_{j=1}^{n} \ell\left(\gamma_j\right).$$

Example

If γ is a line segment from w_1 to w_2, that is,

$$\begin{aligned} \gamma : [0, 1] &\to \mathbb{C} \\ t &\mapsto w_1 + (w_2 - w_1)t \end{aligned}$$

and $\gamma'(t) = w_2 - w_1$. Then trivially, $\ell(\gamma) = \int_0^1 |w_2 - w_1| \,\mathrm{d}t = |w_2 - w_1|$.

Example

$$\begin{aligned} \gamma : [\alpha, \beta] &\to \mathbb{C} \\ t &\to w_0 + r\mathrm{e}^{\mathrm{i}t} \end{aligned}$$

with $w_0 \in \mathbb{C}$ and $r > 0$.

Then $\gamma'(t) = r\mathrm{i}\mathrm{e}^{\mathrm{i}t}$ and $|\gamma'(t)| = r$.

So

$$\begin{aligned} \ell(\gamma) &= \int_\alpha^\beta r\mathrm{d}t \\ &= r(\beta - \alpha). \end{aligned}$$

A whole circle centre 0, radius r has length $2\pi r$.

$\gamma : [0, 4\pi] \to \mathbb{C}$, $\gamma(t) = \mathrm{e}^{\mathrm{i}t}$ has length $\ell(\gamma) = 4\pi$ etc.

Lemma If f is complex-valued then

$$\left|\int_a^b f(t)\mathrm{d}t\right| \leq \int_a^b |f(t)|\,\mathrm{d}t.$$

Proof Let

$$\begin{aligned} I &= \int_a^b f(t)\mathrm{d}t \\ &\in \mathbb{C} \end{aligned}$$

We claim that there is $\alpha = \mathrm{e}^{\mathrm{i}a}$ with $a \in \mathbb{R}$, such that $\alpha I = |I|$.

When $I = 0$, any α works, otherwise $\dfrac{|I|}{I}$ has absolute value so it equals $\mathrm{e}^{\mathrm{i}a}$ with $a \in \mathbb{R}$. Then $|\alpha| = 1$.

$$\begin{aligned} |I| &= \alpha I \\ &= \int_a^\beta \alpha f(t)\mathrm{d}t \\ &= \int_a^\beta u(t)\mathrm{d}t + \mathrm{i}\int_a^\beta v(t)\mathrm{d}t \end{aligned}$$

where $u(t) = \Re\left[\alpha f(t)\right]$, $v(t) = \Im\left[\alpha f(t)\right]$.

Since $|I|$ is real,

$$\begin{aligned} |I| &= \int_a^b u(t)\mathrm{d}t \\ &= \int_a^b \Re\left[\alpha f(t)\right]\mathrm{d}t \\ &\leq \int_a^b |\alpha f(t)|\,\mathrm{d}t \\ &= \int_a^b |f(t)|\,\mathrm{d}t \end{aligned}$$

as $|\alpha| = 1$.

$\square$

2.7.1 Corollary (ML-Bound)

Consider a contour γ and a continuous function f defined on γ. Suppose $|f(z)| \leq M$, for all z on γ. Then

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq ML$$

where $L = \ell(\gamma)$.

Proof Assume γ is a smooth path $\gamma : [a, b] \to \mathbb{C}$. Then

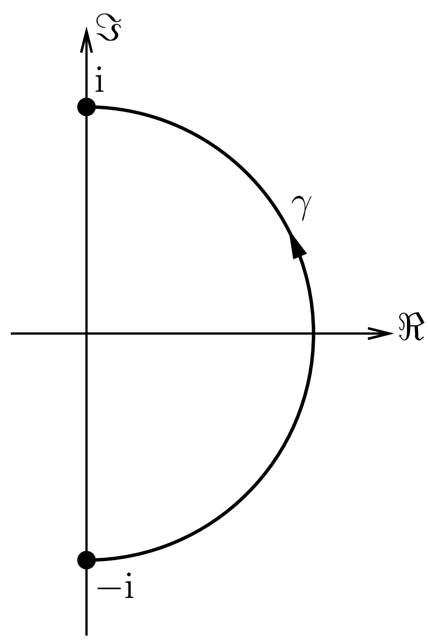
$$\begin{aligned}\left|\int_\gamma f(z)\mathrm{d}z\right| &= \left|\int_a^b f(\gamma(t))\,\gamma'(t)\mathrm{d}t\right| \\ &\leq \int_a^b \left|f(\gamma(t))\,\gamma'(t)\right| \mathrm{d}t \text{ (by the Lemma)} \\ &\leq M\int_a^b \left|\gamma'(t)\right| \mathrm{d}t \text{ (as } |f(\gamma(t))| \leq M) \\ &= ML.\end{aligned}$$

□

Example Showing that

$$\left|\int_\gamma \frac{\mathrm{e}^z}{z^2+2}\mathrm{d}z\right| \leq \mathrm{e}\pi$$

where γ is the arc of the unit circle to the right of the imaginary axis.



On γ,

$$\begin{aligned} |e^z| &= e^{\Re(z)} \\ &\leq e^{|z|} \\ &= e^1 = e \end{aligned}$$

and also

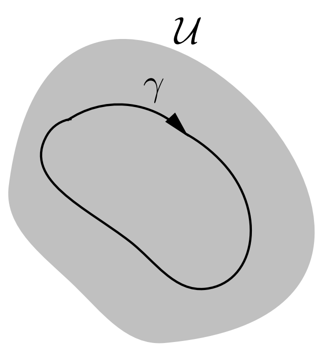
$$\begin{aligned} \left|z^2+2\right| &\geq 2 - \left|z^2\right| \text{ (by the \textit{alternate triangle inequality})} \\ &= 2 - 1 = 1 \\ \implies \left|\frac{1}{z^2+2}\right| &\leq \frac{1}{1} = 1 \\ \therefore \left|\frac{e^z}{z^2+2}\right| &\leq e \text{ on } \gamma. \end{aligned}$$

So by the *ML-bound*,

$$\left|\int_\gamma \frac{e^z}{z^2+2} dz\right| \leq e\pi.$$

2.8 Cauchy's Theorem

Let f be holomorphic on a domain $\mathcal{U}$ and let γ be a closed contour inside $\mathcal{U}$.



By the *Fundamental Theorem of Calculus*, if f has an "antiderivative" g, i.e. $f = g'$, then

$$\boxed{\int_\gamma f(z) dz = \int_\gamma g'(z) dz = 0}$$

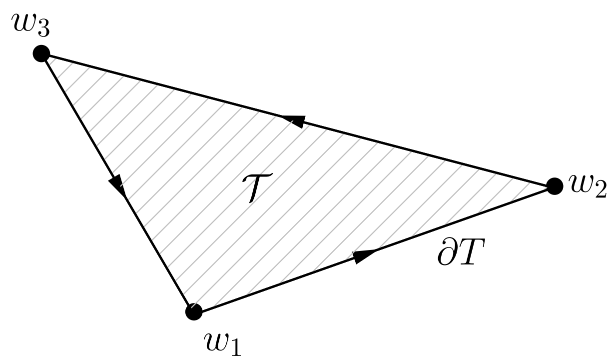
But $\int_\gamma \frac{dz}{z} = 2\pi \neq 0$ where γ is the unit circle.

The function $f(z)$ is holomorphic everywhere **except** at $z = 0$, and this lies **inside** the contour γ.

Vague Cauchy's Theorem If f is holomorphic on and at all points inside a closed contour γ, then $\int_\gamma f(z) dz = 0$.

Cauchy's Theorem for triangles **WARNING:** The proof of this theorem is the hardest in the course!

A triangle $\mathcal{T}$ consists of 3 points its vertices, the line segments joining these 3 points, the edges and the interior points. Points lying inside intervals with endpoints on two different edges of the triangle.



Use $\mathcal{T}$ to denote a typical triangle (set of vertices, edge points and interior points) and $\partial\mathcal{T}$ denotes the boundary of $\mathcal{T}$ (vertices and edge points but not the interior points). We also use $\partial\mathcal{T}$ to denote the contour formed by the edges. For definition's sake, assume that $\partial\mathcal{T}$ goes round anticlockwise.

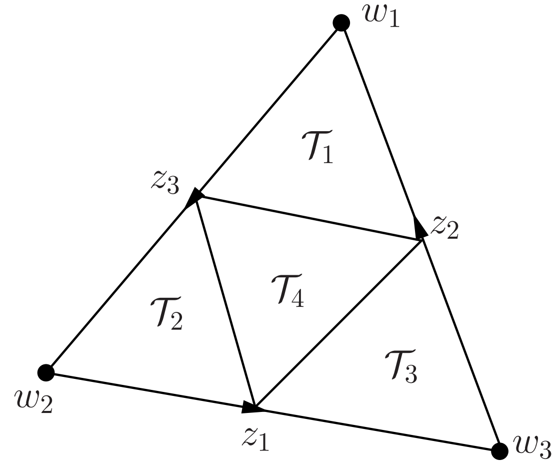
Theorem (Cauchy's Theorem for Triangles)
If f is holomorphic on a domain $\mathcal{U}$ and $\mathcal{T} \subseteq \mathcal{U}$ is a triangle, then $\int_{\partial\mathcal{T}} f(z)\mathrm{d}z = 0$.

Remarks

1. $\int_{\partial\mathcal{T}} f(z)\mathrm{d}z = \int_{w_1}^{w_2} f(z)\mathrm{d}z + \int_{w_2}^{w_3} f(z)\mathrm{d}z + \int_{w_3}^{w_1} f(z)\mathrm{d}z$ (integrals along straight line segments).

2. If we knew f' was continuous on $\mathcal{U}$, then Cauchy's theorem follows quickly from Green's Theorem. In fact f' will be continuous, but proving that relies on this theorem! (To see the Green's theorem proof see William Chem's notes).

Lemma
Let $\mathcal{T}$ be a triangle with vertices w_1, w_2, w_3.



Then

$$\int_{\partial\mathcal{T}} f(z)\mathrm{d}z = \sum_{j=1}^{4} \int_{\partial\mathcal{T}_j} f(z)\mathrm{d}z.$$

Here we take the boundaries in the anticlockwise direction.

$\int_{\partial\mathcal{T}_1} f(z)\mathrm{d}z + \ldots + \int_{\partial\mathcal{T}_4} f(z)\mathrm{d}z$ is the sum of 12 integrals over the sides of the small triangles, 6 of which cancel, e.g.

$\int_{z_3}^{z_2} f(z)\mathrm{d}z$ occurs in $\int_{\partial\mathcal{T}_1} f(z)\mathrm{d}z$
and $\int_{z_2}^{z_3} f(z)\mathrm{d}z$ occurs in $\int_{\partial\mathcal{T}_2} f(z)\mathrm{d}z$
and these cancel.
Also $\int_{w_1}^{z_3} f(z)\mathrm{d}z$ occurs in $\int_{\partial\mathcal{T}_1} f(z)\mathrm{d}z$
and $\int_{z_3}^{w_2} f(z)\mathrm{d}z$ occurs in $\int_{\partial\mathcal{T}_2} f(z)\mathrm{d}z$
and these add to give $\int_{w_1}^{w_2} f(z)\mathrm{d}z$which occurs in $\int_{\partial\mathcal{T}} f(z)\mathrm{d}z$.

The remaining 6 integrals combine in this way to give $\int_{\partial\mathcal{T}} f(z)\mathrm{d}z$.

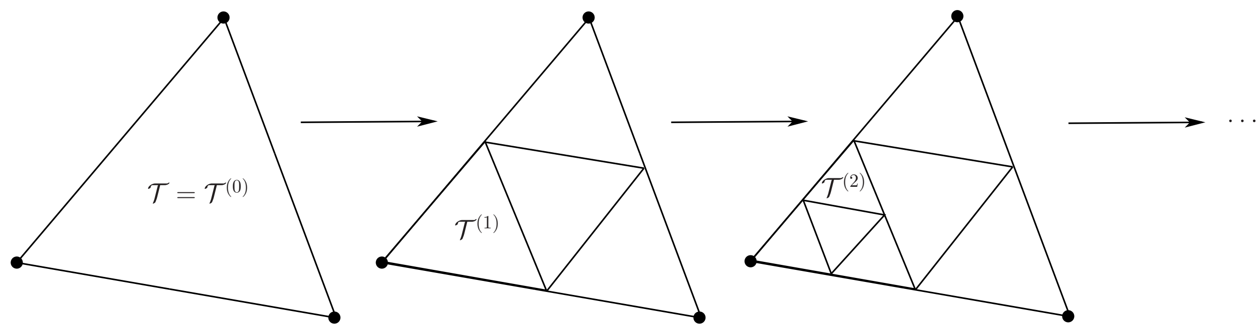
$\square$

Proof of Theorem
Write

$$\begin{aligned}\int_{\partial \mathcal{T}} f(z)\mathrm{d}z &= I(\mathcal{T}) \\ &= \sum_{j=1}^{4} I(\mathcal{T}_j).\end{aligned}$$

One of these $\mathcal{T}_j$ satisfies $|I(\mathcal{T}_j)| \geq \frac{1}{4}\,|I(\mathcal{T}_j)|$. Set $\mathcal{T}^{(0)} = \mathcal{T}$ and $\mathcal{T}^{(1)} = \mathcal{T}$. Note that $\mathcal{T}_j$ has side lengths half of those of $\mathcal{T}$. We get a sequence $\left(\mathcal{T}^{(m)}\right)$ of triangles by repeating the argument, with

$$\mathcal{T}^{(m+1)} \subseteq \mathcal{T}^{(m)}, \mathcal{T}^{(0)} \subseteq \mathcal{T}.$$



The side lengths of $\mathcal{T}^{(m)}$ are $\frac{1}{2^m}$ those of $\mathcal{T}$ and

$$\left|I\left(\mathcal{T}^{(m)}\right)\right| \geq \frac{1}{4^m}\left|I\left(\mathcal{T}\right)\right|.$$

We claim there is a point z_0 with $z_0 \in \mathcal{T}^{(m)}$ for all m.

(The website has a proof that a decreasing sequence of closed bounded non-empty subsets of $\mathbb{C}$ has a point in common)

Take $\varepsilon > 0$. The function f is differentiable at z_0, i.e.

$$\lim_{z \to z_0}\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| = 0.$$

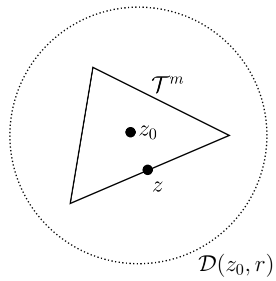
Therefore we can pick a radius such that

- f is holomorphic on $\mathcal{D}(a, r)$
- $\left|\dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon$ for $z \in \mathcal{D}(z_0, r)$.

On $\mathcal{D}(z_0, r)$,

$$\left|f(z) - f(z_0) - f'(z_0)(z - z_0)\right| \leq \varepsilon\,|z - z_0|\,.$$

Let $g(z) = f(z) - f(z_0) - f'(z_0)(z - z_0)$. For large enough m, $\mathcal{T}^{(m)} \subseteq \mathcal{D}(z_0, r)$.



So

$$\begin{aligned}
\left|\int_{\partial\mathcal{T}^{(m)}} g(z)\mathrm{d}z\right| &\leq \varepsilon \cdot \text{length}\left(\partial\mathcal{T}^{(m)}\right) \cdot \max|z - z_0| \\
&= \varepsilon 2^{-m} P d_m \\
&\text{(where } d_m = \text{greatest side length of } \mathcal{T}^{(m)} = \frac{1}{2^m} \times \text{greatest side length of } \mathcal{T} = \frac{P}{2m}\text{)} \\
&= \varepsilon 4^{-m} P^2
\end{aligned}$$

but $g(z) = f(z) + a + bz$ for some constants a and b.

$$\begin{aligned}
\int_{\partial\mathcal{T}^{(m)}} g(z)\mathrm{d}z &= \int_{\partial\mathcal{T}^{(m)}} f(z)\mathrm{d}z + \int_{\partial\mathcal{T}^{(m)}} (a + bz)\,\mathrm{d}z \\
&= I\left(T^{(m)}\right) + \underbrace{\int_{\partial\mathcal{T}^{(m)}} \frac{\mathrm{d}}{\mathrm{d}z}\left(az + \frac{1}{2}bz^2\right)\mathrm{d}z}_{=0} = I\left(\mathcal{T}^{(m)}\right)
\end{aligned}$$

so

$$\begin{gathered}
4^{-m}\left|I\left(\mathcal{T}^{(m)}\right)\right| \leq \varepsilon 4^{-m} P^2 \\
\implies |I(\mathcal{T})| \leq \varepsilon P^2 \\
\therefore I(\mathcal{T}) = 0.
\end{gathered}$$

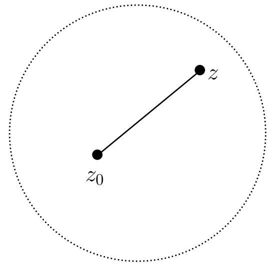
□

2.9 Star Domains

A **star domain** is a domain in $\mathbb{C}$ with a star centre. A point z_0 is a **star centre** of $\mathcal{U}$ if for all $z \in \mathcal{U}$ the line segment from z to z_0 lines inside $\mathcal{U}$.

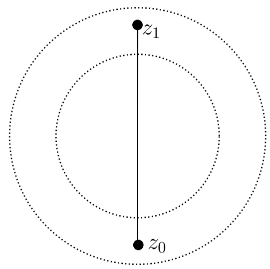
Examples

1. Discs are convex; every line segment with endpoints in the disc are in the disc.



2. An annulus is **not** a star domain.

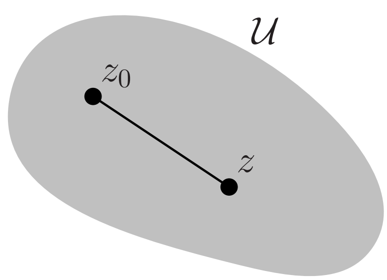
   If z_0 lies in the annulus, let z_1 be a reflection of z_0 about the centre, then the line segment from z_0 to z_1 is not contained in the annulus.



3. The slit plane $\mathcal{U} = \mathbb{C} \setminus \{x \in \mathbb{R} | \, x \leq 0\}$ is a star domain. 1 is a star centre.

Theorem
If f is holomorphic on a star domain $\mathcal{U}$ then $f = g'$ for some g holomorphic on $\mathcal{U}$.

Proof Let z_0 be a star centre of $\mathcal{U}$.



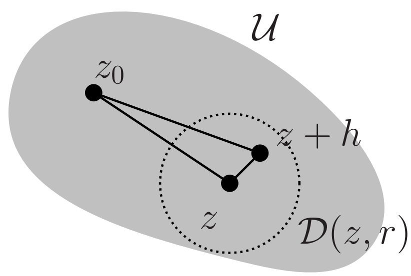
Define

$$g(z) = \int_{z_0}^{z} f(w)\mathrm{d}w.$$

We need to prove that

$$\lim_{h \to 0} \left(\frac{g(z+h) - g(z)}{h} - f(z) \right) = 0.$$

As $\mathcal{U}$ is a domain, $\mathcal{U}$ is open so there is $r > 0$ with $\mathcal{D}(z, r) \subseteq \mathcal{U}$. In considering this limit I can assume $|h| < r$. So $z + h \in \mathcal{D}(a, r)$.



Let $\mathcal{T}$ be the triangle with vertices $z_0, z, z+h$. I claim that $\mathcal{T} \subseteq \mathcal{U}$. The line segment from z to $z+h$ lies in $\mathcal{D}(z,r)$ and so also in $\mathcal{U}$.

For any point z' on this line segment, the line segment from z_0 to z' lies inside $\mathcal{U}$ as $\mathcal{U}$ is a star domain, and z_0 is a star centre. Any point inside $\mathcal{T}$ lies on a line segment from $z_0$to a point on the line segment between z and $z+h$, so $\mathcal{T} \subseteq \mathcal{U}$.

We can apply *Cauchy's theorem* to $\mathcal{T}$, so that $\int_{\partial\mathcal{T}} f(w)\mathrm{d}w = 0$, i.e.

$$\int_{z_0}^{z} f(w)\mathrm{d}w - \int_{z_0}^{z+h} f(w)\mathrm{d}w + \int_{z}^{z+h} f(w)\mathrm{d}w = 0$$

$$\begin{aligned}
\implies g(z+h) - g(z) &= \int_{z}^{z+h} f(w)\mathrm{d}w \\
\implies \frac{g(z+h)-g(z)}{h} - f(z) &= \frac{1}{h}\int_{z}^{z+h} \left(f(w) - f(z)\right)\mathrm{d}w \\
\implies \left|\frac{g(z+h)-g(z)}{h} - f(z)\right| &\leq \frac{1}{|h|}\left|\int_{z}^{z+h} \left(f(w) - f(z)\right)\mathrm{d}w\right|
\end{aligned}$$

Pick $\varepsilon > 0$. For some $s > 0$ ($s < r$ if we like) $\mathcal{D}(z,s)$ has the property that $w \in \mathcal{D}(z,s)$, then $|f(w) - f(z)| < \varepsilon$.

As f is continuous, $\lim\limits_{w\to z}\left(f(w) - f(z)\right) = 0$. If $|h| < s$, then $|f(w) - f(z)| < \varepsilon$ for all w on the line segment from z to $z+h$.

So I can take ε for "M" in the *ML-bound*, with the length of the segment being $|h|$.

So

$$\begin{aligned}
\int_{z}^{z+h} \left(f(w) - f(z)\right)\mathrm{d}w &\leq \varepsilon\,|h| \\
\implies \left|\frac{g(z+h)-g(z)}{h} - f(z)\right| &\leq \varepsilon
\end{aligned}$$

holds if $|h| < s$. So $\forall\varepsilon > 0 \exists s > 0$ such that the above line holds, and therefore from the definition of the limit,

$$\lim_{h\to 0}\left(\frac{g(z+h)-g(z)}{h} - f(z)\right) = 0.$$

Thus g is differentiable at z and $g'(z) = f(z)$.

□

Corollary (Cauchy's theorem for star domains) If $\mathcal{U}$ is a star domain with f holomorphic on $\mathcal{U}$, and γ is a closed contour on $\mathcal{U}$, then $\int_\gamma f(z)\mathrm{d}z = 0.$

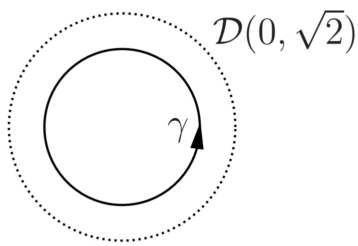
Proof There exists g with $f = g'$ such that $\int_\gamma f(z)\mathrm{d}z = \int_\gamma g'(z)\mathrm{d}z = 0$ by the *Fundamental Theorem of Calculus*.

□

Example

$$\int_\gamma \frac{\mathrm{e}^{z^3+z}\cos(z^2)}{(2+z^2)^2}\mathrm{d}z = 0$$

where γ is the unit circle.



The integrand (however horrible looking it is) is holomorphic on $\mathcal{D}(0,\sqrt{2})$, which contains the unit circle γ, and is a star domain. So the integral is zero.

Example

$$\mathcal{U} = \mathbb{C}\setminus\{x \in \mathbb{R} | x \leq 0\} \text{ (the "slit plane")}$$

is a star domain with star centre 1. Now $\frac{1}{z}$ is holomorphic on $\mathcal{U}$, so $\frac{1}{z} = L'(z)$ for some $L(z)$. We can take $L(z) = \int_1^z \frac{\mathrm{d}w}{w}$.

We claim that on $\mathcal{U}$, $L(z) = \mathrm{Log}(z)$.

To prove this, assume $|z| > 1$, $\Im(z) > 0$ (not essential - slightly different diagrams needed for other cases).

I claim $L(z) = \int_\gamma \frac{\mathrm{d}w}{w}$ where γ consists of the line segment from 1 to $|z|$ and the circular arc γ_2 from $|z|$ to $|z|$ (centred at 0).

γ_1 followed by γ_2 followed by the reverse of the line segment $[1,z]$ is a closed contour, so as $\mathcal{U}$ is a star domain, the integral over this contour is zero, i.e.

$$\int_{\gamma_1}\frac{\mathrm{d}w}{w} + \int_{\gamma_2}\frac{\mathrm{d}w}{w} - \int_1^z \frac{\mathrm{d}w}{w} = 0$$
$$\text{i.e. } L(z) = \int_{\gamma_1}\frac{\mathrm{d}w}{w} + \int_{\gamma_2}\frac{\mathrm{d}w}{w}.$$

$$\text{where } \int_{\gamma_1} \frac{\mathrm{d}w}{w} = \int_1^{|z|} \frac{\mathrm{d}t}{t}$$
$$= \log|z|.$$

γ_2 is parametrised by $\gamma(t) = |z|\,\mathrm{e}^{\mathrm{i}t}$, for $0 \le t \le \theta = \mathrm{Arg}(z)$, $\gamma'(t) = \mathrm{i}\,|z|\,\mathrm{e}^{\mathrm{i}t}$. Then

$$\begin{aligned}\int_{\gamma_2} \frac{\mathrm{d}w}{w} &= \int_0^{\mathrm{Arg}(z)} \frac{\mathrm{i}\,|z|\,\mathrm{e}^{\mathrm{i}t}}{|z|\,\mathrm{e}^{\mathrm{i}t}}\mathrm{d}t \\ &= \mathrm{i}\mathrm{Arg}(z)\end{aligned}$$

Therefore

$$\begin{aligned}L(z) &= \log|z| + \mathrm{i}\mathrm{Arg}(z) \\ &= \mathrm{Log}(z).\end{aligned}$$

This constitutes another proof that $\mathrm{Log}(z)$ is holomorphic on the slit plane, with $L'(z) = 1/z$.

$\square$

2.10 Cauchy's Integral Formula

Theorem (Cauchy's Integral Formula for a Circle) Let $\mathcal{U}$be a domain. Suppose $\overline{\mathcal{D}}(a,r) \subseteq \mathcal{U}$. Let γ be the circle centre a and radius r. If f is holomorphic on $\mathcal{U}$ then

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{w-z}\mathrm{d}w$$

for $z \in \mathcal{D}(a,r)$.
Note that this formula expresses the values of f in the interior of the disc in terms of its values on the boundary of the disc.

Proof (technical claim) There exists $r' > r$ such that $\mathcal{D}(a,r') \subseteq \mathcal{U}$. The proof of this initial remark is in a document on the website.

Assume that f is holomorphic on $\mathcal{D}(a,r')$. Let $z \in \mathcal{D}(a,r)$. Let γ_1 be a small circle, centre z radius $s > 0$ contained inside $\mathcal{D}(a,r)$.

We claim that

$$\int_\gamma \frac{f(w)}{w-z}\mathrm{d}w = \int_{\gamma_1} \frac{f(w)}{w-z}\mathrm{d}w.$$

Draw a line horizontally through z. Use this to define contours γ_2 and γ_3 as shown.

We claim that that

$$\int_{\gamma_2} \frac{f(w)}{w-z}\mathrm{d}w + \int_{\gamma_3} \frac{f(w)}{w-z}\mathrm{d}w = \int_{\gamma} \frac{f(w)}{w-z} - \int_{\gamma_1} \frac{f(w)}{w-z}.$$

In the first sum, the integrals over the horizontal edges cancel.

The integrals over over arcs of the big circle add to $\int_\gamma \frac{f(w)}{w-z}\mathrm{d}w$, and those of the small circle add to $-\int_{\gamma_1} \frac{f(w)}{w-z}\mathrm{d}w$.

Consider $\int_{\gamma_2} \frac{f(w)}{w-z}\mathrm{d}w$. The function $\frac{f(w)}{w-z}$ is not holomorphic on $w = z$, so what we want to do now is consider a contour over which $\frac{f(w)}{w-z}$ is holomorphic and dodges this singularity.

Now consider $\mathcal{U}'$ which is obtained by deleting z and all points vertically below it from $\mathcal{D}(a, r')$; $\mathcal{U}'$ is a star domain.

By *Cauchy's theorem*, $\int_{\gamma_2} \frac{f(w)}{w-z}\mathrm{d}w = 0$ and similarly,

$$\int_{\gamma_3} \frac{f(w)}{w-z}\mathrm{d}w = 0 \implies \int_{\gamma} \frac{f(w)}{w-z}\mathrm{d}w = \int_{\gamma_1} \frac{f(w)}{w-z}\mathrm{d}w.$$

Parametrise γ_1 by $\gamma_1(t) = z + \delta\mathrm{e}^{\mathrm{i}t}$ with $t \in [0, 2\pi]$, and $\gamma_1'(t) = \mathrm{i}\delta\mathrm{e}^{\mathrm{i}t}$.

$$\begin{aligned}\int_{\gamma_1} \frac{f(w)}{w-z}\mathrm{d}w &= \int_0^{2\pi} \frac{f(z+\delta\mathrm{e}^{\mathrm{i}t})}{\delta\mathrm{e}^{\mathrm{i}t}}\mathrm{i}\delta\mathrm{e}^{\mathrm{i}t}\mathrm{d}t \\ &= \mathrm{i}\int_0^{2\pi} f(z+\delta\mathrm{e}^{\mathrm{i}t})\mathrm{d}t \\ &= \int_{\gamma} \frac{f(w)}{w-z}\mathrm{d}w.\end{aligned}$$

Now let $\delta \to 0$. We claim that

$$\lim_{\delta\to 0}\int_0^{2\pi} f(z+\delta\mathrm{e}^{\mathrm{i}t})\mathrm{d}t = 2\pi f(z).$$

So we need to justify this claim. Note that

$$\int_0^{2\pi} f(z+\delta\mathrm{e}^{\mathrm{i}t})\mathrm{d}t - 2\pi f(z) = \int_0^{2\pi} \left[f(z+\delta\mathrm{e}^{\mathrm{i}t}) - f(z)\right]\mathrm{d}t$$

and as f is continuous at z, given $\varepsilon > 0$ there is some $\eta > 0$ such that $|f(u) - f(z)| < \varepsilon$ if $u \in \mathcal{D}(z, \eta)$.

So if $\delta < \eta$,

$$\begin{aligned}\left|\int_0^{2\pi} \left[f(z+\delta\mathrm{e}^{\mathrm{i}t}) - f(z)\right]\mathrm{d}t\right| &\le \int_0^{2\pi} \left|f(z+\delta\mathrm{e}^{\mathrm{i}t}) - f(z)\right|\mathrm{d}t \\ &\le 2\pi\varepsilon.\end{aligned}$$

So this shows that $\lim_{\delta\to 0}\int_0^{2\pi} f(z+\delta\mathrm{e}^{\mathrm{i}t})\mathrm{d}t = 2\pi f(z)$. Therefore

$$\begin{aligned}\int_\gamma \frac{f(w)}{w-z}\mathrm{d}w &= \lim_{\delta\to 0} \mathrm{i}\int_0^{2\pi} f(z+\delta\mathrm{e}^{\mathrm{i}t})\mathrm{d}t \\ &= 2\pi\mathrm{i}f(z)\end{aligned}$$

as required.

□

Example

$$\int_\gamma \frac{\mathrm{e}^{z^2}}{z+1}\mathrm{d}z$$

where γ is the circle centre 0, radius 2.

Start off by parametrising $\gamma(t) = 2\mathrm{e}^{\mathrm{i}t}$ with $t \in [0, 2\pi]$.

Also, note that $z+1 = z-(-1)$. By *Cauchy's integral formula*, as $f(z) = \mathrm{e}^{z^2}$ is holomorphic on $\mathbb{C}$, we have

$$\begin{aligned}\int_\gamma \frac{\mathrm{e}^{z^2}\mathrm{d}z}{z+1} &= 2\pi\mathrm{i}f(-1) \\ &= 2\pi\mathrm{i}\mathrm{e}.\end{aligned}$$

Example

$$\int_\gamma \frac{\mathrm{e}^{2z}}{z^2-1}\mathrm{d}z$$

where γ is the circle centre 1, radius 1.

Note that

$$\begin{aligned}\frac{\mathrm{e}^{2z}}{z^2-1} &= \frac{\mathrm{e}^{2z}}{(z+1)(z-1)} \\ &= \frac{f(z)}{z-1}\end{aligned}$$

where $f(z) = \frac{\mathrm{e}^{2z}}{z+1}$.

$f(z)$ is holomorphic on and inside γ, so the integral is

$$\begin{aligned}2\pi\mathrm{i}f(1) &= 2\pi\mathrm{i}\frac{\mathrm{e}^2}{2} \\ &= \pi\mathrm{i}\mathrm{e}^2.\end{aligned}$$

Cauchy's Integral Formula for other closed contours

The formula holds for f holomorphic on and inside a rectangular contour. The idea is the same, i.e. that integrating over the rectangle is the same as integrating over a small circle, and this reduces to proving that integrals along the two contours are zero etc.

We claim that the formula is true for all **simple closed contours**, if a contour is traversed **anticlockwise.** The idea is to reduce to an integral over a small circle, and break up the different into contours each contained in a star domain.

There is a well-defined notion of anticlockwise on γ. then if f is holomorphic on a domain containing γ and its interior, $\int_\gamma f(z)\mathrm{d}z = 0$ and for z in the interior of γ, $f(z) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f(w)}{w-z}\mathrm{d}w$.

Cauchy's Integral Formula for the Derivative

Theorem Let γ be a simple close contour. Let f be holomorphic on γ and its interior (we always assume γ is taken in an anticlockwise direction).

If z is in the interior of γ, then

$$f'(z) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f(w)}{(w-z)^2}\mathrm{d}w.$$

Proof The idea is to consider *Cauchy's integral formula* and differentiate it with respect to z. But we shall proceed carefully!

Define

$$F(z) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f(w)}{(w-z)^2}\mathrm{d}w.$$

To show that $F(z) = f'(z)$ we need to prove that

$$\lim_{h\to 0}\left(\frac{f(z+h)-f(z)}{h} - F(z)\right) = 0.$$

Let z be in the interior of γ. Pick $\delta > 0$ such that $\mathcal{D}(z,\delta)$ is contained in the interior of γ.

We claim that $|f(w)|$ is bounded on γ. If I have some parametrisation of γ, then $\gamma : [a,b] \to \mathbb{C}$ then $t \mapsto |f(\gamma(t))|$ is a continuous real-valued function on this interval, so it is bounded.

There is $M > 0$ with $|f(w)| \leq M$ for all w on γ. There is $M > 0$ with $|f(w)| \leq M$ for all w on γ.

Let $0 < |h| < \delta$. Then

$$
\begin{aligned}
&\frac{f(z+h)-f(z)}{h} - F(z) \\
&= \frac{1}{2\pi \mathrm{i}} \int_\gamma \left[\frac{1}{h} \left[\underbrace{\frac{1}{(w-z-h)}}_{f(z+h)} - \underbrace{\frac{1}{(w-z)}}_{f(z)} \right] - \frac{1}{(w-z)^2} \right] f(w) \mathrm{d}w \\
&= \frac{1}{2\pi \mathrm{i}} \int_\gamma \left[\frac{1}{(w-z-h)(w-z)} - \frac{1}{(w-z)^2} \right] f(w) \mathrm{d}w \\
&= \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{(w-z)-(w-z-h)}{(w-z)^2(w-z-h)} f(w) \mathrm{d}w \\
&= \frac{h}{2\pi \mathrm{i}} \int_\gamma \frac{f(w)\mathrm{d}w}{(w-z)^2(w-z-h)}
\end{aligned}
$$

$|f(w)| \leq M$, $|w-z| \geq \delta$ for w on γ and $\mathcal{D}(z,\delta)$ doesn't meet γ.

By the *"alternate" triangle inequality*, we have

$$
\begin{aligned}
|w-z-h| &\geq \delta - |h| \\
\therefore \left| \frac{f(z+h)-f(z)}{h} - F(z) \right| &\leq \frac{|h|}{2\pi} \frac{M}{\delta^2(\delta - |h|)} \cdot \mathrm{length}(\gamma) \\
&\to 0 \text{ as } h \to 0.
\end{aligned}
$$

So

$$
\frac{f(z+h)-f(z)}{h} - F(z) \to 0
$$

as $h \to 0$, whence $F(z) = f'(z)$.

□

Using this for z in the interior of γ we get

$$
\lim_{h \to 0} \frac{f'(z+h)-f'(z)}{h} = \frac{2}{2\pi \mathrm{i}} \int_\gamma \frac{f(w)}{(w-z)^3} \mathrm{d}w
$$

The proof is very similar to the above but the algebra is slightly fiddlier (if there was a LaTeXsymbol for "sad face", I'd be obliged to put one here).

Theorem Let $\mathcal{U}$ be a domain, with f holomorphic on $\mathcal{U}$. Then f' is also holomorphic on $\mathcal{U}$. Consequently, if a complex function is differentiable once, then it is infinitely differentiable! **Wow!**

Proof Let $z \in \mathcal{U}$, γ be a circle with centre z with with γ and its interior contained in $\mathcal{U}$ (this is okay because $\mathcal{U}$ is open).

Then

$$
\lim_{h \to 0} \frac{f'(z+h)-f'(z)}{h}
$$

exists (derived above), and then $f''(z)$ exists and so on!

□

2.10.1 Corollary (Cauchy's Formula for the nth Derivative)

If f is holomorphic, the nth derivative of f exists for all $n \in \mathbb{N}$.
We also get Cauchy's integral formula for the nth derivative, under the hypotheses of Cauchy's integral formula.

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} \mathrm{d}w$$

Outline proofs

1. Follow proof for $f'(z)$ making appropriate modifications.
2. Integrating $f^{(n)}(z) \underset{\text{CIF}}{=} \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f^{(n)}(w)}{w-z} \mathrm{d}w$ by parts n times.

Example

$$\int_\gamma \frac{\sin z}{z^4} \mathrm{d}z$$

where γ is the unit circle.

$$\int_\gamma \frac{\sin z}{z^4} \mathrm{d}z = \int_\gamma \frac{f(z)}{z^{3+1}} \mathrm{d}z \text{ where } f(z) = \sin z \text{ (holomorphic on } \mathbb{C})$$

By *Cauchy's integral formula*,

$$\begin{aligned} f'''(0) &= \frac{3!}{2\pi\mathrm{i}} \int_\gamma \frac{f(z)}{(z-0)^{3+1}} \mathrm{d}z \\ &= \frac{6}{2\pi\mathrm{i}} \int_\gamma \frac{f(z)}{z^4} \mathrm{d}z \end{aligned}$$

However, $f'''(0) = -\cos 0 = -1$, so

$$\begin{aligned} \frac{6}{2\pi\mathrm{i}} \int_\gamma \frac{\sin z}{z^4} \mathrm{d}z &= -1 \\ \iff \int_\gamma \frac{\sin z}{z^4} \mathrm{d}z &= -\frac{\pi\mathrm{i}}{3}. \end{aligned}$$

Example

$$\int_\gamma \frac{\mathrm{e}^z}{z^2(z+1)} \mathrm{d}z$$

where γ is the circle with centre 0 and radius 2.

The denominator of the integrand vanishes at $z = 0$ and $z = -1$, both of which are inside γ, so we can't apply Cauchy's integral formula directly.

- **Method 1:** "Break up" γ into two contours each encircling one of the values $z = 0$ or $z = -1$. So

$$\begin{aligned}\int_\gamma f(z)\mathrm{d}z &= \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z \\ \implies \int_\gamma \frac{\mathrm{e}^z}{z^2(z+1)}\mathrm{d}z &= \int_{\gamma_1} \frac{g(z)}{z^2}\mathrm{d}z + \int_{\gamma_2} \frac{h(z)}{z+1}\mathrm{d}z\end{aligned}$$

where $g(z) = \frac{\mathrm{e}^z}{z+1}$ and $h(z) = \frac{\mathrm{e}^z}{z^2}$, holomorphic on and inside γ_1 and γ_2 respectively.
By *Cauchy's integral formula*,

$$\begin{aligned}\int_\gamma \frac{\mathrm{e}^z}{z^2(z+1)}\mathrm{d}z &= 2\pi\mathrm{i}g'(0) + 2\pi\mathrm{i}h(-1) \\ &= 2\pi\mathrm{i}\left[\frac{\mathrm{e}^z}{z+1} - \frac{\mathrm{e}^z}{(z+1)^2}\right]\Bigg|_{z=0} + 2\pi\mathrm{i}\left[\frac{\mathrm{e}^z}{z^2}\right]\Bigg|_{z=-1} \\ &= 0 + 2\pi\mathrm{i}\mathrm{e}^{-1} \\ &= \frac{2\pi\mathrm{i}}{\mathrm{e}}.\end{aligned}$$

- **Method 2:** expand $\frac{1}{z^2(z+1)}$ into partial fractions.

$$\begin{aligned}\frac{1}{z^2(z+1)} &= \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z+1} \\ \text{so } \int_\gamma \frac{\mathrm{e}^z}{z^2(z+1)}\mathrm{d}z &= A\int_\gamma \frac{\mathrm{e}^z}{z}\mathrm{d}z + B\int_\gamma \frac{\mathrm{e}^z}{z^2}\mathrm{d}z + C\int_\gamma \frac{\mathrm{e}^z}{z+1}\mathrm{d}z\end{aligned}$$

- **Method 3:** *Calculus of residues* (later).

Cauchy Estimates

Theorem Let f be holomorphic on a domain containing the closed disc $\mathcal{D}(a, r)$. Let M be an upper bound for $|f(z)|$ for z on the circle centre a, radius r. That is, $|f(z)| \leq M$ for all z with $|z - a| = r$.
Then

$$f^{(n)}(a) \leq \frac{n!M}{r^n}.$$

Proof Let γ be the circle centre a and radius r. Then by *Cauchy's integral formula*,

$$f^{(n)}(a) = \frac{n!}{2\pi\mathrm{i}}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\mathrm{d}w.$$

On γ, $|f(w)| \leq M$ by hypothesis, where $|w - a| = r$.

The length of γ is $2\pi r$. So by the *ML-bound*,

$$\begin{aligned}\left|f^{(n)}(a)\right| &\leq \frac{n!}{2\pi}\frac{M}{r^{n+1}} \cdot 2\pi r \\ &= \frac{n!M}{r^n}.\end{aligned}$$

□

2.11 Liouville's Theorem

Corollary (Liouville's Theorem) Let f be holomorphic on $\mathbb{C}$ (such a function is **entire**).

Then if f is bounded on $\mathbb{C}$, i.e. there exists M with $|f(z)| \leq M \; \forall z \in \mathbb{C}$, then f is constant.

Proof We can apply *Cauchy's estimates* with this bound M, and by any a and any r.

$$\left|f'(a)\right| \leq \frac{M}{r} \; (\forall a \in \mathbb{C}\text{and } r > 0).$$

We can let $r \to \infty$, and we deduce that $f'(a) = 0 \; \forall a \in \mathbb{C}$, i.e. $f' = 0$.

If $z \in \mathbb{C}$,

$$\begin{aligned} f(z) - f(0) &= \int_0^z f'(w)\mathrm{d}w \\ &= 0 \\ \implies f(z) &= f(0) \;\; \forall z \in \mathbb{C}\end{aligned}$$

so f is constant.

□

Corollary (Fundamental Theorem of Algebra) Let

$$P(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1}z + a_n$$

be a polynomials of degree $n \geq 1$. Then $P(z) = 0$ has at least 1 solution in $\mathbb{C}$. **Wow!**

Proof Let $f(z) = \dfrac{1}{P(z)}$. If $P(z)$ has no root, then $f(z)$ is an entire function. We claim that if this is the case then $f(z)$ is bounded.

$$f(z) = z^{-n}\frac{1}{1 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n}}$$

with $z \neq 0$. We see that $|f(z)| \to 0$ as $|z| \to 0$.

There is $R > 0$ such that $|f(z)| < 1$ if $|z| > R$. But $|f(z)|$ is continuous on $\overline{\mathcal{D}}(0, R)$ so $|f(z)|$ is bounded on $\overline{\mathcal{D}}(0, R)$ (a continuous real-valued function on a closed bounded set is bounded there).

So f is bounded by *Liouville's theorem*. Thus f is constant and so $P = 1/f$ is constant.

But then $P(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1}z + a_n$, so $|P(z)| \to \infty$ so P isn't constant! This is our **contradiction**.

Therefore we must have that $P(z)$ has a zero somewhere.

Corollary If $P(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1}z + a_n$ then $P(z) = (z - z_1)(z - z_2)\ldots(z - z_n)$ for some $z_1, \ldots, z_n \in \mathbb{C}$.

Proof By the *Fundamental Theorem of Algebra*,

$$\begin{aligned} & P(z_1) = 0 \text{ for some } z_1 \\ \implies & P(z) = (z - z_1)Q(z) \end{aligned}$$

where $Q(z)$ is a polynomial of degree $n - 1$.

Keep extracting factors to get the desired result.

$\square$

2.11.1 Corollary (Generalised Liouville Theorem)

Let f be entire. Suppose there are n, C, R such that $|f(z)| \leq C\,|z|^n$ for sufficiently large z with $z > R$. Then f is a polynomial of degree at most n.

Proof Let $a \in \mathbb{C}$. Then for each $r > 0$,

$$f^{(n+1)}(a) \leq \frac{(n+1)!}{r^{n+1}} M(r)$$

where $M(r)$ is some upper bound for $|f(z)|$ when $|z - a| < r$ (by *Cauchy estimates*).

If we take r large enough, then

$$\begin{aligned} |f(z)| &\leq C\,|z|^n \\ &\leq C\,(|z-a| + |a|)^n \text{ (by the \textit{triangle inequality})} \\ &= C\,(r + |a|)^n \end{aligned}$$

when $|z - a| = r$.

Take $M(r) = C\,(r + |a|)^n$. Then

$$\begin{aligned} \left|f^{(n+1)}(z)\right| &\leq \frac{(n+1)!}{r^{n+1}}\,(r + |a|)^n \\ &\to 0 \text{ as } r \to \infty. \end{aligned}$$

$$
\begin{aligned}
&\therefore \left|f^{(n+1)}(z)\right| = 0 \\
&\implies f^{(n+1)}(z) = 0 \\
&\implies f(z) = A_0 + A_1 z + \ldots + A_n z^n \text{ (integrating } n+1 \text{ times).}
\end{aligned}
$$

□

Example Suppose $|f(z)| \leq |z| + 1$ for all $z \in \mathbb{C}$ where f is holomorphic. Prove that $f(z) = Az + B$ where $|A| \leq 1$ and $|B| \leq 1$.

Proof If $|z| \geq 1$ then

$$
\begin{aligned}
|f(z)| &\leq |z| + 1 \\
&\leq 2|z|
\end{aligned}
$$

so by the *Generalised Liouville theorem*, $f(z) = Az + B$ where $A, B \in \mathbb{C}$.

Now

$$
\begin{aligned}
|B| &= |f(0)| \\
&\leq |0| + 1 = 1 \\
&\text{by the condition } |f(z)| \leq |z| + 1.
\end{aligned}
$$

By *Cauchy's estimates*,

$$
|A| = \left|f'(0)\right| \leq \frac{1}{r} M(r)
$$

with $z = r > 0$.

On $|z| = r$,

$$
\begin{aligned}
|f(z)| &\leq |z| + 1 \\
&= r + 1.
\end{aligned}
$$

Take $M(r) = r + 1$.Then $|A| \leq \frac{r+1}{r} \to 1$ as $r \to \infty$, so $|A| \leq 1$.

□

2.12 Power Series

2.12.1 Theorem (Taylor's Theorem)

Let f be holomorphic on a domain $\mathcal{U}$, and suppose $\mathcal{D}(a, r) \subseteq \mathcal{U}$. Then

$$
f(z) = \sum_{n=0}^{\infty} f^{(n)}(a) \cdot \frac{(z-a)^n}{n!}
$$

if $z \in \mathcal{D}(a, r)$ (the series converges for such z).

Proof By considering $f(z+a)$ in place of $f(z)$, we can assume $a=0$. So $\mathcal{D}(0,r)\subseteq\mathcal{U}$.

Let $0<R<r$. Then γ, the circle centre 0 radius R is contained in $\mathcal{D}(0,r)$.

If $|z|<R$, $f(z)=\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{f(w)\mathrm{d}w}{w-z}$ by *Cauchy's integral formula.*

Now

$$\begin{aligned}\frac{1}{w-z}&=\frac{1}{w}\frac{1}{1-\frac{z}{w}}\\&=\frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n\\ \therefore\sum_{n=0}^{N}\frac{1}{w}\left(\frac{z}{w}\right)^n&=\frac{1}{w}\frac{1-\left(\frac{z}{w}\right)^{N+1}}{1-\left(\frac{z}{w}\right)}\text{ (geometric series formula)}\\&=\frac{1}{w-z}-\frac{\left(\frac{z}{w}\right)^{N+1}}{w-z}\\ \iff\frac{1}{w-z}&=\frac{\left(\frac{z}{w}\right)^{N+1}}{w-z}+\sum_{n=0}^{N}\frac{1}{w}\left(\frac{z}{w}\right)^n\end{aligned}$$

Therefore

$$\begin{aligned}f(z)&=\frac{1}{2\pi\mathrm{i}}\int_\gamma\left(\frac{\left(\frac{z}{w}\right)^{N+1}}{w-z}+\sum_{n=0}^{N}\frac{1}{w}\left(\frac{z}{w}\right)^n\right)f(w)\mathrm{d}w\\&=\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{\left(\frac{z}{w}\right)^{N+1}}{w-z}f(w)\mathrm{d}w+\sum_{n=0}^{N}\int_\gamma\frac{z^nf(w)}{w^{n+1}}\mathrm{d}w\\&=I_n+\sum_{n=0}^{N}a_nz^n\,(*)\end{aligned}$$

where $a_n=\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{f(w)\mathrm{d}w}{w^{n+1}}=\frac{f^{(n)}(0)}{n!}$ by *Cauchy's integral formula for the nth derivative***.**

Now we apply the *ML-bound* to I_N.

$$|I_N|\leq\frac{1}{2\pi}\frac{(|z|/R)^{N+1}M}{R-|z|}\cdot 2\pi R$$

$|f(w)|$ is bounded by M, say on γ, so that $|w|=R$ on γ.

Since $|z|<R$, $\left(\frac{|z|}{R}\right)^{N+1}\to 0$ as $N\to\infty$, $I_N\to 0$ as $N\to\infty$.

By putting this limit into $(*)$, we have

$$\begin{aligned}f(z)&=\sum_{n=0}^{\infty}a_nz^n\\&=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n\text{ if }z\in\mathcal{D}(0,R).\end{aligned}$$

If $z \in \mathcal{D}(0,r)$, $|z| < r$ so I can choose R with $|z| < R < r$.

Using this particular r, we can get

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

□

Taylor's theorem says that holomorphic functions are "locally" power series.

Remark Power series expansions may not hold for all points of the domain, but certainly in any disc surrounding a given point. For example,

$$f(z) = \frac{1}{z}$$

and $a = 1$. f is holomorphic on $\mathbb{C}\backslash\{0\}$.

f is also holomorphic on $\mathcal{D}(1,1)$. In this disc,

$$\begin{aligned} f(z) &= \frac{1}{z-1+1} \\ &= \frac{1}{1-(-(z-1))} \\ &= \sum_{n=0}^{\infty}(-1)^n(z-1)^n. \end{aligned}$$

However, outside of this disc, the series is divergent. f isn't given by this series outside of the disc.

Again, as in real analysis, there are 3 possibilities.

1. $f(z) = \sum_{n=0}^{\infty} b_n(z-a)^n$ converges $\forall z \in \mathbb{C}$.
2. $f(z) = \sum_{n=0}^{\infty} b_n(z-a)^n$ converges only for $z = 0$.
3. There exists $R \in (0,\infty)$ (the **radius of convergence)** such that

$$f(z) \begin{cases} \text{converges absolutely} & |z-a| < R \\ \text{diverges} & |z-a| > R \end{cases}.$$

Proof: Same as in real analysis.

The radius of convergence makes more sense over $\mathbb{C}$ than $\mathbb{R}$, since the power series converges on a disc of radius R.

Theorem If the power series $\sum_{n=0}^{\infty} b_n(z-a)^n$ is convergent on $\mathcal{D}(a,R)$, then it defines a holomorphic function f there and moreover

$$f'(z) = \sum_{n=1}^{\infty} nb_n(z-a)^{n-1} \text{ on } \mathcal{D}(a,R).$$

Proof Assume $a = 0$. otherwise replace $z - a$ with z.

Let $f_1(z) = \sum_{n=1}^{\infty} nb_n z^{n-1}$. Robin claims that f_1 converges on $\mathcal{D}(0, R)$.

Let $z \in \mathcal{D}(0, R)$. One can choose r with $|z| < r < R$. Then $\sum_{n=0}^{\infty} b_n r^n$ converges, so $b_n r^n \to 0$, so $(b_n r^n)$ is a bounded sequence. There is $M \in \mathbb{R}$ with $|b_n r^n| \leq M$ for all n.

$$\left|nb_n z^{n-1}\right| = \frac{|z|^{n-1}}{r^{n-1}}$$

so

$$\begin{aligned}\left|nb_n r^{n-1}\right| &= \left(\frac{|z|}{r}\right)^{n-1} \frac{n}{r} \left|nb_n r^n\right| \\ &\leq \frac{M}{r} n \left(\frac{|z|}{r}\right)^{n-1}\end{aligned}$$

and so

$$\sum_{n=1}^{\infty} \left|nb_n r^{n-1}\right| \leq \frac{M}{r} \sum_{n=1}^{\infty} \left(\frac{|z|}{r}\right)^{n-1}$$

which is convergent by the *ratio test*.

Therefore $f_1(z) = \sum_{n=1}^{\infty} nb_n z^{n-1}$ is absolutely convergent for $z \in \mathcal{D}(a, R)$.

We repeat the argument for

$$f_2(z) = \sum_{n=2}^{\infty} n(n-1) b_n z^{n-2}$$

which also converges on $\mathcal{D}(a, R)$.

To show that f is differentiable and $f' = f_1$, consider

$$\frac{f(z+h) - f(z)}{h} - f_1(z).$$

We need to prove that this tends to zero as $h \to 0$.

$$\begin{aligned}
\frac{f(z+h)-f(z)}{h} - f_1(z) &= \sum_{n=1}^{\infty} b_n \left[\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right] \text{ (using the fact that } f(z) = \sum_{n=0}^{\infty} b_n z^n\text{)} \\
\text{Now } \frac{(z+h)^n - z^n}{h} &= \sum_{k=1}^{n} \binom{n}{k} z^{n-k} h^{k-1} \text{ (by the \textit{binomial theorem})} \\
\therefore \frac{(z+h)^n - z^n}{h} - nz^{n-1} &= h \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^{k-2} \\
\therefore \left|\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right| &= \left|h \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^{k-2}\right| \\
&\leq |h| \sum_{k=2}^{n} \binom{n}{k} |z|^{n-k} |h|^{k-2} \text{ (by the \textit{triangle inequality})} \\
&= |h| \sum_{k=2}^{n} \frac{n!}{k!(n-k)!} |z|^{n-k} |h|^{k-2} \\
&= |h| \sum_{k=2}^{n} \frac{n(n-1)(n-2)!}{k(k-1)} \frac{|z|^{n-k}}{(n-k)!} \frac{|h|^{k-2}}{(k-2)!} \\
&= |h|\, n(n-1) \sum_{k=2}^{n} \frac{1}{k(k-1)} \frac{(n-2)!}{(k-2)!(n-k)!} |z|^{n-k} |h|^{k-2} \\
&\leq |h|\, n(n-1) \sum_{k=0}^{n} \binom{n-2}{k-2} |z|^{n-k} |h|^{k-2} \text{ (as } k \geq 2, k(k-1) \geq 2 \cdot 1 = 2\text{)} \\
&= \frac{n(n-1)}{2} (|z| + |h|)^{n-2}
\end{aligned}$$

Therefore

$$\left|\frac{f(z+h)-f(z)}{h} - f_1(z)\right| \leq \frac{|h|}{2} \sum_{n=2}^{\infty} n(n-1)\,|b_n|\,(|z|+|h|)^{n-2}.$$

If $|h| < R - |z|$ then $|z| + |h| < R$ so $\sum_{n=2}^{\infty} n(n-1)\,|b_n|\,(|z|+|h|)^{n-2}$ is absolutely convergent.

Let $|z| < r < R$. If $|h| < r$ then $\left|\frac{f(z+h)-f(z)}{h} - f_1(z)\right| \leq \frac{|h|M}{2}$.

Indeed, $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f_1(z)$, so f is differentiable with

$$f'(z) = \sum_{n=1}^{\infty} n b_n z^{n-1}.$$

□

Inside their disc of convergence, power series define functions that are holomorphic.

Remark: Just as with power series over $\mathbb{R}$, the radius of convergence can be computed by *the ratio test*, that is,

$$\text{if } \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right| = R$$

then $\sum b_n(z-a)^n$ has radius of convergence equal to R.

2.13 Zeros of holomorphic functions

If f is holomorphic, then **a zero of** f is some $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.

How does f behave near a zero?

Let f be holomorphic on $\mathcal{U}$, and $f(a) = 0$ ($a \in \mathcal{U}$).

By *Taylor's theorem*, $f(z) = \sum_{n=1}^{\infty} b_n(z-a)^n$ (starting at $n = 1$ as $f(0) = 0$). This is valid in some disc centre a.

- **Case 1 (trivial):** All the $b_n = 0 \implies f(z) = 0$ on a disc $\mathcal{D}(a,r)$, $r > 0$.
- **Case 2:** Some $b_n \neq 0$. Let N be the smallest integer with $b_N \neq 0$. Then $f(z) = \sum_{n=N}^{\infty} b_n(z-a)^n$ on some $\mathcal{D}(a,r)$.
  In this case we say f has a **zero of order** N (this integer is unique) at $z = a$.
  - A zero of order 1 is **simple**
  - A zero of order 2 is **double**
  - A zero of order 3 is **triple**, and so on.

The order of a zero of f is m at $z = a$ if and only if

$$f(a) = \ldots = f^{(m-1)}(a) = 0 \neq f^{(m)}(a).$$

Remarked that we can have $f(z) = 0$ in some $\mathcal{D}(a,r)$ if not then $f(a) \neq 0$ or f has a zero of order m at a.

Then as $\frac{f(z)}{(z-a)^m} \to b_m \neq 0$ as $z \to a$, $f(z) \neq 0$ in some punctured open disc $\mathcal{D}'(a,r) = \{\, z \in \mathbb{C} |\, 0 < |z-a| < r \,\}$.

Zeros of order m are "isolated", surrounded by a disc where $f(z) \neq 0$ (except from at a).

Theorem If f is holomorphic on a domain $\mathcal{U}$, and f is identically zero on some disc $\mathcal{D}(a,r)$ with $\mathcal{D}(a,r) \subseteq \mathcal{U}$, then f is identically zero on $\mathcal{U}$.

Proof (vague outline) We need the fact that $\mathcal{U}$ is connected. Let $\mathcal{V} = \{\, a \in \mathcal{U} |\, f \text{ is identically zero on some } \mathcal{D}(a,r) \,\}$. Then one proves that

1. $\mathcal{V}$ is open
2. $\mathcal{U} \backslash \mathcal{V}$ is open
3. In a connected open set $\mathcal{U}$, the only $\mathcal{V}$ satisfying the first two rules are $\mathcal{V} = \emptyset$ or $\mathcal{V} = \mathcal{U}$.

We omit proofs of these three properties, and say that $\mathcal{V} = \emptyset$ by the hypothesis so $\mathcal{V} = \mathcal{U} \implies f$ is identically zero on $\mathcal{U}$.

2.13.1 Corollary (Identity Theorem)

Let $\mathcal{U}$ be a domain. Let (z_n) be a sequence of points in $\mathcal{U}$ converging to $w \in \mathcal{U}$ with $z_n \neq w \forall n$. Then if f and g are holomorphic on $\mathcal{U}$ and $f(z_n) = g(z_n)$ for all n, then $f(z) = g(z)$ for all $z \in \mathcal{U}$.

Proof Let $h(z) = f(z) - g(z)$. $h(z_n) = 0$ as h is holomorphic and continuous. So also $z_n \to w \implies h(z_n) \to h(w) \implies h(w) = 0.$

By the Theorem above, either h is identically zero on $\mathcal{U}$, i.e. $f = g$ on all of $\mathcal{U}$, or h has a zero of order m at w.

Then w is an isolated zero of h, so that $h(w) \neq 0$ on some $\mathcal{D}(w, r)$.

Let $z_n \to w$ but $z_n \neq w$ for large enough n. Then $z_n \in \mathcal{D}'(w, r)$, the punctured disc. But this means that $h(z_n) \neq 0$ which is a **contradiction**.

$\square$

Examples

1. $f(z) = z^n$ has a zero of order n at $z = 0$.

2. $f(z) = \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$ has a simple zero at $z = 0$ and at $z = \pi$ (and other integer multiples of π);

$$\begin{aligned} f(z+\pi) &= \sin(z+\pi) \\ &= -\sin z \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ f(z) &= -\sum_{n=0}^{\infty} \frac{(-1)^n (z-\pi)^{2n+1}}{(2n+1)!} \\ &= -(z-\pi) + \frac{(z-\pi)^3}{6} - \ldots \end{aligned}$$

so $\sin z$does have a simple zero at $z = \pi$.

3. $f(z) = 1 - \cos z = 1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right) = \frac{z^2}{2!} - \ldots$ so $1 - \cos z$ has a double zero at $z = 0$.

2.14 Laurent Series

2.14.1 Theorem (Laurent's Theorem)

Let f be holomorphic on the annulus $A = \{z \in \mathbb{C} | \, r < |z-a| < R\}$ where $0 \leq r < R \, (\leq \infty)$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z-a)^n$$

where $b_n \in \mathbb{C}$ for each $n \in \mathbb{Z}$.
This series is called a **Laurent series** and equals

$$\ldots + \frac{b_{-2}}{(z-a)^2} + \frac{b_{-1}}{(z-a)} + b_0 + b_1(z-a) + b_2(z-a)^2 + \ldots$$

Proof Let $z \in A$. Pick r_1, r_2 with $r < r_1 < |z-a| < r_2 < R$. Assume without loss of generality that $a = 0$.

Create contours γ_1 and γ_2, defined to be circles (both centre 0) with radii r_1 and r_2 respectively.

I claim that

$$f(z) = \frac{1}{2\pi \mathrm{i}} \left[\int_{\gamma_1} \frac{f(w)}{w-z} \mathrm{d}w - \int_{\gamma_2} \frac{f(w)}{w-z} \mathrm{d}w \right].$$

So

$$\mathcal{I}(\gamma_2) - \mathcal{I}(\gamma_1) = \mathcal{I}(\gamma_3) + \mathcal{I}(\gamma_4)$$

(where in this case, $\mathcal{I}(\gamma)$ is shorthand for $\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(w)}{w-z} \mathrm{d}z$).

Applying *Cauchy's integral formula* to $\mathcal{I}(\gamma_3)$, we get

$$\mathcal{I}(\gamma_3) = \int_{\gamma_3} \frac{f(w)}{w-z} \mathrm{d}w = 2\pi \mathrm{i} f(z).$$

Also, by *Cauchy's theorem*, $I(\gamma_4) = 0$.

This means that

$$f(z) = \mathcal{I}(\gamma_2) - \mathcal{I}(\gamma_1).$$

Now

$$\mathcal{I}(\gamma_2) = \sum_{n=0}^{\infty} b_n z^n$$

where $b_n = \dfrac{1}{2\pi \mathrm{i}} \int_{\gamma_2} \frac{f(w)}{w^{n+1}} \mathrm{d}w$ (from *Taylor's theorem*).

$$
\begin{aligned}
\frac{1}{2\pi\mathrm{i}}\int_{\gamma_1}\frac{f(w)}{w-z}\mathrm{d}w &= \frac{1}{2\pi\mathrm{i}}\left(-\frac{1}{z}\right)\int_{\gamma_1}\frac{f(w)}{1-\frac{w}{z}}\mathrm{d}w \\
&= \frac{1}{2\pi\mathrm{i}}\left(-\frac{1}{z}\right)\int_{\gamma_1} f(w)\sum_{m=0}^{\infty}\frac{w^m}{z^m}\mathrm{d}w \\
&= \left(-\frac{1}{z}\right)\frac{1}{z^m}\sum_{m=0}^{\infty}\underbrace{\left(\frac{1}{2\pi\mathrm{i}}\int_{\gamma_1} w^m f(w)\mathrm{d}w\right)}_{b_{m-1}} \\
&= -\frac{1}{2\pi\mathrm{i}}\sum_{m=0}^{\infty}\frac{b_{-m-1}}{z^{-m-1}}.
\end{aligned}
$$

Technical point: The formulae for b_n involve r_1 or r_2.

But a similar argument to Robin's "circle chopping" contour argument shows that b_n is independent of the radii r_1, r_2.

So the formula

$$f(z) = \sum_{n=-\infty}^{\infty} b_n z^n$$

is true on the annulus.

$\square$

The Laurent series can be split up as follows.

$$\sum_{n=-\infty}^{\infty} b_n(z-a)^n = \sum_{n=0}^{\infty} b_n(z-a)^n + \sum_{n=-\infty}^{-1} b_n(z-a)^n$$

The first term is a power series in $(z-a)$ which converges absolutely for $|z-a| < R$.

The second term is a power series in $(z-a)^{-1}$ which converges absolutely for $\left|\frac{1}{z-a}\right| < R'$.

If $\frac{1}{R'} < R$ then the Laurent series converges absolutely for $\frac{1}{R'} < |z-a| < R$.

Example

$$
\begin{aligned}
f(z) &= \frac{1}{(z-1)(z-2)} \\
&= \frac{1}{z-2} - \frac{1}{z-1}
\end{aligned}
$$

is holomorphic everywhere except for $z = 1, 2$. In particular, f is holomorphic on $A = \{\, z \in \mathbb{C} | \, 1 < |z| < 2 \,\}$.

$$\begin{aligned}
\frac{1}{z-2} &= \frac{1}{2}\left(\frac{1}{\frac{z}{2}-1}\right) \\
&= -\frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right) \\
&= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \text{ (as } \left|\frac{z}{2}\right| < 1) \\
&= -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{z-1} &= \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) \\
&= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} \text{ (as } \left|\frac{1}{z}\right| < 1) \\
&= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} \\
&= \sum_{m=-\infty}^{-1} z^m \ (m = -(n+1))
\end{aligned}$$

Therefore

$$\frac{1}{z-2} - \frac{1}{z-1} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{m=-\infty}^{-1} z^m$$

This means that $\dfrac{1}{(z-2)(z-1)}$ has Laurent series on the annuli

$\{\, z \in \mathbb{C} |\, 0 < |z| < 1\}$ (Taylor series; no terms $z^n$with $n < 0$)
$\{\, z \in \mathbb{C} |\, 2 < |z|\}$ (Laurent series only with terms $z^n$with $n < 0$)

which are different to the ones we calculated!

2.15 Singularities

Definition Let $\mathcal{U}$ be a domain on which f is holomorphic. Let $a \notin \mathcal{U}$ but suppose $\mathcal{D}'(a,r) = \{\, z \in \mathbb{C} |\, 0 < |z-a| < r\} \subseteq \mathcal{U}$ for some $r > 0$.

We say that f has an **isolated singularity** at $z = a$. For example, $\frac{1}{z}$, $\frac{\sin z}{z}$, $\mathrm{e}^{1/z}$ all have isolated singularities at $z = 0$.

If f is holomorphic on $\mathcal{D}'(a,r)$, then Laurent's theorem states that $f(z) = \sum_{-\infty}^{\infty} b_n(z-a)^n$ on $\mathcal{D}'(a,r)$.

$$\begin{aligned}
f(z) &= f_1(z) + f_2(z) \\
&= \sum_{n=0}^{\infty} b_n (z-a)^n + \sum_{n=-\infty}^{-1} b_n (z-a)^n \\
&= \sum_{n=0}^{\infty} b_n (z-a)^n + \underbrace{\sum_{m=1}^{\infty} \frac{b_{-m}}{(z-a)^m}}_{\textbf{principal part} \text{ of } a}
\end{aligned}$$

is convergent for all $z \neq a$.

Definition

- **Case (i):** If $b_{-m} = 0$ for $m > 0$, so $f_2(z) = 0$, then f has a **removable singularity** at a. Then

$$f_1(z) = \begin{cases} f(z) & z \neq a \\ b_0 & z = a \end{cases}$$

  is holomorphic on $\mathcal{D}(a, r)$ and $f_1(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$.

- **Case (ii):** $f_2(z) \neq 0$ but the series has finitely many terms.

$$\begin{aligned}
f_2(z) &= \sum_{n=-N}^{-1} b_n (z-a)^n \\
&= \frac{b_{-N}}{(z-a)^N} + \ldots + \frac{b_{-1}}{z-a}
\end{aligned}$$

  where $b_{-N} \neq 0$. Then $f(z) = \sum_{n=-N}^{\infty} b_n (z-a)^n$.

  We say f has a **pole of order** N **at** $z = a$ (for example, $f(z) = 1/z$ has a pole of order 1 at $z = 0$).

  - Poles of order 1 are **simple poles**
  - Poles of order 2 are **double poles**, and so on.

  If f has a pole of order N are a, then $(z-a)^N f(z)$ has a removable singularity at a, and therefore $(z-a)^N f(z)$ converges towards a finite limit as $z \to a$. This implies that $|f(z)| \to \infty$ as $z \to a$.

- **Case (iii):** $f_2(z)$ has infinitely many terms.

  We say f has an **essential singularity** at a.

  For example,

$$\begin{aligned}
f(z) &= \mathrm{e}^{1/z} \\
&= \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{m=-\infty}^{0} \frac{z^m}{(-m)!}
\end{aligned}$$

  has an essential singularity at $z = 0$.
  Essential singularities are complicated!

Example

$$f(z) = \frac{\sin z}{z}$$

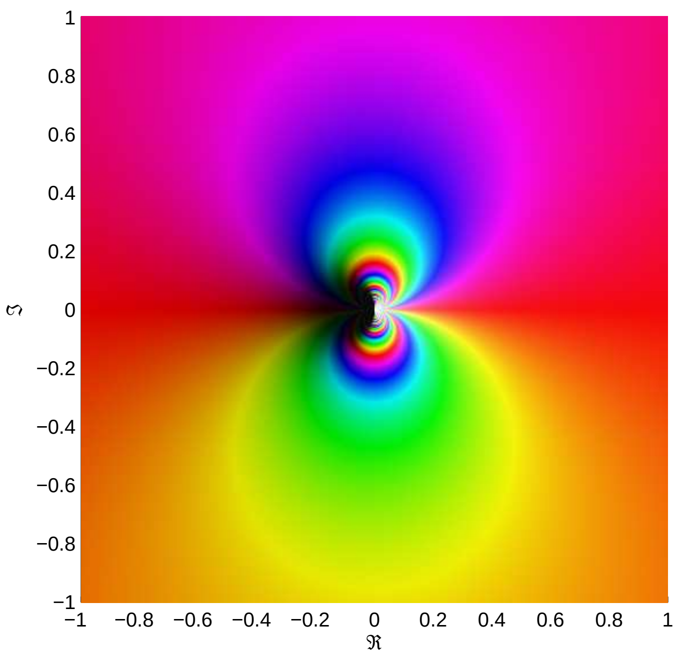
has a removable singularity at $z = 0$.

$$\begin{aligned} f(z) &= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots}{z} \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \end{aligned}$$

There are no terms of negative exponent in this series. f has a removable singularity.

Theorem (Picard's Great Theorem) Suppose f has an essential singularity at a. If f is defined on $\mathcal{D}'(a, r)$ then f takes all complex values with at most one exception on $\mathcal{D}'(a, r)$.
At an isolated singularity a, $f(z) = \sum_{n=-\infty}^{\infty} b_n (z-a)^n$.

Here is an example to illustrate this. The complex function that has been plotted is $f(z) = \mathrm{e}^{1/z}$.



$f(z) = \mathrm{e}^{1/z}$ has an essential singularity at $z = 0$. We can see that every possible complex value is reached (with all brightnesses and colours) in an arbitrarily large disc about $z = 0$. This illustrates the fascinating (and rather nasty) behaviour of essential singularities.

The **principal part** of f at a is

$$\begin{aligned} &\sum_{n=-\infty}^{-1} b_n (z-a)^n \\ &= \sum_{n=1}^{\infty} \frac{b_{-n}}{(z-a)^n} \end{aligned}$$

which converges for all $z \neq a$.

The **residue** of f at a is

$$\operatorname*{Res}_{z=a} f(z) = b_{-1}.$$

If f has a removable singularity at a, then $\operatorname*{Res}_{z=a} f(z) = 0$.

If f has a simple pole then $\operatorname*{Res}_{z=a} f(z) \neq 0$.

2.16 Calculus of Residues

Theorem (Calculus of Residues) Suppose γ is a closed, simple contour traversed anticlockwise. Let f be a function which is holomorphic on and inside γ except for a finite number of isolated singularities inside γ (call them $a_1, \dots, a_k$).
Then

$$\int_\gamma f(z)\mathrm{d}z = 2\pi\mathrm{i} \sum_{j=1}^{k} \operatorname*{Res}_{z=a_j} f(z).$$

This theorem subsumes Cauchy's theorem, Cauchy's integral formula and Cauchy's integral formula for the nth derivative.

Proof At each singularity a_j, let $g_j(z)$ be its principal part. $F(z) = f(z) = \sum_{j=1}^{k} g_j(z)$ will have only removable singularities and so can be considered holomorphic.

$\int_\gamma F(z)\mathrm{d}z = 0$ by Cauchy's theorem (as the contour γ is closed) so

$$\int_\gamma f(z)\mathrm{d}z = \sum_{j=1}^{k} \int_\gamma g_j(z)\mathrm{d}z.$$

$$\begin{aligned}
g_j(z) &= \frac{b_{-1}}{z - a_j} + h_j(z) \\
\implies \int_\gamma g_j(z)\mathrm{d}z &= b_{-1} \int_\gamma \frac{1}{z - a_j} + \int_\gamma h_j(z)\mathrm{d}z \\
&= \operatorname*{Res}_{z=a} f(z) \cdot 2\pi\mathrm{i} + \int_\gamma h_j(z)\mathrm{d}z \text{ by Cauchy's integral formula}
\end{aligned}$$

where $h_j(z) = \sum_{n=2}^{\infty} \dfrac{b_{-n}}{(z - a_j)^n}$.

But $h_j(z) = \frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=2}^{\infty} \dfrac{-b_{-n}}{(n-1)(z - a_j)^{n-1}}$ so $\int_\gamma h_j(z)\mathrm{d}z = 0$ by the Fundamental Theorem of Calculus.

Putting it all together,

$$\int_\gamma f(z)\mathrm{d}z = \sum_{j=1}^{k} 2\pi\mathrm{i} \cdot \operatorname*{Res}_{z=a_j} f(z)$$

as required.

□

Calculating Residues

- **Case (i): f has a simple pole at $z = a$.**

$$
\begin{aligned}
f(z) &= \frac{b_{-1}}{z-a} + b_0 + b_1(z-a) + \ldots \text{ near } z = a. \\
\implies (z-a)f(z) &= b_{-1} \\
&= \lim_{z \to a} f(z)(z-a)
\end{aligned}
$$

- **Case (ii): f has multiplie poles (order $N \geq 2$).**

$$f(z) = \frac{b_{-N}}{(z-a)^N} + \ldots + \frac{b_{-1}}{z-a} + b_0 + \ldots \text{ near } z = a.$$

  - **Method 1 (NOT recommended)**

$$
\begin{aligned}
(z-a)^N f(z) &= b_{-N} + \ldots + b_{-1}(z-a)^{N-1} + b_0(z-a)^N + \ldots \\
\implies \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{N-1} \left((z-a)^N f(z)\right) &= (N-1)! b_{-1} + N! b_0 (z-a) + \ldots
\end{aligned}
$$

The residue is then

$$b_{-1} = \lim_{z \to a} \frac{1}{(N-1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{N-1} \left((z-a)^N f(z)\right).$$

  - **Method 2 (Robin's favourite)**

$w = z - a$ or $z = w + a$ so

$$f(w+a) = \frac{b_{-N}}{w^N} + \ldots + \frac{b_{-1}}{w} + b_0 + \ldots$$

Expand $f(w+a)$ as a Laurent series as far as the w^{-1} term.

- **Case (iii): Essential singularity**

  - Have to compare Laurent series. These are usually rather nasty, and are rare on exercises and problem sheets.

Example

$$\operatorname*{Res}_{z=\mathrm{i}} \frac{1}{z^2+1}$$

This is where $f(z) = \dfrac{1}{z^2+1}$ has two simple poles at $z = \pm\mathrm{i}$.

Note that

$$\begin{aligned} \frac{z-\mathrm{i}}{z^2+1} &= \frac{z-\mathrm{i}}{(z+\mathrm{i})(z-\mathrm{i})} \\ &= \frac{1}{z+\mathrm{i}} \text{ provided } z \neq \pm\mathrm{i}. \end{aligned}$$

So

$$\begin{aligned} \operatorname*{Res}_{z=\mathrm{i}} \frac{1}{z^2+1} &= \lim_{z\to\mathrm{i}} \frac{z-\mathrm{i}}{z^2+1} \\ &= \frac{1}{2\mathrm{i}} = \underline{-\frac{\mathrm{i}}{2}} \end{aligned}$$

In practice we are usually given $f(z) = \frac{g(z)}{h(z)}$ where g, h are holomorphic, $g(a) \neq 0$ and h has a simple pole at $z = a$

$$\begin{aligned} \operatorname*{Res}_{z=a} \frac{g(z)}{h(z)} &= \lim_{z\to a} \frac{(z-a)g(z)}{h(z)} \\ &= \lim_{z\to a} g(z) \lim_{z\to a} \frac{z-a}{h(z)-h(a)} \text{ as } h(a) = 0 \\ &= \frac{g(a)}{h'(a)}. \end{aligned}$$

So for the example above, we would have

$$\operatorname*{Res}_{z=\mathrm{i}} \frac{1}{z^2+1} = \frac{g(a)}{h'(a)} = \frac{1}{2\mathrm{i}}$$

so $g(z) = 1$ and $h'(z) = 2\mathrm{i}$.

Example

$$\begin{aligned} \operatorname*{Res}_{z=\frac{\pi}{2}} \tan z &= \frac{\sin\frac{\pi}{2}}{\cos'\left(\frac{\pi}{2}\right)} \\ &= \frac{\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = \underline{-1}. \end{aligned}$$

Example

$$\operatorname*{Res}_{z=\mathrm{i}} \frac{1}{(z^2+1)^2}$$

$f(z) = \dfrac{1}{(z^2+1)^2}$ has double poles at $z = \pm\mathrm{i}$.

$$\begin{aligned} f(w+\mathrm{i}) &= \frac{1}{\left(w^2+2\mathrm{i}w\right)^2} \\ &= \frac{1}{(2\mathrm{i}w)^2\left(1-\frac{\mathrm{i}w}{2}\right)^2} \\ &= -\frac{1}{4w^2}\frac{1}{\left(1-\mathrm{i}w-\frac{w^2}{4}\right)} \\ &= -\frac{1}{4}w^{-2}\left(c_0+c_1w+c_2w^2+\ldots\right) \\ \therefore \left(1-\mathrm{i}w-\frac{w^2}{4}\right)&\left(c_0+c_1w+c_2w^2+\ldots\right) = 1 \end{aligned}$$

The residue is the w^{-1} coefficient, which is $-\dfrac{c_1}{4}$.

By comparing constants, we have $c_0 = 1$.

By comparing w coefficients, we have $c_1 - \mathrm{i}c_0 = 0 \iff c_1 = \mathrm{i}c_0 = \mathrm{i}$.

So

$$\operatorname*{Res}_{z\to\mathrm{i}}\frac{1}{(z^2+1)^2} = -\frac{\mathrm{i}}{4}.$$

Example

$$\int_\gamma \frac{\mathrm{e}^z}{1+z^2}\mathrm{d}z$$

where γ is the circle centre 0 radius 2.

$f(z) = \dfrac{\mathrm{e}^z}{1+z^2}$ has singularities at $z = \pm\mathrm{i}$, both of which are inside γ (and are simple poles). Then

$$\int_\gamma \frac{\mathrm{e}^z}{1+z^2}\mathrm{d}z = 2\pi\mathrm{i}\left(\operatorname*{Res}_{z=\mathrm{i}}f(z) + \operatorname*{Res}_{z=-\mathrm{i}}f(z)\right)$$

Both residues can be calculated by method of writing $f(z) = \frac{g(z)}{h(z)}$ where h has a simple zero at the singular point.

$$\begin{aligned} \operatorname*{Res}_{z=\mathrm{i}}f(z) &= \left.\frac{\mathrm{e}^z}{(1+z^2)'}\right|_{z=\mathrm{i}} = \left.\frac{\mathrm{e}^z}{2z}\right|_{z=\mathrm{i}} \\ &= \frac{\mathrm{e}^{\mathrm{i}}}{2\mathrm{i}} \end{aligned}$$

and similarly

$$\operatorname*{Res}_{z=-\mathrm{i}} f(z) = \frac{\mathrm{e}^{-\mathrm{i}}}{-2\mathrm{i}}$$

So

$$\begin{aligned}\int_\gamma \frac{\mathrm{e}^z}{1+z^2}\mathrm{d}z &= 2\pi\mathrm{i}\left(\frac{\mathrm{e}^{\mathrm{i}}}{2\mathrm{i}} + \frac{\mathrm{e}^{-\mathrm{i}}}{-2\mathrm{i}}\right)\\ &= \pi\left(\mathrm{e}^{\mathrm{i}} - \mathrm{e}^{-\mathrm{i}}\right)\\ &= 2\pi\mathrm{i}\sin 1.\end{aligned}$$

Example

$$\int_\gamma \frac{\mathrm{d}z}{z^2(\mathrm{e}^z-1)}$$

where γ is the circle centre radius $2\mathrm{i}$ radius 3.

Let $f(z) = \dfrac{1}{z^2(\mathrm{e}^z-1)}$. This function has singularities at $z=0$ (triple pole) and not forgetting $z = 2n\pi\mathrm{i}$, $n\in\mathbb{Z}$ (as $\mathrm{e}^{2n\pi\mathrm{i}}=1$). However, only $z=0$ is inside γ.

$$\begin{aligned} f(z) &= \frac{z^{-2}}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots}\\ &= \frac{z^{-3}}{1+\frac{z}{2!}+\frac{z^2}{3!}+\dots} = z^{-3} + az^{-2} + bz^{-1} + \dots\\ \iff z^{-3} &= \left(1+\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)\left(z^{-3}+az^{-2}+bz^{-1}+\dots\right)\end{aligned}$$

Comparing z^{-3} coefficients gives $1=1$.

Comparing z^{-2} coefficients gives $0=\frac{1}{2}+a \iff a=-\frac{1}{2}$.

Comparing z^{-1} coefficients gives $0=\frac{1}{6}+\frac{1}{2}a+b$ so $b=\frac{1}{6}-\frac{1}{2}a = -\frac{1}{6}+\frac{1}{4}=\frac{1}{12}$.

Hence

$$\begin{aligned}\int_\gamma \frac{\mathrm{d}z}{z^2(\mathrm{e}^z-1)} &= \frac{2\pi\mathrm{i}}{12}\\ &= \underline{\frac{\pi\mathrm{i}}{6}}.\end{aligned}$$

2.16.1 Integrals of the form $\int_0^{2\pi} F(\cos t, \sin t)\mathrm{d}t$ (where F is a nice 2-variable function)

When γ is the unit circle centre 0, with parametrisation $\gamma(t)=\mathrm{e}^{\mathrm{i}t}$, this integral becomes $\int_\gamma F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2\mathrm{i}}\right)\frac{\mathrm{d}z}{\mathrm{i}z}$.

Example

$$\int_0^{2\pi} \frac{\mathrm{d}t}{a+\cos t}$$

where $a > 1$.

This equals

$$\begin{aligned} I &= \int_\gamma \frac{1}{a + \dfrac{z+z^{-1}}{2}} \frac{\mathrm{d}z}{\mathrm{i}z} \\ &= \frac{2}{\mathrm{i}} \int_\gamma \frac{\mathrm{d}z}{z^2+2az+1} \text{ where } \gamma \text{is the unit circle.} \end{aligned}$$

Let $f(z) = \dfrac{1}{z^2+2az+1}$. This has poles at the roots of $z^2+2az+1=0$, i.e. at $z = -a \pm \sqrt{a^2-1} \in \mathbb{R}$.

Clearly because $a > 1$, $\sqrt{a^2-1} > 0$ so $-a - \sqrt{a^2-1} < -a < -1$ so the negative root isn't in the unit circle.

The product of the two roots is 1 so $-a + \sqrt{a^2-1} = \dfrac{1}{-a-\sqrt{a^2-1}} \in (-1, 0)$ so the positive root is inside the unit circle.

Hence

$$\begin{aligned} I &= \frac{2}{\mathrm{i}} \cdot 2\pi\mathrm{i} \operatorname*{Res}_{z=-a+\sqrt{a^2-1}} f(z) \\ &= 4\pi \frac{1}{h'\left(-a+\sqrt{a^2-1}\right)} \end{aligned}$$

where $h(z) = z^2+2az+1$ as $f(z) = \frac{1}{h'(z)}$ has a simple pole at $z = a$.

$$\begin{aligned} h'(z) &= 2z+2a \\ &= 2(z+a) \\ h'(b) &= 2(a+b) \\ &= 2\sqrt{a^2-1} \end{aligned}$$

So

$$I = \underline{\frac{2\pi}{\sqrt{a^2-1}}}.$$

Example

$$\int_0^{2\pi} \frac{\cos 2t}{5-3\cos t} \mathrm{d}t$$

Now

$$
\begin{aligned}
\cos 2t &= \frac{\mathrm{e}^{\mathrm{i}(2t)} + \mathrm{e}^{-\mathrm{i}(2t)}}{2} \\
&= \frac{z^2 + z^{-2}}{2} \text{ where } \gamma \text{ is the unit circle.}
\end{aligned}
$$

So

$$
\begin{aligned}
I &= \int_\gamma \frac{\frac{z^2+z^{-2}}{2}}{5 - 3 \cdot \frac{z+z^{-1}}{2}} \frac{\mathrm{d}z}{z\mathrm{i}} \\
&= \frac{1}{\mathrm{i}} \int_\gamma \frac{z^2 + z^{-2}}{-3z^2 + 10z - 3} \mathrm{d}z \\
&= -\frac{1}{\mathrm{i}} \int_\gamma \frac{1 + z^4}{z^2\left(3z^2 - 10z + 3\right)} \mathrm{d}z
\end{aligned}
$$

$$
f(z) = \frac{1 + z^4}{z^2\left(3z^2 - 10z + 3\right)} = \frac{1 + z^4}{z^2(3z - 1)(z - 3)}
$$

This function has a double pole at $z = 0$ and also simple poles at $z = 3$ and $z = \frac{1}{3}$. Only $z = 0$ and $z = \frac{1}{3}$ are inside the unit circle.

Therefore

$$
I = -\frac{1}{\mathrm{i}} \cdot 2\pi\mathrm{i} \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=\frac{1}{3}} f(z) \right)
$$

$z = 0$ is a double pole, so write

$$
\begin{aligned}
f(z) &= \frac{z^2 + z^{-2}}{3z^2 - 10z + 3} \\
&= \frac{z^{-2}}{3} + az^{-1} + \ldots \\
\therefore z^{-2}\left(1 + z^4\right) &= \left(3 - 10z + 3z^2\right)\left(\frac{z^{-2}}{3} + az^{-1} + \ldots\right)
\end{aligned}
$$

Comparing z^{-1} coefficients gives $0 = 3a - \frac{10}{3} \iff a = \frac{10}{9}$.

So

$$
\begin{aligned}
\operatorname*{Res}_{z=\frac{1}{3}} f(z) &= \left. \frac{1 + z^4}{\frac{1}{3^2}\left(3z^2 - 10z + 3\right)'} \right|_{z=\frac{1}{3}} \\
&= \frac{82}{9} \cdot \frac{1}{\left.(6z - 10)\right|_{z=\frac{1}{3}}} \\
&= \frac{82}{9(2 - 10)} = -\frac{41}{36}.
\end{aligned}
$$

and

$$\operatorname*{Res}_{z=0} f(z) = \frac{10}{9}$$

so

$$\begin{aligned} I &= -2\pi\left(\frac{10}{9} - \frac{41}{36}\right) \\ &= \underline{\frac{\pi}{18}}. \end{aligned}$$

2.16.2 The Semicircle Method

Take a contour γ of the form as shown below. γ_1 is the interval $[-R, R]$ and γ_2 is the semicircular arc of radius R going from R to $-R$.

INSERT DIAGRAM

Let $R \to \infty$. Then $\int_{\gamma_1} f(z)\mathrm{d}z = \int_{-R}^{R} f(x)\mathrm{d}x \to \int_{-\infty}^{\infty} f(x)\mathrm{d}x$. We hope that $|f(z)| \to 0$ sufficiently quickly as $|z| \to \infty$ to ensure that $\int_{\gamma_2} f(z)\mathrm{d}z \to 0$.

If these conditions are satisfied, then

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{R\to\infty} \int_{\gamma} f(z)\mathrm{d}z.$$

Example

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2}$$

Let $f(z) = \dfrac{1}{1+z^2}$. On γ_2, $|z| = R$ and so

$$\begin{aligned} & \left|z^2+1\right| \geq R^2 - 1 \\ & \qquad \text{(by the “alternate” triangle inequality, } |z+w| \geq |z| - |w| \, \forall z, w \in \mathbb{C}) \\ \iff & \left|\frac{1}{z^2+1}\right| \leq \frac{1}{R^2-1}. \end{aligned}$$

Then if $R > 1$,

$$\left|\frac{1}{1+z^2}\right| \leq \frac{1}{R^2-1}.$$

The arc γ_2 has length πR, so by the *ML-bound*,

$$
\begin{aligned}
\left|\int_{\gamma_2} f(z)\mathrm{d}z\right| &\leq \frac{\pi R}{R^2-1} \text{ (if } R > 1\text{)} \\
&\to 0 \text{ as } R \to \infty \\
\implies \int_{\gamma_2} f(z)\mathrm{d}z &\to 0 \text{ as } R \to \infty.
\end{aligned}
$$

Therefore we can use the semicircle method (therefore calculus of residues) to evaluate the integral.

$f(z)$ is holomorphic except at $z = \pm\mathrm{i}$ (two simple poles). Only one of these lies inside γ for large R, which is $z = \mathrm{i}$.

Therefore

$$
\begin{aligned}
\int_\gamma f(z)\mathrm{d}z &= 2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}}\frac{1}{1+z^2} \\
&= 2\pi\mathrm{i}\left.\frac{1}{\frac{\mathrm{d}}{\mathrm{d}z}\left(1+z^2\right)}\right|_{z=\mathrm{i}} \\
&= 2\pi\mathrm{i}\cdot\frac{1}{2\mathrm{i}} \\
&= \pi.
\end{aligned}
$$

Confirming that this is true, we could use traditional integration methods:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{1+x^2} &= [\arctan x]_{-\infty}^{\infty} \\
&= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\
&= \pi.
\end{aligned}
$$

Example

$$
I = \int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(1+x^2)^2}
$$

since the integrand is an even function of x and is therefore symmetrical about $x = 0$.

Let $f(z) = \dfrac{1}{(1+x^2)^2}$. Then

$$
|f(z)| \leq \frac{1}{(R^2-1)^2} \text{ if } z \text{ is on } \gamma_2
$$

from the previous example. The same inequality holds because the integrand and limit are the square of those in the previous example, and squaring both sides of the inequality preserves the sign.

Then by the *ML-bound* again,

$$\begin{aligned}\left|\int_{\gamma_2} f(z)\mathrm{d}z\right| &= \frac{\pi R}{(R^2-1)^2} \\ &\to 0 \text{ as } R \to \infty.\end{aligned}$$

Therefore the condition to use the semicircle method is met.

$f(z)$ has the same singularities as in the previous example ($z = \pm\mathrm{i}$, double poles this time!) and once again $z = \mathrm{i}$ is inside the contour γ for large R.

From a previous lecture, $\operatorname*{Res}_{z=\mathrm{i}} \frac{1}{(1+z^2)^2} = \frac{1}{4\mathrm{i}}$, so

$$\begin{aligned}\int_\gamma \frac{\mathrm{d}z}{(1+z^2)^2} &= 2\pi\mathrm{i}\cdot\frac{1}{4\mathrm{i}} \\ &= \frac{\pi}{2} = \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(1+x^2)^2} \\ \therefore \int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^2} &= \frac{\pi}{4}.\end{aligned}$$

Again this can be verified using traditional integration:

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^2} &= \int_0^{\pi/2}\frac{\sec^2\theta\mathrm{d}\theta}{(1+\tan^2\theta)^2} \\ &\text{(setting } x=\tan\theta \text{so } \mathrm{d}x = \sec^2\theta\mathrm{d}\theta) \\ &= \int_0^{\pi/2}\frac{\sec^2\theta\mathrm{d}\theta}{\sec^4\theta} \\ &= \int_0^{\pi/2}\cos^2\theta\mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,\mathrm{d}\theta \\ &= \frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2} \\ &= \frac{1}{2}\cdot\frac{\pi}{2} \\ &= \frac{\pi}{4}.\end{aligned}$$

Example

$$I = \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} \text{ with } 0 < a < b.$$

Let $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$On γ_2 (the semicircular arc) where $|z| = R$, we have

$$|f(z)| \le \frac{1}{(R^2-a^2)(R^2-b^2)}$$

and using similar reasoning to the previous examples and the *ML-bound*, we have

$$\begin{aligned}\left|\int_\gamma f(z)\mathrm{d}z\right| &\leq \frac{\pi R}{(R^2-a^2)(R^2-b^2)} \\ &\to 0 \text{ as } R\to\infty.\end{aligned}$$

$f(z)$ has 4 singularities, at $z=\pm a\mathrm{i}$ and $z=\pm b\mathrm{i}$ (each simple poles). For large $R>b$, the contour γ encloses 2 of these poles ($z=a\mathrm{i}$ and $z=b\mathrm{i}$). Therefore

$$\begin{aligned}\int_\gamma f(z)\mathrm{d}z &= 2\pi\mathrm{i}\left[\operatorname*{Res}_{z=a\mathrm{i}} f(z) + \operatorname*{Res}_{z=b\mathrm{i}} f(z)\right] \\ &= 2\pi\mathrm{i}\left[\left.\frac{1}{\frac{\mathrm{d}}{\mathrm{d}z}\left((z^2+a^2)(z^2+b^2)\right)}\right|_{z=a\mathrm{i}} + \left.\frac{1}{\frac{\mathrm{d}}{\mathrm{d}z}\left((z^2+a^2)(z^2+b^2)\right)}\right|_{z=b\mathrm{i}}\right]\end{aligned}$$

and $\frac{\mathrm{d}}{\mathrm{d}z}\left[(z^2+a^2)(z^2+b^2)\right] = 2z(z^2+a^2)+2z(z^2+b^2) = 2z\left(2z^2+a^2+b^2\right)$

so the integral becomes

$$\begin{aligned}\int_\gamma f(z)\mathrm{d}z &= 2\pi\mathrm{i}\left[\frac{1}{2a\mathrm{i}\left(-2a^2+a^2+b^2\right)} + \frac{1}{2b\mathrm{i}\left(-2b^2+a^2+b^2\right)}\right] \\ &= 2\pi\mathrm{i}\left[\frac{1}{2a\mathrm{i}\left(b^2-a^2\right)} + \frac{1}{2b\mathrm{i}\left(a^2-b^2\right)}\right] \\ &= \pi\left[\frac{1}{a\left(b^2-a^2\right)} - \frac{1}{b\left(b^2-a^2\right)}\right] \\ &= \pi\left[\frac{b-a}{ab\left(b^2-a^2\right)}\right] \\ &= \frac{\pi}{ab(a+b)}\end{aligned}$$

Example

$$I = \int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\mathrm{d}x$$

This is a wolf in sheep's clothing! The snag is that $\left|\frac{\cos x}{1+x^2}\right| \not\to 0$ as $R\to 0$, so we need to use a more strict bound.

We could try using $g(z) = \dfrac{\mathrm{e}^{\mathrm{i}z}}{1+z^2}$ and then apply the semicircle method.

For z in the upper half-plane, $\left|\mathrm{e}^{\mathrm{i}z}\right| \leq 1$ since $\left|\mathrm{e}^{\mathrm{i}z}\right| = \mathrm{e}^{\Re(\mathrm{i}z)} = \mathrm{e}^{-\Im(z)} \leq 1$ if $y=\Im(z)\geq 0$.

On γ_2 (the semi-circular arc),

$$\begin{aligned}\left|\frac{\mathrm{e}^{\mathrm{i}z}}{1+z^2}\right| &\leq \frac{1}{|z^2+1|} \\ &\leq \frac{1}{R^2-1}\ (R>1)\end{aligned}$$

so by the *ML-bound*, we have

$$\begin{aligned}\left|\int_\gamma g(z)\mathrm{d}z\right| &\leq \frac{\pi R}{R^2-1} \\ &\to 0 \text{ as } R\to\infty.\end{aligned}$$

So the condition to apply the semicircle method holds.

However,

$$\begin{aligned}\int_{-\infty}^{\infty} g(x)\mathrm{d}x &= \int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}x}}{1+x^2}\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\frac{\cos x+\mathrm{i}\sin x}{1+x^2}\mathrm{d}x \\ \therefore \int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\mathrm{d}x &= \Re\left[\int_{-\infty}^{\infty} g(x)\mathrm{d}x\right].\end{aligned}$$

$g(z)$ has a simple pole, $z=\mathrm{i}$, which lies inside γ. Therefore

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\mathrm{d}x &= \Re\left[\int_{-\infty}^{\infty} g(x)\mathrm{d}x\right] \\ &= \Re\left[2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}} g(z)\right] \\ &= \Re\left[2\pi\mathrm{i}\frac{\mathrm{e}^{-1}}{2\mathrm{i}}\right] \text{ (using an adaptation of a previous example)} \\ &= \Re\left[\frac{\pi}{\mathrm{e}}\right] \\ &= \frac{\pi}{\mathrm{e}}.\end{aligned}$$

2.16.3 "Slice of pie" Method

$$I = \int_0^\infty \frac{\mathrm{d}x}{1+x^{100}}$$

To avoid adding 50 residues I use a contour enclosing just one pole. The poles of $f(z)=\dfrac{1}{1+z^{100}}$ are the hundredth roots of unity, i.e. $\mathrm{e}^{\mathrm{i}\left[\frac{\pi}{100}+\frac{2\pi k}{100}\right]}$.

Consider $\alpha=\mathrm{e}^{\frac{\pi\mathrm{i}}{100}}$. Integrate over a "slice of pie" contour γ, the boundary of a sector of a circle of centre 0 and radius R, and angle $\frac{2\pi}{100}$.

INCLUDE DIAGRAM

γ only contains one pole in its interior. Poles of f are located at $\alpha, \alpha^3, \alpha^5, \dots, \alpha^{199} = \alpha^{-1}$ with only α enclosed in γ.

Fortunately though, $z = \alpha$ is a simple pole, so by the calculus of residues,

$$\begin{aligned}
\int_\gamma f(z)\mathrm{d}z &= 2\pi\mathrm{i}\operatorname*{Res}_{z=a} f(z) \\
&= 2\pi\mathrm{i}\, \frac{1}{\frac{\mathrm{d}}{\mathrm{d}z}(1+z^{100})}\bigg|_{z=\alpha} \\
&= 2\pi\mathrm{i} \cdot \frac{1}{100\alpha^{99}} \\
&= \frac{2\pi\mathrm{i}\alpha}{100\alpha^{100}} \\
&= -\frac{2\pi\mathrm{i}\alpha}{100} \text{ (as } \alpha^{100} = -1\text{).}
\end{aligned}$$

Clearly $\int_{\gamma_1} f(z)\mathrm{d}z \to \int_0^\infty f(x)\mathrm{d}x$ as $R \to \infty$.

On γ_2 (the arc), $|z| = R$, so $\left|z^{100} + 1\right| = R^{100} - 1$. If $R > 1$,

$$|f(z)| \le \frac{1}{R^{100} - 1}.$$

Also, $\mathrm{length}(\gamma_2) = \frac{2\pi R}{100}$. Hence

$$\begin{aligned}
\left|\int_{\gamma_2} f(z)\mathrm{d}z\right| &\le \frac{\frac{2\pi R}{100}}{R^{100} - 1} \\
&\to 0 \text{ as } R \to \infty \\
\implies \int_{\gamma_2} f(z)\mathrm{d}z &\to 0 \text{ as } R \to \infty.
\end{aligned}$$

Now we need to integrate over γ_3 (the line segment from 0 to $\alpha^2 R$). Parametrise this by $z = \alpha^2 t$ for $0 \le t \le R$.

Then

$$\begin{aligned}
\int_{\gamma_3} f(z)\mathrm{d}z &= -\int_{\gamma_3^-} f(z)\mathrm{d}z \\
&= -\int_0^R f(\alpha^2 t)\alpha^2 \mathrm{d}t \\
&= -\alpha^2 \int_0^R \frac{1}{1 + \alpha^{200} t^{100}}\mathrm{d}t \\
&= -\alpha^2 \int_0^R \frac{1}{1 + t^{100}}\mathrm{d}t \text{ (as } \alpha^{200} = \left(\alpha^{100}\right)^2 = (-1)^2 = 1\text{)} \\
&\to -\alpha^2 I \text{ as } R \to \infty\text{, where } I \text{is the original integral that we wanted to find.}
\end{aligned}$$

Therefore as $R \to \infty$,

$$\begin{aligned}\int_\gamma f(z)\mathrm{d}z &\to (1-\alpha^2)I \\ &= -\frac{2\pi \mathrm{i}\alpha}{100} \text{ for } R > 1 \\ \therefore I &= -\frac{2\pi\mathrm{i}\alpha}{100(1-\alpha^2)} \\ &= \frac{2\pi\mathrm{i}}{100(\alpha-\alpha^{-1})} \\ &= \frac{2\pi\mathrm{i}}{100\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{100}} - \mathrm{e}^{-\frac{\mathrm{i}\pi}{100}}\right)} \\ &= \frac{\pi}{100\sin\frac{\pi}{100}} \\ &= \frac{\pi}{100}\operatorname{cosec}\frac{\pi}{100}.\end{aligned}$$

The “slice of pie” method is often useful if the integrand depends only on x^n where n is a large integer.

This cannot be verified using traditional integration!

Example

$$I = \int_0^\infty \frac{\sin x}{x}\mathrm{d}x$$

Now since $\dfrac{\sin x}{x} \to 1$ as $x \to 0$ we have no problems with any sort of poles.

However, $\dfrac{\sin x}{x}$ is **not absolutely integrable** on $[0, \infty)$, i.e. $\int_0^\infty \left|\frac{\sin x}{x}\right| \mathrm{d}x$ diverges! So if we interpret

$$I \text{ as } \lim_{R\to\infty} \int_0^R \frac{\sin x}{x}\mathrm{d}x$$

then it will emerge that this limit exists. We would like to attack this integral using the semicircle method...

Note that

$$\begin{aligned}2I &= \int_{-\infty}^\infty \frac{\sin x}{x}\mathrm{d}x \\ &\stackrel{?}{=} \Im\left[\int_{-\infty}^\infty \frac{\mathrm{e}^{\mathrm{i}x}}{x}\mathrm{d}x\right]\end{aligned}$$

although this is somewhat dubious as the integral goes right through the singularity at $z = 0$. So we have to use a modified version of the semicircle method where the contour omits $z = 0$ somehow.

We create a new contour γ that circles around $z = 0$ with radius ε, and then consider the limit as $R \to \infty$ and $\varepsilon \to 0$.

$f(z) = \mathrm{e}^{\mathrm{i}z}/z$ only has the one singularity, at $z = 0$, which is outside γ, so

$$\int_{\gamma} f(z)\mathrm{d}z = 0$$

by *Cauchy's theorem*.

Now let $R \to \infty$ and $\varepsilon \to 0$. Then

$$\begin{aligned}
\int_{\gamma_1} f(z)\mathrm{d}z &= \int_{\varepsilon}^{R} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\mathrm{d}x \\
\text{and } \int_{\gamma_3} f(z)\mathrm{d}z &= \int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\mathrm{d}x \\
&= -\int_{\varepsilon}^{R} \frac{\mathrm{e}^{-\mathrm{i}x}}{x}\mathrm{d}x \\
\therefore \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_3} f(z)\mathrm{d}z &= \int_{\varepsilon}^{R} \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{x} \\
&= 2\mathrm{i}\int_{\varepsilon}^{R} \frac{\sin x}{x} \\
&= 2\mathrm{i}I \text{ as } R \to \infty \text{ and } \varepsilon \to 0.
\end{aligned}$$

Consider $\int_{\gamma_4} f(z)\mathrm{d}z = -\int_{\gamma_4^-} f(z)\mathrm{d}z$, and the parametrisation $z = \varepsilon\mathrm{e}^{it}$ on $t \in [0, \pi]$.

Now by using Laurent series,

$$\begin{aligned}
f(z) &= \frac{1}{z} + \mathrm{i} - \frac{z}{2!} - \frac{\mathrm{i}z^2}{3!} + \frac{z^3}{4!} + \dots \\
&= \frac{1}{z} + g(z) \text{ where } g \text{is holomorphic on } \mathbb{C} \\
&= \frac{1}{z} + G'(z) \text{ where } G \text{is holomorphic on } \mathbb{C}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\gamma_4} g(z)\mathrm{d}z &= \int_{\gamma_4} \frac{\mathrm{d}z}{z} + \int_{\gamma_4} G'(z)\mathrm{d}z \\
&= \int_{\gamma_4} \frac{\mathrm{d}z}{z} + G(\varepsilon) - G(-\varepsilon) \text{ by the Fundamental Theorem of Calculus} \\
\text{Now } \int_{\gamma_4} \frac{\mathrm{d}z}{z} &= -\int_{\gamma_4^-} \frac{\mathrm{d}z}{z} \\
&= -\int_0^{\pi} \frac{\varepsilon\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{\varepsilon\mathrm{e}^{\mathrm{i}t}}\mathrm{d}t \\
&= -\pi\mathrm{i}
\end{aligned}$$

Therefore

$$\int_{\gamma_4} f(z)\mathrm{d}z \to -\pi\mathrm{i} + G(0) - G(0) \text{ as } \varepsilon \to 0.$$

We need to prove that $\int_{\gamma_2} f(z)\mathrm{d}z \to 0$ as $R \to \infty$. The *ML-bound* isn't enough to prove this, as $\left|\frac{\mathrm{e}^{\mathrm{i}z}}{z}\right| \leq \frac{1}{R}$ on the large arc, but γ_2 has length πR so this bound implies

$$\left|\int_{\gamma_2} f(z)\mathrm{d}z\right| \leq \pi$$

which is not good enough! So we need to work harder and use a different bound.

Jordan's Inequality

$$\sin t \geq \frac{2t}{\pi}$$

for $t \in \left[0, \frac{\pi}{2}\right]$.

So we can use Jordan's inequality on a parametrisation of $\int_{\gamma_2} f(z)\mathrm{d}z$. Let $z = R\mathrm{e}^{\mathrm{i}t}$ with $t \in [0, \pi]$. Then

$$\begin{aligned}
\int_{\gamma_2} f(z)\mathrm{d}z &= \int_0^\pi \frac{\mathrm{e}^{\mathrm{i}R\mathrm{e}^{\mathrm{i}t}}}{R\mathrm{e}^{\mathrm{i}t}}\mathrm{i}R\mathrm{e}^{\mathrm{i}t}\mathrm{d}t \\
&= \mathrm{i}\int_0^\pi \mathrm{e}^{\mathrm{i}R\mathrm{e}^{\mathrm{i}t}}\mathrm{d}t \\
&= \mathrm{i}\int_0^\pi \mathrm{e}^{-R\sin t + \mathrm{i}R\cos t}\mathrm{d}t \\
\therefore \left|\int_{\gamma_2} f(z)\mathrm{d}z\right| &\leq \int_0^\pi \mathrm{e}^{-R\sin t}\mathrm{d}t \\
&= 2\int_0^{\pi/2} \mathrm{e}^{-R\sin t}\mathrm{d}t \text{ (as the integrand is symmetrical about the line } t = \frac{\pi}{2}\text{)} \\
&\leq 2\int_0^{\pi/2} \mathrm{e}^{-\frac{2R}{\pi}t}\mathrm{d}t \\
&\text{(by Jordan's inequality and the fact that } \exp \text{is an increasing function, so the inequality is preserved)} \\
&< 2\int_0^{\infty} \mathrm{e}^{-\frac{2R}{\pi}t}\mathrm{d}t \\
&= \frac{\pi}{R} \\
&\to 0 \text{ as } R \to \infty. \\
\therefore \int_{\gamma_2} f(z)\mathrm{d}z &\to 0 \text{ as } R \to \infty.
\end{aligned}$$

So

$$\begin{aligned}
\int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z + \int_{\gamma_3} f(z)\mathrm{d}z + \int_{\gamma_4} f(z)\mathrm{d}z = 0 \\
\implies 0 = 2\pi I - \pi\mathrm{i} + 0 \text{ (letting } R \to \infty \text{and } \varepsilon \to 0\text{)}
\end{aligned}$$

Therefore

$$\boxed{\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}}$$

which is a very important integral in Fourier analysis.

2.16.4 Rectangle Method

Example (Rectangle Method)

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh x}$$

The **rectangle method** is useful for $f(z)$ periodic with period $\mathrm{i}\lambda$ for $\lambda \in \mathbb{R}$. Remember that $\cosh z$ is periodic over $\mathbb{C}$ but not $\mathbb{R}$.

Let $f(z) = \dfrac{1}{\cosh z}$. Integrate this over the contour γ, which traverses the rectangle with vertices $\pm R$ and $\pm R+\pi\mathrm{i}$ anticlockwise. γ_1 is the path from $-R$ to R.

The poles of $\frac{1}{\cosh z} = \frac{1}{\cos \mathrm{i}z}$ occur when $\mathrm{i}z = (2n+1)\frac{\pi}{2}$ for $n \in \mathbb{Z}$, therefore when $z = -\mathrm{i}\,(2n+1)\,\frac{\pi}{2}$. Inside γ there is only one simple pole, which is $z = \frac{\pi\mathrm{i}}{2}$.

Therefore

$$\begin{aligned}
\int_\gamma f(z)\mathrm{d}z &= 2\pi\mathrm{i}\operatorname*{Res}_{z=\frac{\pi\mathrm{i}}{2}} f(z) \\
&= 2\pi\mathrm{i}\cdot \left.\frac{1}{\frac{\mathrm{d}}{\mathrm{d}z}(\cosh z)}\right|_{z=\frac{\pi\mathrm{i}}{2}} \\
&= 2\pi\mathrm{i}\cdot \left.\frac{1}{\sinh z}\right|_{z=\frac{\pi\mathrm{i}}{2}} \\
&= 2\pi\mathrm{i}\cdot \frac{1}{\mathrm{i}\sin\frac{\pi}{2}} \\
&= 2\pi
\end{aligned}$$

Also, note that

$$\begin{aligned}
\int_{\gamma_1} f(z)\mathrm{d}z &= \int_{-R}^{R} \frac{\mathrm{d}x}{\cosh x} \\
\text{and } \int_{\gamma_3} f(z)\mathrm{d}z &= -\int_{\gamma_3^-} f(z)\mathrm{d}z \\
&= -\int_{-R}^{R} \frac{\mathrm{d}x}{\cosh(x+\pi\mathrm{i})} \\
&= \int_{-R}^{R} \frac{\mathrm{d}x}{\cosh x}
\end{aligned}$$

because $\cosh(x+\pi\mathrm{i}) = \cos(\mathrm{i}x-\pi) = -\cos\mathrm{i}x = \cosh x$.

Therefore

$$\begin{aligned}\int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_3} f(z)\mathrm{d}z &\to 2I \text{ as } R \to \infty \\ &= 2\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh x}.\end{aligned}$$

Consider $\int_{\gamma_2} f(z)\mathrm{d}z$. On γ_2, $z = R + \mathrm{i}t$ with $t \in [0, \pi]$.

$$\begin{aligned}\left|\cosh^2 z\right| &= \left|\cos^2 \mathrm{i}z\right| \\ &= \left|\cos^2(\mathrm{i}R - t)\right| \\ &= \cos^2 t + \sinh^2 R \\ &\geq \sinh^2 R \text{ (as } \cos^2 t > 0 \forall t).\end{aligned}$$

So on γ_2,

$$\left|\frac{1}{\cosh z}\right| \leq \frac{1}{\sinh R}.$$

Therefore by the *ML-bound*, with $\mathrm{length}(\gamma_2) = \pi$, we have

$$\begin{aligned}\left|\int_{\gamma_2} f(z)\mathrm{d}z\right| &\leq \frac{\pi}{\sinh R} \\ &\to 0 \text{ as } R \to \infty. \\ \therefore \int_{\gamma_2} f(z)\mathrm{d}z &\to 0 \text{ as } R \to \infty.\end{aligned}$$

And by similar reasoning, $\int_{\gamma_4} f(z)\mathrm{d}z \to 0$ as $R \to \infty$.

Letting $R \to \infty$ in $\int_{\gamma} f(z)\mathrm{d}z = 2\pi$, we get

$$\begin{aligned}2\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh x} &= 2\pi \\ \iff \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh x} &= \pi.\end{aligned}$$

Example (Summing a series using contour integration)

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Let

$$f(z) = \frac{\cot z}{z^2}$$

where z has poles where $\tan z = 0$, i.e. at $z = n\pi$ for $n \in \mathbb{Z}$. All these poles are simple, because $\tan'(n\pi) = \sec^2 n\pi = 1$ so all these poles are simple. So $f(z)$ has simple poles at $z = n\pi$ for $n \in \mathbb{Z}\backslash\{0\}$, but $z = 0$ is a triple pole.

Let $N \in \mathbb{N}$. Let γ be the contour which is the perimeter of a square with vertices $\pm\left(N+\frac{1}{2}\right)\pi$ and $\pm\left(N+\frac{1}{2}\right)\pi\mathrm{i}$. This means that γ encloses all poles $-N\pi, \ldots, -\pi, 0, \pi, \ldots, N\pi$. Then

$$\int_\gamma f(z)\mathrm{d}z = 2\pi\mathrm{i}\cdot\left[a_0 + \sum_{n=1}^{N}(a_n + a_{-n})\right]$$

where $a_n = \operatorname*{Res}_{z=n\pi} f(z)$

When $n \neq 0$, $n\pi$ is a simple pole, so

$$\begin{aligned}
a_n &= \operatorname*{Res}_{z=n\pi} f(z) \\
&= \operatorname*{Res}_{z=n\pi}\left(\frac{\cos z}{z^2 \sin z}\right) \\
&= \left.\frac{\cos z}{\frac{\mathrm{d}}{\mathrm{d}z}\left(z^2\sin z\right)}\right|_{z=n\pi} \\
&= \left.\frac{\cos z}{2z\sin z + z^2\cos z}\right|_{z=n\pi} \\
&= \frac{\cos n\pi}{2n\pi\sin n\pi + (n\pi)^2\cos n\pi} \\
&= \frac{(-1)^n}{(n\pi)^2(-1)^n} \\
&= \frac{1}{n^2\pi^2}.
\end{aligned}$$

0 is a triple pole of $f(z)$:

$$\begin{aligned}
f(z) &= \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots}{z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\right)} \\
&= z^{-3}\left(1 + c_2 z^2 + c_4 z^4 + \ldots\right)
\end{aligned}$$

$$\begin{aligned}
\iff \left(1 + c_2 z^2 + \ldots\right)\left(1 - \frac{z^2}{3!} + \ldots\right) &= 1 - \frac{z^2}{2!} + \ldots \\
\implies \left(1 - \frac{z^2}{3!} + \ldots\right) + \left(c_2 z^2 - \frac{c_2 z^4}{3!} + \ldots\right) &= 1 - \frac{z^2}{2!} + \ldots
\end{aligned}$$

so by equating z^2 coefficients, we obtain $-\frac{1}{3!} + c_2 = -\frac{1}{2!} \iff c_2 = \frac{1}{6} - \frac{1}{2} = -\frac{1}{3}$. This is the z^{-1} coefficient in the Laurent series of $f(z)$ and therefore

$$a_0 = \operatorname*{Res}_{z=0} f(z) = -\frac{1}{3}.$$

Therefore

$$
\begin{aligned}
\int_\gamma f(z)\mathrm{d}z &= 2\pi\mathrm{i}\left[a_0 + \sum_{n=1}^{N}(a_n + a_{-n})\right] \\
&= 2\pi\mathrm{i}\left[-\frac{1}{3} + \sum_{n=1}^{N}\frac{2}{n^2\pi^2}\right] \\
&= \frac{4\pi\mathrm{i}}{\pi^2}\left[-\frac{\pi^2}{6} + \sum_{n=1}^{N}\frac{1}{n^2}\right].
\end{aligned}
$$

To prove that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ we can show that $\int_\gamma f(z)\mathrm{d}z = 0$ as $N \to \infty$.

Note that

$$
\begin{aligned}
\left|\cot^2(x+\mathrm{i}y)\right| &= \frac{|\cos(x+\mathrm{i}y)|^2}{|\sin(x+\mathrm{i}y)|^2} \\
&= \frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y}.
\end{aligned}
$$

Suppose that $z = x + \mathrm{i}y$ lies on γ.

- **Case (i):** z is on a horizontal side of γ, i.e. $y = \pm\left(N+\frac{1}{2}\right)\pi$ (with $x \in \left[-\left(N+\frac{1}{2}\right)\pi, \left(N+\frac{1}{2}\right)\pi\right]$). Then

$$
\begin{aligned}
|\cot z|^2 &\le \frac{1 + \sinh^2\left(N+\frac{1}{2}\right)\pi}{\sinh^2\left(N+\frac{1}{2}\right)\pi} \\
&= \frac{\cosh^2\left(N+\frac{1}{2}\right)\pi}{\sinh^2\left(N+\frac{1}{2}\right)\pi} \\
&= \coth^2\left(N+\frac{1}{2}\right)\pi
\end{aligned}
$$

- **Case (ii):** z is on a vertical side of γ, i.e. $x = \pm\left(N+\frac{1}{2}\right)\pi$ (with $y \in \left[-\left(N+\frac{1}{2}\right)\pi, \left(N+\frac{1}{2}\right)\pi\right]$). Then

$$
\begin{aligned}
\cos x = 0 &\implies \cos^2 x = 0 \\
&\iff \sin^2 x = 1
\end{aligned}
$$

So

$$
\begin{aligned}
|\cot z|^2 &= \frac{\sinh^2 y}{1 + \sinh^2 y} \\
&= \tanh^2 y \\
&\le 1
\end{aligned}
$$

as $1 < \coth\left(N+\frac{1}{2}\right)\pi$.

On γ, $|\cot z| \leq \coth\left(N+\frac{1}{2}\right)\pi$, and $|z| \geq \left(N+\frac{1}{2}\right)\pi$ so

$$|f(z)| \leq \frac{\coth\left(N+\frac{1}{2}\right)\pi}{\left(N+\frac{1}{2}\right)^2\pi^2}.$$

γ has length $4(2N+1)\pi$.

Therefore by the *ML-bound*,

$$\begin{aligned}
\left|\int_\gamma \frac{\cot z}{z^2}\mathrm{d}z\right| &\leq \frac{4(2N+1)\coth\left(N+\frac{1}{2}\right)\pi}{\left(N+\frac{1}{2}\right)^2\pi^2} \\
&= \frac{8\left(N+\frac{1}{2}\right)\coth\left(N+\frac{1}{2}\right)\pi}{\left(N+\frac{1}{2}\right)^2\pi^2} \\
&= \frac{8\coth\left(N+\frac{1}{2}\right)\pi}{\left(N+\frac{1}{2}\right)\pi^2} \\
&\to 0 \text{ as } N \to \infty. \\
\therefore \int_\gamma \frac{\cot z}{z^2}\mathrm{d}z &\to 0 \text{ as } N \to \infty.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{4\pi\mathrm{i}}{\pi^2}\left[-\frac{\pi^2}{6}+\sum_{n=1}^{N}\frac{1}{n^2}\right] &\to 0 \text{ as } N \to \infty \\
\therefore \sum_{n=1}^{N}\frac{1}{n^2} &= \frac{\pi^2}{6}
\end{aligned}$$

as required!

2.17 Rouché's Theorem

Recall that f is holomorphic on a domain $\mathcal{U}$ if f is differentiable there.

We call f **meromorphic** on a domain $\mathcal{U}$ if f is holomorphic on $\mathcal{U}$ except possibly for poles.

For example:

- **All** rational functions are meromorphic on $\mathbb{C}$.
- $f(z) = \mathrm{e}^{1/z}$ is **not** meromorphic on $\mathbb{C}$ (because 0 is not a pole of $f(z)$) but it is on $\mathbb{C}\backslash\{0\}$.

Rouché's theorem is about containing zeros inside a simple closed contour.

Let $f(z)$ be a meromorphic function and consider $f'(z)/f(z)$. If a is in the domain of f (assuming f is not constant) its Laurent series is

$$
\begin{aligned}
f(z) &= \sum_{n=N}^{\infty} a_n (z-a)^n \quad (a_N \neq 0) \\
\text{with } N &= \operatorname*{ord}_{z=a} f(z) \\
\implies f'(z) &= \sum_{n=N}^{\infty} n a_n (z-a)^{n-1}
\end{aligned}
$$

If $N \neq 0$,

$$
\begin{aligned}
\frac{f'(z)}{f(z)} &= \frac{N a_N (z-a)^{N-1} + (N+1) a_{N+1} (z-a)^N + \dots}{a_N (z-a)^N + a_{N+1} (z-a)^{N+1} + \dots} \\
&= \frac{1}{z-a} \left(\frac{N a_N + (N+1) a_{N+1} (z-a) + \dots}{a_N + a_{N+1} (z-a) + \dots} \right)
\end{aligned}
$$

So $f'(z)/f(z)$ has a simple pole of residue N at $z = a$.

If $N = 0$, then $f(a) \neq 0$, and so $\dfrac{f'(z)}{f(z)}$ is holomorphic at $z = a$. $\dfrac{f'(z)}{f(z)}$ is also meromorphic, as it has at most simple poles precisely at the poles and zeros of f. Moreover,

$$
\operatorname*{Res}_{z=a} \frac{f'(z)}{f(z)} = \operatorname*{ord}_{z=a} f(z).
$$

Now we apply the *residue theorem*.

If f is meromorphic on and inside γ (a simple closed contour) and f has no zeros or poles on γ, then

$$
\begin{aligned}
\int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z &= 2\pi \mathrm{i} \cdot \sum \left(\text{residues of } \frac{f'}{f} \text{in } \gamma \right) \\
&= 2\pi \mathrm{i} \cdot \sum \left(\text{orders of } f \text{at } 0 \text{and poles in } \gamma \right) \\
\implies \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z &= Z - P
\end{aligned}
$$

where $Z = \sum$(order of **zeros** of $f$in γ) and $P = \sum$(orders of **poles** of $f$in γ).

If f is holomorphic inside γ, then $P = 0$ and

$$
\begin{aligned}
\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z &= Z \\
&= \#(\text{zeros of } f \text{in } \gamma \text{countered according to multiplicity})
\end{aligned}
$$

Theorem (Rouché's Theorem) Let γ be a simple closed contour. Let f and g be holomorphic on and inside γ.
If

$$
|f(z) - g(z)| < |f(z)| + |g(z)|
$$

for all z on γ, then $Z_f = Z_g$, where $Z_f =$ total order of zeros of $f$inside $\gamma$and similarly for Z_g.
(A more usual statement and stronger hypothesis is that $|f(z) - g(z)| < |f(z)|$ on γ)

Proof The hypothesis implies that f and g are nonzero on γ . If $f(z) = 0$ then

$$\begin{aligned} |0 - g(z)| &< 0 + |g(z)| \\ \text{i.e. } |g(z)| &< |g(z)| \end{aligned}$$

is false so $f(z) \neq 0$ on γ and similarly for $g(z)$.

We claim also that $\dfrac{f(z)}{g(z)}$ can't be a negative real number.

If $\dfrac{f(z)}{g(z)} = -t$, for $t \in \mathbb{R}^+$ then

$$\begin{aligned} f(z) &= -tg(z) \\ \implies f(z) - g(z) &= (1+t)g(z) \\ \implies |f(z) - g(z)| &= (1+t)\,|g(z)| \\ \text{and } |f(z)| + |g(z)| &= |g(z)| + t\,|g(z)| \\ &= (1+t)\,|g(z)| \end{aligned}$$

which contradicts that $|f(z) - g(z)| < f(z) + g(z)$.

For z on γ, $\dfrac{f(z)}{g(z)} \in \mathcal{U} = \mathbb{C}\backslash\{-t|\, t \in \mathbb{R}, t \geq 0\}$ (the slit plane). On $\mathcal{U}$, Log is holomorphic. So $\mathrm{Log}\dfrac{f(z)}{g(z)}$ is holomorphic on a domain containing γ.

$$\begin{aligned} h(z) &= \mathrm{Log}\frac{f(z)}{g(z)} \\ \implies h'(z) &= \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}. \end{aligned}$$

As γ is a closed contour, $\int_\gamma h'(z)\mathrm{d}z = 0$, i.e.

$$\begin{aligned} \int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z &= \int_\gamma \frac{g'(z)}{g(z)}\mathrm{d}z \\ \implies Z_f &= Z_g \end{aligned}$$

as required.

$\square$

Example Prove that all roots of $z^5 + 7z + 12$ lie in the annulus $\{z \in \mathbb{C}|\, 1 < |z| < 2\}$.

We consider the numbers of possible roots inside each disc and then subtract one from the other.

- **Large disc:** let γ be the circle with $|z| = 2.$, and $f(z) = z^5 + 7z + 12.$
  Then let $g(z) = z^5$ (the term in f with greatest modulus). Then on γ,

$$\begin{aligned} |f(z) - g(z)| &= |7z + 12| \\ &\le 7|z| + 12 \text{ (by the \textit{triangle inequality})} \\ &= 26 \text{ (as } |z| = 2 \text{on } \gamma) \end{aligned}$$

On γ, $|g(z)| = \left|z^5\right| = |z|^5 = 32.$

Now since

$$\begin{aligned} |f(z) - g(z)| &< g(z) \\ &\le |f(z)| + |g(z)| \end{aligned}$$

on γ, then by *Rouché's theorem*, we conclude that the number of solutions of $f(z) = 0$ in the annulus is equal to the number of solutions of $g(z) = 0$ in the disc with $|z| < 2$, which is 5 (counted with multiplicity) and z^5 has a quintuple zero at $z = 0$ and no others.

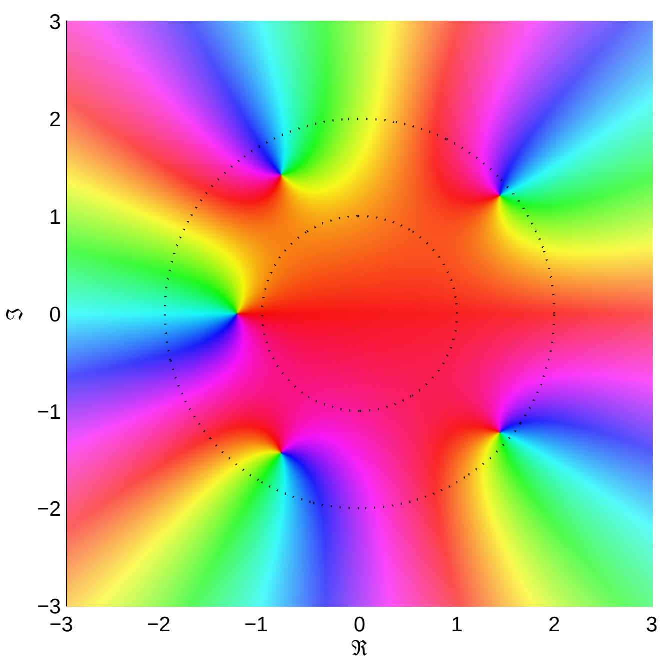
- **Small disc:** let γ be the unit circle, and this time we take $g(z) = 12$, because $\left|z^5\right| < |z| < 12$ when $|z| < 1$.

  On γ, we have $|z| = 1$ and so

$$\begin{aligned} |f(z) - g(z)| &= \left|z^5 + 7z\right| \\ &\le |z|^5 + 7|z| \\ &= 8 < |g(z)|. \end{aligned}$$

  By *Rouché's theorem*, f and g have the same number of zeros inside γ, i.e. none!

Therefore all zeros of f lie in the annulus $\{z \in \mathbb{C} | 1 < |z| < 2\}$. To visualise this, see the diagram below.



Example (From 2009 exam) How many zeros (counted with multiplicity) has $f(z) = z^5 - 5z^4 + z^2 + 9z - 1$ in the annulus $\{z \in \mathbb{C} | 1 < |z| < 3\}$?

- **Large disc:** let γ be the circle centre 0 radius 3. Let $g(z) = -5z^4$ (**not** z^5, as $\left|-5z^4\right| > \left|z^5\right|$ when $|z| = 3$, as on γ).

  Hence on γ, we have

$$\begin{aligned} |f(z) - g(z)| &= \left|z^5 + z^2 + 9z - 1\right| \\ &\leq |z|^5 + |z|^2 + 9|z| + 1 \text{ (by the \textit{triangle inequality})} \\ &= 243 + 9 + 27 + 1 \\ &= 280 \end{aligned}$$

  and

$$\begin{aligned} |g(z)| &= 5|z|^4 \\ &= 405 \end{aligned}$$

  on γ, so $|f(z) - g(z)| < |g(z)|$. Rouché's theorem applies. f and g have the same number of zeros inside γ, i.e. 4 (counted with multiplicity).

- **Small disc:** let γ be the unit circle, i.e. where $|z| = 1$. This time, let $g(z) = 9z$ (as all the other terms have smaller moduli than $|9z|$ when $|z| = 1$).

  Then

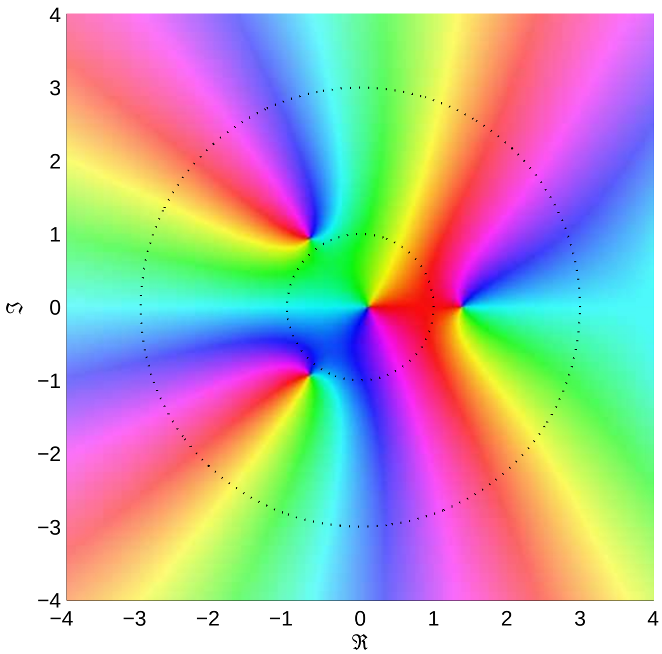
$$\begin{aligned} |f(z) - g(z)| &= \left|z^5 - 5z^4 + z^2 - 1\right| \\ &\leq |z|^5 + 5|z|^4 + |z|^2 + 1 \text{ (by the \textit{triangle inequality})} \\ &= 1 + 5 + 1 + 1 \\ &= 8 \text{ on } \gamma. \end{aligned}$$

  Also,

$$\begin{aligned} |g(z)| &= 9 \\ &> 8 \text{ on } \gamma. \end{aligned}$$

  Rouché's theorem strikes again! $f(z)$ has the same number of zeros as $g(z)$ inside γ, i.e. 1.

Therefore in the annulus $\{z \in \mathbb{C} | 1 < |z| < 3\}$, $f(z)$ has $4 - 1 = 3$ zeros (counted with multiplicity). You can see this below.



Proof of Fundamental Theorem of Calculus (Using Rouché's Theorem) An important thing to note is that Rouché's theorem implies the Fundamental Theorem of Calculus!

Let

$$f(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a$$

and $g(z) = z^n$, which is term in $f(z)$ of greatest modulus. We consider these functions on the contour γ, the circle centre 0 radius R.

Then

$$\begin{aligned} |f(z) - g(z)| &= \left| a_1 z^{n-1} + \ldots + a_{n-1} z + a \right| \\ &\leq |a_1| \, |z|^{n-1} + \ldots + |a_{n-1}| \, |z| + |a| \text{ (by the \textit{triangle inequality})} \end{aligned}$$

Then

$$\begin{aligned} \frac{|f(z) - g(z)|}{|g(z)|} &\leq \frac{|a_1| \, R^{n-1} + |a_2| \, R^{n-2} + \ldots + |a_n|}{R^n} \\ &= \frac{|a_1|}{R} + \frac{|a_2|}{R^2} + \ldots + \frac{|a_n|}{R^n} \end{aligned}$$

There is R_0 such that $\frac{|f(z)-g(z)|}{|g(z)|} < 1$ if z is on γ and $R > R_0$. For such R, $|f(z) - g(z)| < g(z)$ holds on γ so *Rouche's theorem* applies.

So f and g have the same number of zeros (counted with multiplicity) inside γ, i.e. n zeros.

So $f(z)$ has exactly n zeros on $\mathbb{C}$.

□

2.18 Maximum Modulus Theorem

Theorem (Maximum Modulus Theorem) Let f be holomorphic on a disc $\mathcal{D}(a,r)$. Then if $|f|$ is maximised on $\mathcal{D}(a,r)$ at a, then f is constant on $\mathcal{D}(a,r)$.
That is, if $|f(z)| \leq f(a)$ on $\mathcal{D}(a,r)$ then f is constant on the disc.

Proof Apply *Cauchy's integral formula*.

- **Case (i):** $f(a)=0$. Then $|f(z)| \leq 0$ for all $z \in \mathcal{D}(a,r)$ unless f is identically zero.
- **Case (ii):** $f(a) \neq 0$. Consider $g(z) = f(z)/f(a)$. Then $g(a)=0$ and $|g(z)| \leq 1$ for all $z \in \mathcal{D}(a,r)$.

  Apply *Cauchy's integral formula* to g. Let γ the the circle centre a radius r' where $0 < r' < r$. We can parametrise this by using $z = a + r'\mathrm{e}^{\mathrm{i}t}$ with $t \in [0, 2\pi]$.

  Then

$$\begin{aligned} 1 &= g(a) \\ &= \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{g(z)}{z-a}\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_0^{2\pi} \frac{g(a+r\mathrm{e}^{\mathrm{i}t})}{r'\mathrm{e}^{\mathrm{i}t}} \cdot \mathrm{i}r'\mathrm{e}^{\mathrm{i}t}\mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(a+r\mathrm{e}^{\mathrm{i}t})\mathrm{d}t \\ &= \text{mean value of } g(z) \text{for } z \text{on } \gamma. \end{aligned}$$

  If we take real parts, we have

$$\begin{aligned} 1 &= \frac{1}{2\pi} \int_0^{2\pi} \Re\left[g(a+r'\mathrm{e}^{\mathrm{i}t})\right]\mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} \phi(t)\mathrm{d}t \end{aligned}$$

  so

$$\begin{aligned} \left|g\left(a+r'\mathrm{e}^{\mathrm{i}t}\right)\right| \leq 1 &\implies \Re\left[g(a+r'\mathrm{e}^{\mathrm{i}t})\right] \leq 1 \text{ (as } \Re(z) \leq |z| \, \forall z \in \mathbb{C}) \\ &\iff \phi(t) \leq 1 \end{aligned}$$

  Therefore

$$\begin{aligned} \int_0^{2\pi} (1-\phi(t))\,\mathrm{d}t &= 2\pi - \int_0^{2\pi} \phi(t)\mathrm{d}t \\ &= 0 \text{ (by \textit{Cauchy's theorem})} \end{aligned}$$

so $1-\phi(t) \geq 0$. If for some t_0, $1-\phi(t) > 0$, then $1-\phi(t) > 0$ in an interval containing t_0, so $\int_0^{2\pi}(1-\phi(t)) > 0$. This is a **contradiction**, as the integral can't be both zero and greater than zero.

But $\left|g(a+r'\mathrm{e}^{\mathrm{i}t})\right| \leq 1$ and $\Re\left[g(a+r'\mathrm{e}^{\mathrm{i}t})\right] = 1$, so $g(a+r\mathrm{e}^{\mathrm{i}t}) = 1$, i.e. $g(z) = 1$ on γ.

By the *identity theorem*, $g(z) = 1$ on all $z \in \mathcal{D}(a, r) \implies f(z) = f(a)$ on $\mathcal{D}(a, r)$.

□

2.19 Winding numbers

Consider a path $\gamma \in \mathbb{C}$ not containing 0. Let $\gamma : [a, b] \to \mathbb{C}\backslash\{0\}$.

A fact is that there are continuous functions

$$
\begin{aligned}
r &: [a, b] \to \mathbb{R}^+ \\
\theta &: [a, b] \to \mathbb{R}
\end{aligned}
$$

with $\gamma(t) = r(t)\mathrm{e}^{\mathrm{i}\theta(t)}$.

Obviously $r(t) = |\gamma(t)|$, but the existence of $\theta(t)$ is more delicate to prove - see *Stewart & Tall*.

If f is smooth, then

$$
\theta(t) = \mathrm{Arg}\left(\gamma(a)\right) + \int_a^t \frac{\gamma'(u)}{\gamma(u)}\mathrm{d}u
$$

which is related to an exercise on the problem sheet.

If γ is a closed path then $\gamma(a) = \gamma(b) \implies \theta(b) = \theta(a) = 2\pi m$ with $m \in \mathbb{Z}$.

For example, on the unit circle, $\gamma(t) = \mathrm{e}^{\mathrm{i}t} \implies \theta(t) = t \implies \theta(2\pi) - \theta(0) = 2\pi$.

We write m as $w(\gamma; 0)$ and this is called the **winding number** of γ about 0 and this is an integer when γ is closed.

- When γ is smooth and closed, its winding number is

$$
\begin{aligned}
w(\gamma; 0) &= \frac{1}{2\pi}\int_a^b \frac{\gamma'(u)}{\gamma(u)}\mathrm{d}u \\
&= \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{d}z}{z}.
\end{aligned}
$$

- If γ is a closed smooth path not containing z_0, then

$$
\begin{aligned}
w(\gamma; z_0) &= \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{d}z}{z - z_0} \\
&= \frac{1}{2\pi\mathrm{i}}\left(\theta(b) - \theta(a)\right)
\end{aligned}
$$

  where $\gamma(t) = z_0 + r(t)\mathrm{e}^{\mathrm{i}\theta(t)}$ with r, θ continuous.

Another fact is that a closed smooth path divides its complement in $\mathbb{C}$ into domains. On each of these, the winding number is constant. One domain is unbounded, so it has winding number zero.

Examples See the diagrams below.

The **interior** of a closed contour γ is the set of z_0 not on γ with $w(\gamma; z_0) \neq 0$.

And now we can provide a more rigorous statement of Cauchy's theorem!

Rigorous statement of Cauchy's theorem Let γ be a closed contour, in a domain $\mathcal{U}$. Suppose $\mathcal{U}$ contains all of γ's interior points. Then $\int_\gamma f(z)\mathrm{d}z = 0$ for all holomorphic f on the set $\mathcal{U}$.