Complex analysis: Problems

1. Find the real part, the imaginary part, the absolute value, the principal argument and the complex conjugate of the following complex numbers:

$$\text{(a) } z_1 = 5 - 12i, \qquad \text{(b) } z_2 = \frac{19+8i}{3-4i}, \qquad \text{(c) } z_3 = e^{2+11\pi i/6}.$$

2. (a) Sketch all solutions of $z^6 = -8$ in the complex plane and write them in the form $a + bi$.
   (b) Find all solutions of $2z^2 - (1+i)/z^2 = 0$ in polar form and sketch them in the complex plane.

3. Prove that $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$ for each complex number z.

4. Prove that if z, $w \in \mathbf{C}$ then

$$|z + w| \geq |z| - |w|.$$

   You may either prove this from scratch, or deduce it from the traingle inequality. (This is an incredibly useful inequality which I urge you to remember.)

5. (a) Let w be a complex number with $w \notin \mathbf{R}$. Mark the complex numbers 0, w, $|w|$ and $w + |w|$ on the complex plane, and prove that they form the vertices of a rhombus. Deduce, geometrically, that $\operatorname{Arg}(w + |w|) = \frac{1}{2}\operatorname{Arg}(w)$ and that there is a positive real number t such that $z = t(w + |w|)$ is a solution of $z^2 = w$.
   (b) Solve the following equations

$$\text{(a) } z^2 = 5 - 12i, \qquad \text{(b) } z^2 = 4 - 3i, \qquad \text{(c) } z^2 = 3 + 3i$$

   in the form $z = \pm(x + yi)$.
   (c) Solve the quadratic equation $z^2 - iz - 7 - 9i = 0$.
   (d) Find all complex solutions of

$$z^3 + (3 - 2i)z - 6 + 18i = 0$$

   given that $z = 3i$ is one solution.

6. A *root of unity* is a complex number z satisfying $z^n = 1$ for some positive integer n. Prove that if z and w are roots of unity then so are z^{-1} and zw.

   Show that for each n there are precisely n solutions to $z^n = 1$; these are the *n-th roots of unity*. What is their sum? What is their product? (The answers will depend on n.)

7. Prove that there is no function $f : \mathbf{C} \to \mathbf{C}$ satisfying both the following properties

   (i) $f(z)^2 = z$ for all $z \in \mathbf{C}$,

   (ii) $f(zw) = f(z)f(w)$ for all z, $w \in \mathbf{C}$.

   However prove that there is more than one function which satisfies (i), and there is more than one function which satisfies (ii).

8. Prove that for nonzero complex numbers z and w,

$$\mathrm{Arg}(zw) - \mathrm{Arg}(z) - \mathrm{Arg}(w) \in \{-2\pi, 0, 2\pi\}$$

   and that each of these three values can occur.

9. (Cocycle relation for arguments.) For nonzero complex numbers z and w define

$$c(z, w) = \mathrm{Arg}(zw) - \mathrm{Arg}(z) - \mathrm{Arg}(w).$$

   Prove that for nonzero comples z, w and u that

$$c(z, w) + c(zw, u) = c(z, wu) + c(w, u)$$

   holds.

10. Prove that for real x and y,

$$|\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y$$

   and find a similar formula for $|\cos(x + iy)|^2$.

11. Prove that the set

$$\left\{ z \in \mathbf{C} : \left| \frac{z-1}{z-i} \right| = 3 \right\}$$

   is a circle, and find its centre and radius.

12. Let n be a positive integer, and set $\zeta = \exp(2\pi i/n)$. Prove that

$$z^n - 1 = \prod_{j=0}^{n-1} (z - \zeta^j).$$

Deduce that

$$n = \prod_{j=1}^{n-1} (1 - \zeta^j).$$

13. (Only look at this if you've done my Number Theory course.) Let p be an odd prime, and let $\zeta = \exp(2\pi i/p)$. Define

$$G(z) = \prod_{j=1}^{(p-1)/2} (z^j - z^{-j}).$$

Prove that

$$|G(\zeta)|^2 = p$$

and deduce that

$$G(\zeta) = i^{(p-1)/2}\sqrt{p}.$$

Also prove that

$$G(\zeta^a) = \left(\frac{a}{p}\right) G(\zeta)$$

whenever $a \in \mathbf{Z}$. (This is the start of a complex analytic proof of the law of quadratic reciprocity.)

14. Which of the following subsets of $\mathbf{C}$ are open, which are closed? Justify your answers.

(a) $\{z \in \mathbf{C} \ : \ 0 < \mathrm{Re}(z) \leq 1, \ -1 \leq \mathrm{Im}(z) < 0\}$,

(b) $\{z \in \mathbf{C} \ : \ 1 \leq |z| < 2\}$,

(c) $\{(1+i)t : t \in \mathbf{R}\}$.

15. Let U and V be open subsets of $\mathbf{C}$. Prove that $U \cup V$ and $U \cap V$ are also open subsets of $\mathbf{C}$. Also prove that if (U_n) is a sequence of open subsets of $\mathbf{C}$ then $\bigcup_{n=1}^{\infty} U_n$ is open. If all the U_n are open is it necessarily the case that $\bigcap_{n=1}^{\infty} U_n$ is open?

16. For each of the following subsets of $\mathbf{C}$ decide whether it is open, and if so whether it is a domain. Justify your answers.

(a) $D(-i, 1) \cup D(i, 1)$,

(b) $\overline{D}(-i,1) \cup D(i,1)$,

(c) $D(-i,1) \cup D(i,1) \cup \{0\}$,

(d) $D(-i,1) \cup D(i,1) \cup D(0,1/1000)$.

(NB here $D(a,r)$ denotes the open disc (and $\overline{D}(a,r)$ the closed disc) with centre a and radius r.)

17. Let A be a closed subset of $\mathbf{C}$, and let (a_n) be a sequence of points of A converging to a complex number a. Prove that $a \in A$. (Hint: show that a cannot be an exterior point of A).

18. Let U be an open subset of $\mathbf{C}$, and let $a \in U$. Let V be the set of all $w \in U$ that can be joined to a by a sequence of line segments contained within U. Prove that V is also an open subset of $\mathbf{C}$.

19. Recall the definition of convergence in $\mathbf{C}$: the sequence (z_n) converges to z if $|z_n - z| \to 0$. Prove directly from the definition that (z_n) converges to z if and only if both $(\operatorname{Re} z_n)$ converges to $\operatorname{Re} z$ and $(\operatorname{Im} z_n)$ converges to $\operatorname{Im} z$. You may appeal to theorems regarding convergence of real series, but if you appeal to theorems regarding convergence of complex series, you should prove them.

20. For each of the following sequences $z_1, z_2, z_3, \ldots$, find its limit or prove that it does not converge.

    (a) $z_n = \dfrac{2n^2 + in - 1}{(2+i)n^2 - (1+i)n - 1 + 2i}$,

    (b) $z_n = ((3+4i)/5)^n + (\sin n)/n$,

    (c) $z_n = ((1-2i)/3)^n$.

21. (a) Determine
$$\lim_{z \to i} \frac{z^2 - 3iz - 2}{z^2 + 1}.$$

    (b) Prove that the function $f(z) = |z|$ is continuous on $\mathbf{C}$.

    (c) Prove that the function $f : \mathbf{C} \setminus \{0\} \to \mathbf{C}$, $f(z) = z/|z|$ is continuous. Describe the function (which values does it take? Which z give the same value for $f(z)$?). Does the limit $\lim_{z \to 0} f(z)$ exist?

22. (Continuity via open sets.) Let U be an open subset of $\mathbf{C}$ and let $f : U \to \mathbf{C}$ be a function. Prove that f is continuous if and only if for every open subset V of $\mathbf{C}$, the set $f^{-1}(V)$, which is defined by
$$f^{-1}(V) = \{z \in U : f(z) \in V\}$$

is an open subset of $\mathbf{C}$.

23. Let
$$f(z) = \begin{cases} z^5/|z|^4 & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$
Prove that the Cauchy-Riemann equations for f hold at the point $z = 0$, but that f is **not** differentiable at $z = 0$.

24. Let $f(z) = f(x + iy) = xy$.
    (a) For which values of z do the the Cauchy-Riemann equations hold for f?
    (b) Is f differentiable at any point $z \neq 0$?
    (c) Show that f is differentiable at 0 and find $f'(0)$.

25. Let f be a holomorphic function on $\mathbf{C}$ such that $\operatorname{Re} f(z) = \operatorname{Im} f(z)$ for all z. Show that f is constant.

26. As ever, write $f(x + iy) = u(x, y) + iv(x, y)$ and suppose u and v have continuous second partial derivatives in a domain D. If f is holomorphic on D, show that u satisfies *Laplace's equation*
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$
A function satisfying Laplace's equation is called *harmonic*. Show that v is also harmonic.

27. In each case find a holomorphic function f on the domain D with $\operatorname{Re}(f(x + iy)) = u(x, y)$:
    (a) $D = \mathbf{C}$, $u(x, y) = x^3 + y^3 - 3xy^2 - 3x^2y$;
    (b) $D = \mathbf{C}$, $u(x, y) = \sin x \sinh y$;
    (c) $D = \mathbf{C} \setminus \{0\}$, $u(x, y) = (x + y)/(x^2 + y^2)$.

28. Determine where the following functions are holomorphic and calculate their derivatives:
$$\text{(a) } f(z) = \frac{z^2 + 2iz - 3 + 2i}{z - i}; \qquad \text{(b) } f(z) = \cos(e^{1/z});$$
$$\text{(c) } f(z) = e^{z-1/z}; \qquad \text{(d) } f(z) = \frac{z}{e^z + 1}.$$

29. Define
$$f(z) = \exp\left(\frac{1}{2}\operatorname{Log}(z)\right)$$
where $\operatorname{Log}(z)$ denotes the principal logarithm. Prove that $f(z)^2 = z$ for all $z \in \mathbf{C}$ but that f is not continuous at -1. However, show that on the "slit plane"
$$U = \mathbf{C} \setminus \{x \in \mathbf{R} : x \leq 0\}$$
f is holomorphic. (You may assume that U is a domain, and that Log is holomorphic on U.)

(This illustrates that a "principal square root" function is holomorphic on U but can't be extended holomorphically to the whole complex plane. The same is true for "power" functions $z \mapsto z^a$ when a is not an integer.)

30. Sketch the following paths in $\mathbf{C}$:

    (a) $\gamma(t) = t^2 + it, \;\; t \in [-1, 1]$;

    (b) $\gamma(t) = t^2 + it^2$, $t \in [0, 1]$;

    (c) $\gamma(t) = 2\cos(t) + 3i\sin(t), \;\; t \in [0, 2\pi]$.

31. Consider the path
$$\gamma_a(t) = e^{(-1/5+2i)t}$$
for $t \in [0, a]$. Sketch it for $a = 3\pi$. What is the common name for this kind of curve? What is its length. What is the the limit of its length as a tends to ∞?

32. (Reparameterization) Let $\gamma : [a, b] \to \mathbf{C}$ be a smooth path, and f a holomorphic function defined on γ. Let $\phi : [c, d] \to [a, b]$ be a function with a continuous derivative satisfying $\phi(c) = a$ and $\phi(d) = b$. Define $\delta = \gamma \circ \phi : [c, d] \to \mathbf{C}$. Then δ is a path in $\mathbf{C}$ with the same image as γ. Prove that
$$\int_\delta f(z)\, dz = \int_\gamma f(z)\, dz.$$

33. (Integration by parts) Let C be a closed contour, and let f and g be holomorphic functions defined on C. Prove that
$$\int_C f(z)g'(z)\, dz = -\int_C f'(z)g(z)\, dz.$$
What if C is a contour that isn't closed?

34. Evaluate $\int_\gamma \mathrm{Re}(z)\,dz$ for the following contours γ from 0 to $1+i$:

(a) the straight line from 0 to $1+i$;

(b) the straight line from 0 to 1 followed by the straight line from 1 to $1+i$.

35. Evaluate the integral $\int_\gamma f(z)\,dz$ where:

(a) $f(z) = \overline{z}$ and γ is the unit circle taken anticlockwise;

(b) $f(z) = \mathrm{Log}\, z$ and γ is the semicircular arc $\gamma(t) = 2e^{it}$, $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$;

(c) $f(z) = \sin z$ and γ is the line segment from $1-2i$ to $1+2i$;

(d) $f(z) = (z-1)^2/z$ and γ is the unit circle taken anticlockwise;

(e) $f(z) = (z-1)^2/z$ and γ is the circle with centre 2 and radius 1, traversed anticlockwise.

36. Let $\gamma : [a,b] \to \mathbf{C} \setminus \{0\}$ be a path. Define

$$G(u) = \int_a^u \frac{\gamma'(t)}{\gamma(t)}\,dt.$$

Prove that $\gamma(u)\exp(-G(u))$ is a constant (independent of u), and deduce that

$$\exp\left(\int_\gamma \frac{dz}{z}\right) = \frac{\gamma(b)}{\gamma(a)}.$$

Conclude that if γ is a closed contour in $\mathbf{C} \setminus \{0\}$ then

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$$

is an integer. More generally, if γ is a constant and w is a point not on γ then prove that

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w}$$

is an integer (this is the *winding number* of γ about w).

37. Calculate $\int_\gamma \frac{dz}{z}$ for the following paths from $3+4i$ to $3-4i$:

(a) the major arc of the circle with centre 0 and radius 5 going anticlockwise from $3+4i$ to $3-4i$;

(b) the straight line from $3+4i$ to $3-4i$.

38. Let C be the circle with centre 0 and radius R. Show that if $R > 1$ then
$$\left|\int_C \frac{e^z}{z^3+z}dz\right| \le \frac{2\pi e^R}{R^2-1}.$$

39. Let C be the circle with centre 0 and radius R. Prove that
$$\left|\int_C \frac{z}{z^4+3z^2+2}\,dz\right| \le \frac{2\pi R^2}{(R^2-1)(R^2-2)}$$
provided that R is sufficently large.

40. Which of the following domains are star domains? Justify your answers, giving a star centre for each domain that is a star domain.

 (a) $\{z \in \mathbf{C} : |z| > 1\}$;

 (b) $\{x+iy : x, y \in \mathbf{R},\ \text{either } y \neq 0 \text{ or } |x| < 1\}$;

 (c) $\{z \in \mathbf{C} : \mathrm{Re}(z) > 0\} \cup \{z \in \mathbf{C} : |z| < 1\}$.

41. Is there a star domain with only one star centre?

42. Evaluate
$$\int_C \frac{\sin \pi z}{(z+1/2)^6}\,dz$$
where C is the circle with centre 0 and radius 1, traversed once anti-clockwise.

43. Evaluate
$$\int_C \frac{z^2+z+1}{z^2(z-1)}\,dz$$
where C is the circle with centre 0 and radius 2, traversed once anti-clockwise. (Hint: use partial fractions.)

44. (The Poisson integral formula) Let f be holomorphic on a domain containing the set $\overline{D}(0,1)$ and let γ be the unit circle. Let $w \in D(0,1)$. Prove that
$$f(w) = \frac{1-|w|^2}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)(1-\overline{w}z)}\,dz$$
and deduce that
$$f(re^{i\theta}) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{f(e^{it})}{1-2r\cos(\theta-t)+r^2}\,dt$$
whenever $0 \le r < 1$ and $\theta \in \mathbf{R}$.

45. (Liouville's theorem and Cauchy's estimates)

   (a) Let f be an entire function with $|f(z)| > 2014$ for all $z \in \mathbf{C}$. Prove that f is constant.

   (b) Let f be an entire function. Suppose that there exist constants A, $B \geq 0$ such that
   $$|f(z)| \leq A + B|z|^{\frac{1}{3}}$$
   for all $z \in \mathbf{C}$. Prove that f must be constant. Deduce that there is no entire function f such that $f(z)^3 = z$ for all z.

   (c) Let f be an entire function. Suppose that s
   $$|f(z)|^3 \leq |z|^7$$
   for all z with $|z| \geq 1$. Prove that f is a polynomial of degree at most 2. Also prove that $f(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2$ where each $|\alpha_j| \leq 1$.

46. (a) Let $P(X) = \sum\limits_{k=0}^{m} a_k X^k$ be a polynomial with **real** coefficients. Prove that if $z_0 \in \mathbf{C}$ satisfies $P(z_0) = 0$ then $P(\overline{z_0}) = 0$.

   (b) Deduce that every nonzero real polynomial $P(X)$ is a product of real polynomials each of degree at most 2.

47. From `www.jstor.org` download the paper:
R. P. Boas, Jr., 'Yet Another Proof of the Fundamental Theorem of Algebra', *The American Mathematical Monthly*, Vol. 71, No. 2 (Feb., 1964), p. 180.
(You may need to be on campus to do this). Explain in detail what Boas means by the polynomial $\overline{P}(z)$ and prove that $P(z)\overline{P}(z)$ has real coefficients. Also prove that if $P(z)\overline{P}(z) = 0$ has a complex solution then so does $P(z) = 0$. Explain in detail why
$$\int_0^{2\pi} \frac{d\theta}{P(2\cos\theta)} = \frac{1}{i}\int_\gamma \frac{dz}{zP(z+1/z)}.$$
Why is $Q(z)$ a polynomial, and why has $Q(z) = 0$ no solution with $z \neq 0$?

48. Determine the radius of convergence of each of the following power series:

(a) $\displaystyle\sum_{n=0}^{\infty} n! z^n$; (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^n} z^n$; (c) $\displaystyle\sum_{n=0}^{\infty} i^n \frac{(3n)!^3}{(2n)!^2} z^n$; (d) $\displaystyle\sum_{n=0}^{\infty} \frac{z^{2n}}{(2+i)^n}$.

49. What is the largest open disc on which the series

$$\sum_{n=1}^{\infty} \frac{(z-2)^n}{3^n n}$$

converges? Let $f(z)$ be the holomorphic function given by this power series on that disc. Determine $f'(0)$ and $f'(1)$.

50. Find the Taylor series around $z = 0$ of the following functions. Also find the radius of convergence of each.

$$\text{(a) } f(z) = \sin((1+i)z); \qquad \text{(b) } f(z) = \frac{1}{1-2iz};$$

$$\text{(c) } f(z) = \frac{z^2}{(1+z)^3}.$$

51. For any complex number α, define

$$f_\alpha(z) = \exp(\alpha \operatorname{Log}(z)).$$

Prove that f_α is holomorphic on the slit plane $U = \mathbf{C} \setminus \{x \in \mathbf{R} : x \le 0\}$ and that $f'_\alpha(z) = \alpha f_{\alpha-1}(z)$. Also find the coefficients c_n in the Taylor series:

$$f_\alpha(z+1) = \sum_{n=0}^{\infty} c_n z^n.$$

52. Find the first few terms (up to the z^3 term) in the Laurent series of

$$f(z) = \frac{\sin z}{1 - \cosh z}$$

valid in some annulus $\{z \in \mathbf{C} : 0 < |z| < r\}$. What is the residue of f at 0?

53. Prove that there is no non-constant function f holomorphic on $U = \mathbf{C} \setminus \{0\}$ with the property that $f(z) = f(2z)$ for all $z \in U$. However show that there is a nonzero holomorphic function g on U satisfying $g(2z) = \sqrt{2}\, z g(z)$. (Hint: consider Laurent series.)

54. Find the Laurent expansions of the function f in the stated annular domains:

(a) $f(z) = (z+2)^{-2}(z-1)^{-1}$ for $0 < |z| < 1$;

(b) $f(z) = (z+2)^{-2}(z-1)^{-1}$ for $1 < |z| < 2$;

(c) $f(z) = (z+2)^{-2}(z-1)^{-1}$ for $|z| > 2$;

(d) $f(z) = \cos(2/z)$ for $|z| > 0$;

(e) $f(z) = e^{-z}/(1+z)$ for $0 < |z| < 1$.

55. Determine the nature of the singularity (removable, pole, or essential) at 0 of the following functions. In case of a pole determine its order.

(a) $f(z) = \dfrac{e^z - 1}{\tan z}$; (b) $f(z) = \sin(1/z)$;

(c) $\dfrac{z^3}{2\cosh(z^2) - z^4 - 2}$; (d) $f(z) = \dfrac{1 - \cosh z}{z - \sinh z}$.

56. Let f be a holomorphic function with an isolated singularity at $a \in \mathbf{C}$. Suppose also that

$$\lim_{z \to a} (z-a) f(z) = 0.$$

Without using Laurent's theorem, prove that g defined by

$$g(z) = \begin{cases} (z-a)^2 f(z) & \text{if } z \neq a \\ 0 & \text{if } z = a \end{cases}$$

is holomorphic at a. Applying Taylor's theorem to g at a, prove that f has a removable singularity at a.

57. Let f be holomorphic in $\mathbf{C} \setminus \{0\}$. Assume that $\int_C f(z)\,dz = 0$ where C is the unit circle. Decide whether the following statements are true or false. For each one give a proof or a counterexample.

(a) f has a removable singularity at 0;

(b) f cannot have a simple pole at 0;

(c) f has a multiple pole at 0;

(d) f cannot have an essential singularity at 0.

58. Prove Jordan's inequality:

$$\sin t > \frac{2t}{\pi}$$

for $0 < t < \frac{\pi}{2}$.

59. Evaluate the following integrals using the calculus of residues:

(a) $\displaystyle\int_0^{2\pi} \sin^8 t\, dt$ (b) $\displaystyle\int_0^{\pi} \frac{\cos^2 t}{5+4\cos t}\, dt,$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2-8x+20},$ (d) $\displaystyle\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+9)},$

(e) $\displaystyle\int_0^{\infty} \frac{x^4\, dx}{(x^2+1)^3},$ (f) $\displaystyle\int_0^{\infty} \frac{\sin x}{x(x^2+1)}\, dx,$

(g) $\displaystyle\int_0^{\infty} \frac{x\sin x}{x^2+1}\, dx,$ (h) $\displaystyle\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\, dx,$

(i) $\displaystyle\int_0^{\infty} \frac{dx}{x^{2014}+1},$ (j) $\displaystyle\int_0^{\infty} \frac{x^{67}\, dx}{x^{100}+1}.$

60. In the lectures, I evaluate $\int_0^\infty \sin x\, dx/x$ by integrating e^{iz}/z over an indented semicircular contour in the upper half plane. What would have gone wrong if instead I'd tried to prove it by integrating e^{-iz}/z over the same contour?

61. In the lectures, I evaluate $\int_0^\infty \sin x\, dx/x$ by integrating e^{iz}/z over a large semicircle indented to avoid the pole at zero. This necessitated using Jordan's inequality to estimate the integral over the large semi-circular arc. Show that one can avoid Jordan's lemma by using a different contour: take the contour to be the rectangle with vertices $\pm R$ and $\pm R + iR$ but again indented to avoid zero.

62. Evaluate $\sum_{n=1}^\infty 1/n^4$ by the same method used for $\sum_{n=1}^\infty 1/n^2$ in the lectures. Why cannot the same method be used to evaluate $\sum_{n=1}^\infty 1/n^3$?

63. By integrating

$$f(z) = \frac{1}{z^3\cos(\pi z)}$$

over a square S_N (where $N \in \mathbf{N}$) with vertices at $N(1+i)$, $N(-1+i)$, $N(-1-i)$ and $N(1-i)$ prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

64. Calculate

$$\int_0^\infty x^3 e^{-x}\cos x\, dx \qquad \text{and} \qquad \int_0^\infty x^3 e^{-x}\sin x\, dx$$

by integrating $f(z) = z^3 e^{-z}$ along the boundary of the sector $|z| \le R$, $0 \le \arg(z) \le \frac{\pi}{4}$.

65. (Maximum Modulus Theorem, weak version) Let f be holomorphic in a domain U. Let $a \in U$. Show that there cannot exist $\varepsilon > 0$ such that $D(a, \varepsilon) \subseteq U$ and such that $|f(z)| < |f(a)|$ whenever $0 < |z - a| < \varepsilon$. In other words, $|f|$ cannot have a strict local maximum.

66. (Rouché's theorem: here we count all zeros according to multiplicity.)

    (a) How many zeroes has the polynomial

    $$p(z) = z^5 + 3iz^4 + 3iz^2 + 3$$

    in the disc $D(0, 2)$?

    (b) How many zeroes has the polynomial

    $$q(z) = z^5 - 5z^4 + z^2 - 9z + 1$$

    in the annulus $1 < |z| < 2$.

    (c) How many zeroes has

    $$f(z) = e^z - 3z^{2014}$$

    in the disc $D(0, 1)$.

67. (Only look at this if you've done my Combinatorics course.) Let q be a complex number with $0 < |q| < 1$ and define

    $$f(z) = \sum_{m=-\infty}^{\infty} q^{m^2} z^m.$$

    (You may assume that f is holomorphic on $\mathbf{C} \setminus \{0\}$.) Prove that

    $$f(q^2 z) = q^{-1} z^{-1} f(z)$$

    and deduce that $-q^a$ is a zero of f whenever a is an odd integer. Also prove that each $-q^a$ is a simple zero of f and that f has no other zeros in $\mathbf{C} \setminus \{0\}$. (This is the start of a complex analytic proof of the Jacobi triple product formula.)

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