Fisher's inequality

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This was proved by R.A. Fisheer in 1940 and states that

> if $(X, \mathcal{B})$ is a 2-(v, k, λ) design with $v > k \geq 2$ and having b blocks then $b \geq v$.

The usual approach is to assume the contrary, that is, that $b < v$ and derive a contradiction. This we do.

Label the points as $P_1, \ldots, P_v$ and the blocks as $B_1, \ldots, B_b$. We define the *incidence matrix* M as follows: it is a b-by-v matrix whose (i, j)-entry is

$$m_{i,j} = \begin{cases} 1 & \text{if } P_j \in B_i, \\ 0 & \text{if } P_j \notin B_i. \end{cases}$$

Define $N = M^t M$ where M^t is the transpose of M. Then N is a v-by-v matrix with (j, k)-entry

$$n_{j,k} = \sum_{i=1}^{b} m_{i,j} m_{i,k}.$$

Now $n_{j,k} = 1$ if both $P_j \in B_i$ and $P_k \in B_i$, and $n_{j,k} = 0$ otherwise. Therefore $n_{j,k}$ is the number of blocks containing both the points P_j and P_k. When $j \neq k$, then $n_{j,k} = \lambda$ by the definition of design. On the other hand $n_{j,j}$ is the number of blocks containing P_j. By an earlier theorem,

$$n_{j,j} = b' = \lambda \frac{\binom{v-1}{1}}{\binom{k-1}{1}} = \lambda \frac{v-1}{k-1}.$$

Therefore $b' > \lambda > 0$.

We have shown that

$$N = (b' - \lambda)I + \lambda J$$

where I is the v-by-v identity matrix and J is the v-by-v matrix consisting entirely of 1s. We claim that N is a non-singular matrix. Most texts prove this by computing its determinant. I'll do it by instead finding an inverse for N. You might want to look up the determinant computation in the literature (or do it yourself!),

Fairly obviously,

$$J^2 = vJ$$

and so

$$JN = (b' - \lambda)J + v\lambda J = (b' - \lambda + v\lambda)J.$$

As $b' > \lambda$ and $\lambda > 0$ then $b' - \lambda + v\lambda > 0$ and so

$$J = \frac{1}{b' - \lambda + v\lambda} JN.$$

Then

$$(b' - \lambda)I = N - \lambda J = \left(I - \frac{\lambda}{b' - \lambda + v\lambda} J \right) N.$$

As $b' - \lambda > 0$, then N has the inverse

$$N^{-1} = \frac{1}{b' - \lambda} \left(I - \frac{\lambda}{b' - \lambda + v\lambda} J \right).$$

Recall we are assuming that $b < v$. This means there is a nonzero vector $\mathbf{x}$ with $M\mathbf{x} = 0$ (reduce M to echelon form...). Then $N\mathbf{x} = M^t M\mathbf{x} = 0$ and so $\mathbf{x} = I\mathbf{x} = N^{-1}N\mathbf{x} = 0$. This contradiction shows that the hypothesis $b < v$ is untenable. Therefore we have proved Fisher's inequality: $b \geq v$.

I set Fisher's inequality as an exam question in 2003.