Differentiation of power series

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I provide a proof that power series are differentiable inside their disc of convergence. More precisely, I prove that if

$$f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$$

is a power series, with radius of convergence $R > 0$, then f is holomorphc on $D(a, R)$ (or on all of $\mathbf{C}$ when $R = \infty$) and that

$$f'(z) = \sum_{n=1}^{\infty} b_n n (z-a)^{n-1}.$$

I have given this proof in the lectures in previous years, but it is rather fiddly, so I want to avoid it in order to deal with more genuinely complex-analytic topics in a more leisurely manner.

By replacing $f(z)$ by $f(z+a)$ we can assume that $a = 0$. Define the power series

$$f_1(z) = \sum_{n=1}^{\infty} b_n n z^{n-1} \qquad \text{and} \qquad f_2(z) = \sum_{n=2}^{\infty} b_n n(n-1) z^{n-2}.$$

We want to prove that f_1 is the derivative of f; also f_2 should be the second derivative of f, but at the moment f_1 and f_2 are just two more power series.

If $z \in D(0, R)$ (we think of $D(0, \infty)$ as $\mathbf{C}$) then $|z| < R$ and so there is a slightly smaller radius r, with $|z| < r < R$ and $z \in D(0, r)$. Then $\sum_{n=0}^{\infty} |b_n r^n| = \sum_{n=0}^{\infty} |b_n| r^n$ is convergent. The terms of a convergent series tend to zero, so they are bounded: there is a positive real B with $|b_n| r^n < B$ for all n. If $z \in D(0, r)$ then $|z/r| < 1$. Now

$$|n b_n z^{n-1}| = n r^{n-1} |b_n| |z/r|^n \le r^{-1} n B |z/r|^n.$$

The series $\sum_{n=1}^{\infty} nB|z/r|^n$ is convergent, by the ratio test, and so the series $f_1(z)$ converges absolutely (by comparison) for $z \in D(0, r)$. A similar argument (omitted) shows that the series $f_2(z)$ converges absolutely in $D(0, r)$.

Consider $z \in D(0,r)$. To show that $f'(z) = f_1(z)$ we need to prove that

$$\lim_{h\to 0}\left|\frac{f(z+h)-f(h)}{h} - f_1(z)\right| = 0.$$

For small enough h, $z+h \in D(0,r)$ so that

$$f(z+h)-f(z) = \sum_{n=0}^{\infty} b_n((z+h)^n - z^n) = \sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} \binom{n}{k} h^k z^{n-k}$$

and so

$$\frac{f(z+h)-f(z)}{h} - f_1(z) = \sum_{n=1}^{\infty} b_n \sum_{k=2}^{n} \binom{n}{k} h^{k-1} z^{n-k}.$$

For $k \geq 2$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n(n-1)}{k(k-1)}\binom{n-2}{k-2} \leq n(n-1)\binom{n-2}{k-2}.$$

Therefore

$$\begin{aligned}
\left|\frac{f(z+h)-f(z)}{h} - f_1(z)\right| &\leq \sum_{n=2}^{\infty} |b_n| \sum_{k=2}^{\infty} \binom{n}{k} |h|^{k-1}|z|^{n-k} \\
&\leq |h| \sum_{n=2}^{\infty} n(n-1)|b_n| \sum_{k=2}^{\infty} \binom{n-2}{k-2} |h|^{k-2}|z|^{n-k} \\
&= |h| \sum_{n=2}^{\infty} n(n-1)|b_n|(|z|+|h|)^{n-2}.
\end{aligned}$$

This last series is convergent for small h since the series $f_2(|z|+|h|)$ is convergent. More precisely, if $|z| < s < r$ then for $|h| < s - |z|$ we have

$$\left|\frac{f(z+h)-f(z)}{h} - f_1(z)\right| \leq |h| \sum_{n=2}^{\infty} n(n-1)|b_n| s^{n-2} = C|h|$$

say, since the series is convergent. By the sandwich principle,

$$\lim_{h\to 0}\left|\frac{f(z+h)-f(z)}{h} - f_1(z)\right| = 0$$

and so $f'(z) = f_1(z)$, as required.